The Link Between the Infinite Mapping Class Group of the Disk and the Braid Group on Infinitely Many Strands

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Abstract

For all finite $n \in \mathbb{N}$, there is a well-known isomorphism

$$\pi_0\overline{\varphi}_n : \pi_0 H_n \xrightarrow{\cong} B_n$$

between the standard braid group B_n and the mapping class group $\pi_0 H_n$. This isomorphism has been exhaustively studied in literature, and generalized in many ways. For some basic topological reason, this strong link between finite braid groups and finite mapping class groups can-not be extended to the infinite case in a straightforward way, and, in particular, is not yet well studied in literature.
In our work, we define the infinite braid group B_∞ to be the group of braids with infinitely many strands, all of which can be possibly nontrivial, i.e., not straight. In particular, this definition does not correspond to the group of finitary infinite braids, which is just the union of all finite braid groups. Similar to the maps $\pi_0\overline{\varphi}_n$ for finite n, we introduce a map

$$\pi_0\overline{\varphi}_\infty : \pi_0 H_\infty \to B_\infty$$

that, in particular, turns out not to be an isomorphism. However, we prove its injectivity, and identify its image in B_∞.
The study of the link between mapping class groups and braid groups in the infinite case is motivated by the study of homeomorphisms in H_∞ that give rise to a homoclinic tangle. In fact, the map $\pi_0\overline{\varphi}_\infty$ attributes to each isotopy class of such a homeomorphism an element of the infinite braid group B_∞, and so, allows us to describe the isotopy classes of these homeomorphisms in terms of their image in B_∞. Using the fact that the map $\pi_0\varphi_\infty$ is injective, we prove a result that can be applied to the study of the topological structure of homoclinic tangles.

Keywords: Infinite braid group, infinite mapping class group, infinite permutation group, homoclinic tangles.

Version abrégée

Il est bien connu que, pour tout $n \in \mathbb{N}$, il existe un isomorphisme

$$\pi_0\overline{\varphi}_n : \pi_0 H_n \xrightarrow{\cong} B_n$$

entre le groupe de tresses B_n et le mapping class group $\pi_0 H_n$. Cet isomorphisme est etudié en profondeur dans la littérature, et largement généralisé dans divers contextes. Pour des raisons de topologie de base, il n'existe pas une façon directe détendre ce lien entre les groupes de tresses et mapping class groups finis au cas infini, et, en particulier, n'a pas encore été étudié dans littérature.
Dans notre travail, nous définissons le groupe de tresses infini B_∞ par le groupe de tresses d'une infinité de brins, qui peuvent être simultanément nontriviaux, c'est à dire non droits. En particulier, cette définition ne correspond pas au groupe de tresses finitairement infini, qui est simplement la réunion de tous les groupes de tresses finis. Semblable aux applications $\pi_0\overline{\varphi}_n$ pour n fini, nous introduisons une application

$$\pi_0\overline{\varphi}_\infty : \pi_0 H_\infty \to B_\infty$$

qui, en particulier, n'est pas un isomorphisme. Toutefois, nous prouvons son injectivité, et nous identifions son image dans B_∞.
L'étude du lien entre le mapping class groupe du disque et le groupe de tresses infinis est motivé par l'étude des homéomorphismes dans H_∞ qui donnent lieu à un entrelacement homocline. En effet, l'application $\pi_0\overline{\varphi}_\infty$ attribue à chaque classe d'isotopie d'un tel homéomorphisme un élément du groupe de tresses infini B_∞. De cette manière, l'application $\pi_0\overline{\varphi}_\infty$ permet de décrire les classes d'isotopie des ces homeomorphismes en termes de leur image dans B_∞. En utilisant l'injectivité de $\pi_0\varphi_\infty$, nous démontrons un résultat qui peut être appliqué à l'étude de la structure topologique des enchevêtrements homoclines.

Mots clés: Groupe de tresses infini, mapping class group infini, groupe de permutations infinies, enchevêtrements homoclines.

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Contents

Chapter 1

Foundations

1.1 Introduction

The foundation of the mathematical theory of braids goes back to 1925, when E. Artin introduced in [1] the classical braid groups B_n for finite $n \in \mathbb{N}$.
For each $n \in \mathbb{N}$, B_n is defined by

$$B_n := \pi_1 C_n \quad \forall n \in \mathbb{N},$$

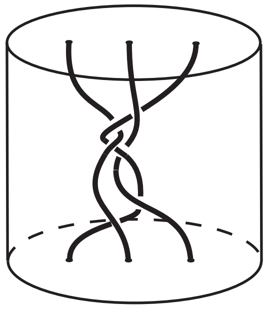
for some given basepoint in C_n, which is the space of unordered sequences of n pairwise distinct points in the interior of the disk $\overset{\circ}{D}{}^2$. More precisely, writing F_n for the space of ordered sequences of pairwise distinct points in $\overset{\circ}{D}{}^2$, C_n is the orbit space

$$C_n := F_n/\Sigma_n,$$

where Σ_n is the group of $n-$permutations, which acts on F_n by permutation of ordered sequences. A representative in ΩC_n of an element of B_n can thus be seen as a set of n strands in the cylinder $\overset{\circ}{D}{}^2 \times I$ that connects a given set $\big\{(\tau_i, 1)\big\}_{i\in[1,n]}$ of n pairwise distinct points on $\overset{\circ}{D}{}^2 \times \{1\}$ to the corresponding point set on

$$\big\{(\tau_i, 0)\big\}_{i\in[1,n]} \in \overset{\circ}{D}{}^2 \times \{0\},$$

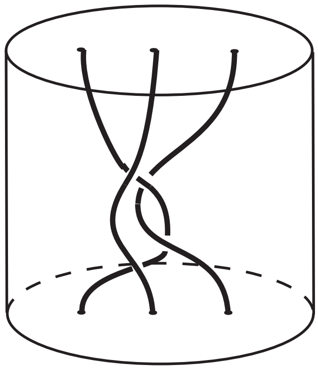
without intersecting each other, where, in particular, $(\tau_i)_{i\in[1,n]}$ is the basepoint of the space F_n.



Similarly to the definition of the braid group B_n, the *pure* braid group PB_n is defined by

$$PB_n := \pi_1 F_n \quad \forall n \in \mathbb{N}.$$

In particular, each element $b \in PB_n$ is represented by a *pure* braid $(\beta_i)_{i\in[1,n]} \in \Omega F_n$, which can be seen as a braid in a cylinder for each strand for which each strand has equal initial- and endpoint.



Moreover, the composition of elements of the groups B_n and PB_n is given by the concatenation of representative braids.
An important feature of the braid groups is their close connection to mapping class groups, which was already observed by Artin in [1, 2]. The mapping class group of the $n-$punctured unit disk D^2 is the group of path connected components $\pi_0 H_n$, where H_n is the topological group of all homeomorphisms $h : D^2 \to D^2$, that fix the boundary pointwise,

$$h|_{\partial D^2} = \mathrm{Id},$$

and that satisfy

$$h\big(\{\tau_i\}_{i\in[1,n]}\big) = \{\tau_i\}_{i\in[1,n]},$$

where the ordered sequence $(\tau_i)_{i\in[1,n]}$ corresponds to the basepoint of F_n. Similarly, for all $n \in \mathbb{N}$, the space PH_n is given by all homeomorphisms $h \in H_0$, that fix the set $\{\tau_i\}$ pointwise, i.e.,

$$h(\tau_i) = \tau_i \quad \forall i \in [1, n].$$

The group $\pi_0 PH_n$ of pathwise connected components of PH_n is called the *pure* mapping class group of the $n-$punctured disk.
For each $n \in \mathbb{N}$, there are maps

$$\overline{\varphi}_n : H_n \to \Omega C_n, \qquad \varphi_n : PH_n \to \Omega F_n,$$

that induce isomorphisms

$$\pi_0 \overline{\varphi}_n : \pi_0 H_n \xrightarrow{\cong} B_n, \qquad \pi_0 \varphi_n : \pi_0 PH_n \xrightarrow{\cong} PB_n,$$

respectively. A detailed introduction to this close connection between finite braid groups and finite mapping class groups is given in [3]
In our work, we consider the link between braid theory and mapping class groups in the infinite case. Similarly to the finite case, we define the infinite pure braid group by

$$PB_\infty := \pi_1 F_\infty,$$

where F_∞ is the space of ordered sequences of infinitely many pairwise distinct points in $\overset{\circ}{D}{}^2$. As in the finite case, we define the space C_∞ of unordered sequences of pairwise distinct points in $\overset{\circ}{D}{}^2$ by the orbit space

$$C_\infty := F_\infty / \Sigma_\infty,$$

where Σ_∞ is the group of bijections from $\mathbb{N}$ to itself that acts on F_∞ by permutation of ordered sequences of points. We have not yet been able to determine whether one can associate an unordered sequence of strands in $\overset{\circ}{D}{}^2$ to any loop in C_∞. To avoid this problem, we consider instead $\mathcal{O}C_\infty \subseteq \Omega C_\infty$, the space of those loops in C_∞ that to which one can associate such an unordered sequence of paths in $\overset{\circ}{D}{}^2$, and define the infinite braid group by

$$B_\infty := \pi_0 \mathcal{O} C_\infty.$$

Unlike the finitary infinite braid group, which is simply the union of all finite braid groups, the infinite braid group B_∞, i.e., the group of braids with infinitely many strands, all of which can be nontrivial simultaneously, seems rarely to have been considered in the literature, in particular not in the context of mapping class groups.
On the other hand, the infinite mapping class group and the infinite pure mapping class group of the disk are given by the groups of path connected components $\pi_0 PH_\infty$ and $\pi_0 H_\infty$ of the spaces

$$\begin{aligned} PH_\infty &:= \Big\{ h \in \mathcal{H}(D^2, D^2) \mid h|_{\partial D^2} = \mathrm{Id}, \quad h(\tau_i) = \tau_i \quad \forall i \in \mathbb{N} \Big\}, \\ H_\infty &:= \Big\{ h \in \mathcal{H}(D^2, D^2) \mid h|_{\partial D^2} = \mathrm{Id}, \quad h\big(\{\tau_i\}_{i \in \mathbb{N}}\big) = \{\tau_i\}_{i \in \mathbb{N}} \Big\}, \end{aligned}$$

respectively.
For all finite $n \in \mathbb{N}$, the groups PB_n, B_n, $\pi_0 PH_n$ and $\pi_0 H_n$ do not depend

on the choice of the point set $(\tau_i)_{i\in[1,n]}$. In the infinite case, this is is still the case for the (pure) braid group (PB_∞) B_∞, whereas, given two choices $(\tau_i)_{i\in\mathbb{N}}, (\widetilde{\tau}_i)_{i\in\mathbb{N}} \in F_\infty$ of the the basepoint of F_∞ where the underlying point sets $\{\tau_i\}_{i\in\mathbb{N}}$ and $\{\widetilde{\tau}_i\}_{i\in\mathbb{N}}$ have different numbers of accumulation points in D^2, the resulting (pure) mapping class groups $(\pi_0 PH_\infty, \pi_0\widetilde{PH}_\infty)$ $(\pi_0 H_\infty, \pi_0\widetilde{H}_\infty)$ are not isomorphic.
As in the finite case, there are maps

$$\overline{\varphi}_\infty : H_\infty \to \mathcal{O}C\infty, \qquad \varphi_\infty : PH_\infty \to \Omega F_\infty$$

where H_∞ is the space of homeomorphisms $h : D^2 \xrightarrow{\cong} D^2$ that fix the boundary of D^2 pointwise, and that fix a given point set $\{\tau_i\}_{i\in\mathbb{N}}$, and PH_∞ is the subspace of H_∞ of homeomorphisms that fix the set $\{\tau_i\}_{i\in\mathbb{N}}$ pointwise. The main objective of our work is to investigate the induced maps

$$\pi_0\overline{\varphi}_\infty : \pi_0 H_\infty \to B_\infty, \qquad \pi_0\varphi_\infty : \pi_0 PH_\infty \to PB_\infty.$$

For basic topological reasons, the approach to prove that the maps $\pi_0\overline{\varphi}_n$ and $\pi_0\varphi_n$ are isomorphisms for finite n cannot be extended to the infinite case. In particular, it turns out that, in contrast to the finite case, the maps $\pi_0\overline{\varphi}_\infty$ and $\pi_0\varphi_\infty$ are injective, but not surjective.
For the generalization of the maps $\pi_0\overline{\varphi}_n$ and $\pi_0\varphi_n$ to $n = \infty$, it would seem natural to use inverse systems of topological spaces and of groups. However, as we point out in section 1.1.3, there isn't any natural way to define an inverse system of braid groups

$$\cdots \to B_{n+1} \to B_n \to \ldots,$$

nor of mapping class groups

$$\cdots \to \pi_0 H_{n+1} \to \pi_0 H_n \to \ldots.$$

On the other hand, there is an inverse system of projection maps

$$PB_\infty \to \cdots \to PB_{n+1} \to PB_n \to \ldots$$

with limit PB_∞, and an inverse system of subspace inclusions

$$PH_\infty \hookrightarrow \ldots \hookrightarrow PH_{n+1} \hookrightarrow PH_n \hookrightarrow \ldots$$

with limit PH_∞.
Moreover, we show that there is a commutative diagram (see Theorem 2.19).

$$\begin{array}{ccc} \pi_0 H_\infty & \xrightarrow{\pi_0\overline{\varphi}_\infty} & B_\infty \\ \uparrow\cong & & \uparrow\cong \\ \Sigma_\infty \ltimes \pi_0 PH_\infty & \xrightarrow{\mathrm{Id}\ltimes\pi_0\varphi_\infty} & \Sigma_\infty \ltimes PB_\infty. \end{array}$$

Indeed, this diagram allows us to reduce the study of the map $\pi_0\overline{\varphi}_\infty$ to the study of the map $\pi_0\varphi_\infty$, which is easier to handle than $\pi_0\overline{\varphi}_\infty$, because its source and target are limits of inverse systems.
To prove the isomorphisms $B_\infty \cong \Sigma_\infty \ltimes PB_\infty$ and $\pi_0 H_\infty \cong \Sigma_\infty \ltimes \pi_0 PH_\infty$, which is given by Propositions 2.17, and 2.16, 2.18, respectively, requires knowledge of the group Σ_∞. In particular, Σ_∞ is not equal to the union of all finite permutation groups, and does not seem to have been well studied in the literature. We show how to canonically attribute to each element $\sigma \in \Sigma_\infty$ an infinite sequence of natural numbers $(s_{\sigma,i})_{i\in\mathbb{N}}$, such that, within a given topology of Σ_∞,

$$\sigma = \lim_{n\to\infty} \big[(n, s_n) \circ \cdots \circ (1, s_1)\big],$$

where, for each $i \in \mathbb{N}$, (i, s_i) is the transposition of i and s_i. In other words, we canonically decompose the elements of Σ_∞ into infinite sequences of transpositions (see section 2.1).
Thereafter, in chapter 3, we show that the map $\pi_0\varphi_\infty$ is injective (Theorem 3.7), which, by the above diagram, means that the map $\pi_0\overline{\varphi}_\infty$, too, is injective. In chapter 4, we identify the image of the map $\pi_0\varphi_\infty$. First, this is done by giving characteristic representatives in ΩF_∞ of the elements of the image of $\pi_0\varphi_\infty$ in PB_∞ (Corollary 4.6). Furthermore, we work towards an algebraic characterization of Im $\pi_0\varphi_\infty$ within a suitable codification of the group PB_∞ (see Section 4.3). In particular, for the codification of PB_∞, we make use of the braid groups of the punctured disk $\mathring{D}^2 \setminus 0$, where 0 is the center of D^2.
Once the image of $\pi_0\varphi_\infty$ is known, the above diagram allows us again to deduce that

$$\text{Im } \pi_0\overline{\varphi}_\infty \cong \Sigma_\infty \ltimes \text{Im } \pi_0\varphi_\infty.$$

The original motivation for the research in this thesis was its possible application to a particular branch of dynamical systems theory: the study of homoclinic tangles (see section 1.1.4 and chapter 5 for more details). A homoclinic tangle associated to a self-homeomorphism h of the unit disk is given by two one-dimensional manifolds in D^2 that intersect each other. Their intersection is a union of non-periodic, biassymptotic orbits of the homeomorphism in D^2, i.e., orbits $\big(h^i(\widehat{x})\big)_{i\in\mathbb{N}}$ with

$$\lim_{i\to\pm\infty} h^i(\widehat{x}) = x,$$

for some $x \in D^2$. When we define H_∞ such that the point set in $\mathring{D}^2$ that is fixed by the elements in H_∞ corresponds to such a non-periodic orbit of h, then, the map

$$\overline{\varphi}_\infty : H_\infty \to \Omega C_\infty,$$

associates an infinite braid to h. As the topological structure of a homoclinic tangle depends to a large extent on these non-periodic orbits, the study of the underlying homeomorphism in terms of infinite braid might be very useful. More precisely, knowledge of the map

$$\pi_0\overline{\varphi}_\infty : \pi_0 H_\infty \to B_\infty$$

may provide interesting information about self-homeomorphisms of the disk that give rise to homoclinic tangles, or, more generally, to non-periodic orbits. A first approach to such an application is given in chapter 5.

1.1.1 Basic definitions and elementary results

Let D^2 be the **unit disk** with interior $\mathring{D}^2$. We usually write

$$[1, n] := \{1, \ldots, n\}, \quad \text{and, for convenience,} \quad [1, \infty] := \mathbb{N}.$$

Definition 1.1. *For all $n \in \mathbb{N} \cup \infty$ and any space X, endow $\prod_{i=1}^{n} X$ with the product topology, and define a subspace $F_n(X)$ by*

$$F_n(X) := \Big\{ (x_i)_{i \in [1,n]} \subset \prod_{i=1}^{n} X \mid x_i \neq x_j \forall i \neq j \Big\}.$$

*This space is called the **configuration space of** n **points in** X.*

Definition 1.2. *For all $n \in \mathbb{N}$, let Σ_n be the symmetric group, and, as a set, define Σ_∞ to be given by the **bijections** of the underlying set of $\mathbb{N}$. Endow the mapping space $\mathbb{N}^\infty$ with the topology of **pointwise convergence**, and topologize Σ_∞ as a subspace of $\mathbb{N}^\infty$. For all $n \in \mathbb{N} \cup \infty$, define the group structure on Σ_n as usual by*

$$\big(\sigma_1 \cdot \sigma_2\big)(i) = \sigma_2\big(\sigma_1(i)\big) \quad \forall i \in [1, n], \;\; \forall \sigma_1, \sigma_2 \in \Sigma_n.$$

Observe that, for any space X and for each $n \in \mathbb{N} \cup \infty$, the symmetric group Σ_n acts on the right of $\prod_{i \in [1,n]} X$ by permutation of components, which, in particular, induces a right action of Σ_n on the subspace $F_n(X) \subset \prod_{i \in [1,n]} X$.

Definition 1.3. *Let X be a topological space, and write, for all $n \in \mathbb{N} \cup \infty$,*

$$C_n(X) := F_n(X)/\Sigma_n$$

*for the **orbit space** by factoring out the right action of the group Σ_n. Moreover, endow $C_n(X)$ with the quotient topology, and write*

$$p_n : F_n(X) \to C_n(X)$$

for the quotient map. As we often work with the space $\mathring{D}^2$, we write

$$F_n := F_n(\mathring{D}^2), \quad \text{and} \quad C_n(\mathring{D}^2) := C_n(\mathring{D}^2) \quad \forall n \in \mathbb{N} \cup \infty$$

for notational convenience.

Notation 1.4. *Choose an arbitrary basepoint $\mathcal{T}_\infty = (\tau_i)_{i \in \mathbb{N}}$ of F_∞, and let $\overline{\mathcal{T}}_\infty := p_\infty(\mathcal{T}_\infty)$ be the basepoint of C_∞. Moreover, define*

$$\mathcal{T}_n := (\tau_1, \ldots, \tau_n) \quad \text{and} \quad \overline{\mathcal{T}}_n := p_n(\mathcal{T}_n)$$

to be the basepoints of the spaces F_n and C_n for all finite n.

Later in this text, we make a particular choice for $\mathcal{T}_\infty$ in order to simplify the proof of certain results (see Definition 2.1). Thereafter, these results are generalized to a basepoint $\mathcal{T}_\infty = (\tau_i)_{i\in\mathbb{N}}$ such that, in D^2, the set $\{\tau_i\}_{i\in\mathbb{N}}$ has a single accumulation point $\tau_\infty \in \overset{\circ}{D}{}^2$.
Generally, F_n means the pointed space $(F_n, \mathcal{T}_n)$. When we endow F_n with a different basepoint $\widetilde{\mathcal{T}}$, we explicitly write $(F_n, \widetilde{\mathcal{T}}_n)$. Note that, by the path-connectedness of these spaces (see Proposition A.1), a change of the basepoint induces an isomorphism on their homotopy groups.

Proposition 1.5. *(Birman [3, Prop 1.1]) For every $n \in \mathbb{N}$, the quotient map*

$$p_n : F_n \to C_n$$

is a covering map with fiber Σ_n.

This means in particular that the projection $p_n : F_n \to C_n$ has the *path lifting property*. Thus, to any given braid $\beta \in \Omega C_n$, we can associate a unique path $(\beta_i)_{i\in[1,n]} \in \mathcal{C}(I, F_n)$, such that

$$\beta = p_n \circ (\beta_i)_{i\in[1,n]}, \quad \beta_i(0) = \tau_i \quad \forall i \in [1, n]$$

Moreover, writing

$$\big(\tau_{i_1}, \ldots, \tau_{i_n}\big) := \big(\beta_1(1), \ldots, \beta_n(1)\big),$$

we can associate to β the permutation $\sigma_\beta \in \Sigma_n$ defined by

$$\sigma_\beta := \begin{pmatrix} 1, & \ldots, & n \\ i_1, & \ldots, & i_n \end{pmatrix}.$$

Moreover, by the uniqueness of the lifting $(\beta_i)_{i\in[1,n]}$, this defines a well defined map

$$\begin{array}{rcl} \Omega C_n & \to & \Sigma_n \\ \beta & \mapsto & \sigma_\beta \end{array}$$

for any fixed choice of $(\tau_i)_{i\in[1,n]}$.
Note that Proposition 1.5 does not extend to $n = \infty$. Moreover, it seems that the projection $p_\infty : F_\infty \to C_\infty$ does not have the path lifting property, although we didn't yet find a counter example. On the other hand, if $\overline{\beta} \in \Omega C_\infty$ that lifts to a path β in F_∞, it is clear that

$$\beta \in \mathcal{C}\big((I, 0, 1), (F_\infty, \mathcal{T}_\infty\sigma_0, \mathcal{T}_\infty\sigma_1)\big)$$

for some $\sigma_0, \sigma_1 \in \Sigma_\infty$. Moreover, every $\beta \in \Omega F_\infty$ can be seen as a list $(\beta_i)_{i\in\mathbb{N}}$ of paths in $\overset{\circ}{D}{}^2$, whereas an element $\overline{\beta} \in \Omega C_\infty$ can be seen as the Σ_∞-orbit of a list of paths only if it lifts to a path β in $\mathcal{C}\big((I, 0, 1), (F_\infty, \mathcal{T}_\infty\sigma_0, \mathcal{T}_\infty\sigma_1)\big)$. In order to bypass this difficulty, we introduce a space $\mathcal{O}C_\infty$ as follows.

Definition 1.6. *Writing* $\mathcal{T}_\infty\Sigma_\infty$ *for the corresponding right coset of* $\mathcal{T}_\infty$*, introduce a space*

$$\mathcal{O}C_\infty := \mathcal{C}\big((I,\dot{I}),(F_\infty,\mathcal{T}\Sigma_\infty)\big)\Big/\Sigma_\infty,$$

where $\dot{I} := \{0,1\}$*, and equip it with the quotient topology.*

This definition allows for the following proposition, that is used repeatedly in the sequel.

Proposition 1.7. *Each element* $\overline{\beta} \in \mathcal{O}C_\infty$ *lifts to a unique path* $\beta =: (\beta_i)_{i\in\mathbb{N}} \in \mathcal{C}(I, F_\infty)$*, such that*

$$\overline{\beta} = p_\infty \circ \beta, \quad \textit{and} \quad \beta_i(1) = \tau_i \quad \forall i \in \mathbb{N}.$$

Proof. Pick an element $\overline{\beta} \in \mathcal{O}C_\infty$, and let $\widehat{\beta} =: (\widehat{\beta}_i)_{i\in\mathbb{N}}$ be a coset representative in $\mathcal{C}\big((I,\dot{I}),(F_\infty,\mathcal{T}_\infty\Sigma_\infty)\big)$, i.e., $\overline{\beta} = \widehat{\beta}\Sigma_\infty$. In particular,

$$\{\widehat{\beta}_i(0)\}_{i\in\mathbb{N}} = \{\widehat{\beta}_i(1)\}_{i\in\mathbb{N}} = \{\tau_i\}_{i\in\mathbb{N}}.$$

Thus, there is a unique sequence of natural numbers $(j_i)_{i\in\mathbb{N}}$ such that

$$\widehat{\beta}_{j_i}(1) = \tau_i \quad \forall i \in \mathbb{N},$$

which means that $(\beta_i)_{i\in\mathbb{N}} := (\widehat{\beta}_{j_i})_{i\in\mathbb{N}}$ is the unique coset representative of $\overline{\beta}$ that satisfies

$$\beta_i(1) = \tau_i \quad \forall i \in \mathbb{N}.$$

□

Notation: The fundamental group $\pi_1 C_n$ ($\pi_1 F_n$), for all $n \in \mathbb{N}\cup\infty$, is called the **(pure) braid group on** n **strands in the disk**. We introduce the common notation

$$PB_n := \pi_1 F_n, \qquad B_n := \pi_1 C_n \qquad \forall n \in \mathbb{N}$$

Moreover,

$$PB_\infty := \pi_1 F_\infty, \qquad B_\infty := \pi_0 \mathcal{O}C_\infty.$$

For all $n \in \mathbb{N}\cup\infty$, the loop space ΩC_n (ΩF_n) is called the **space of (pure) braids on** n **strands**.

Remark 1.8. *Notice that, in [3] and [8], the braid group* B_n *is defined by*

$$B_n := \pi_1 F_n\big(\mathbb{E}^2, (\tau_i)_{i\in[1,n]}\big)$$

where $\mathbb{E}^2$ *is the euclidean plane, and* $(\tau_i)_{i\in[1,n]}$ *is arbitrary. As* $\mathbb{E}^2$ *is homeomorhpic to* $\overset{\circ}{D}{}^2$*, the resulting braid groups are isomorphic. We prefer to work with configurations in* $\overset{\circ}{D}{}^2$ *rather than with configurations in* $\mathbb{E}$ *because the closure of* $\overset{\circ}{D}{}^2$ *is simply* D^2*, which makes it technically easy to work with infinite configurations, and with infinite sequences. On the other hand, as the sequence*

$(\tau_i)_{i\in\mathbb{N}}$ is in $\overset{\circ}{D}{}^2$ and doesn't accumulate on ∂D^2, one can show that, for all $n \in \mathbb{N} \cup \infty$,

$$B_n \cong \pi_1\big(F_n(D^2), (\tau_i)_{i\in[1,n]}\big).$$

However, it is preferable to work with $F_n(\overset{\circ}{D}{}^2)$ rather than with $F_n(D^2)$, because Theorem 1.12 doesn't hold when replacing $F_n(\overset{\circ}{D}{}^2)$ by $F_n(D^2)$.

To fix the notation, write $\mathcal{C}(X,Y)$ ($\mathcal{H}(X,Y)$) for the space of continuous functions (homeomorphisms) from X to Y, where X and Y are arbitrary spaces, and endow both spaces with the compact-open topology.

Definition 1.9. *For all $n \in \mathbb{N} \cup \infty$, define*

$$\begin{array}{rcl} H_0 & := & \big\{f \in \mathcal{H}(D^2, D^2) \mid f|_{\partial D^2} = Id_{\partial D^2}\big\}, \\ H_n & := & \big\{f \in \mathcal{H}(D^2, D^2) \mid f|_{\partial D^2} = Id_{\partial D^2},\ f\big(\{\tau_i\}_{i\in[1,n]}\big) = \{\tau_i\}_{i\in[1,n]}\big\}, \\ PH_n & := & \big\{f \in \mathcal{H}(D^2, D^2) \mid f|_{\partial D^2} = Id_{\partial D^2},\ f(\tau_i) = \tau_i \ \ \forall\, i \in [1,n]\big\} \end{array}$$

equipped with the subspace topology.

Note that

$$PH_n \subset H_n \qquad \forall n \in \mathbb{N} \cup \infty,$$

and, furthermore,

$$PH_n \subseteq PH_m \qquad \forall n \geq m. \tag{1.1}$$

On the other hand, H_n is not a subspace of H_m for any $n \neq m$.

Proposition 1.10. *(Birman [3, Thm 4.4]) The spaces H_0 and H_1 are contractible.*

This is not true if an arbitrary closed surface replaces D^2. For example, consider the torus $(\mathbb{R} \mod 2\pi) \times (\mathbb{R} \mod 2\pi)$. The homeomorphism defined by

$$\big([t_1],[t_2]\big) \mapsto \big([t_1],[-t_2]\big)$$

is not homotopic to the identity.

Definition 1.11. *For every $n \in \mathbb{N} \cup \infty$, define evaluation maps*

$$\begin{array}{rccc} ev_n: & H_0 & \to & F_n \\ & f & \mapsto & \big(f(\tau_i)\big)_{i\in[1,n]} \end{array} \qquad \begin{array}{rccc} \overline{ev}_n: & H_0 & \to & C_n \\ & g & \mapsto & p_n\big(g(\tau_i)_{i\in[1,n]}\big), \end{array}$$

Theorem 1.12. *(Birman [4]) For all $n \in \mathbb{N}$, the maps*

$$ev_n : H_0 \to F_n, \quad \overline{ev}_n : H_0 \to C_n$$

are fiber bundles with fiber PH_n and H_n, respectively.

Note that this result does not hold for $n = \infty$. See the subsection 1.1.4 for more comments.

Definition 1.13. *According to Proposition 1.10, let*

$$K : H_0 \times I \to H_0, \quad K(f,0) = f,\ K(f,1) = Id_{D^2} \ \ \forall f \in H_0$$

be an arbitrary contracting homotopy of the space H_0*. For all* $n \in \mathbb{N} \cup \infty$*, define maps*

$$\begin{array}{rcl} \varphi_n : PH_n & \to & \Omega F_n \\ h & \mapsto & ev_n(K(h,\cdot)) = \big(K(h,\cdot)(\tau_i)\big)_{i\in[1,n]}, \\ \overline{\varphi}_n : H_n & \to & \Omega C_n \\ h & \mapsto & \overline{ev}_n(K(h,\cdot)) = \big[\big(K(h,\cdot)(\tau_i)\big)_{i\in[1,n]}\big]. \end{array}$$

Remark 1.14. *Notice that the definition of the maps* φ_n *and* $\overline{\varphi}_n$ *depends on the contracting homotopy* $K : H_0 \times I \to H_0$*. However, by Lemma A.3, the induced maps*

$$\pi_0\overline{\varphi}_n : \pi_0 H_n \to B_n \quad \text{and} \quad \pi_0\varphi_n : \pi_0 PH_n \to PB_n$$

do not depend on K*.*

Observe that, for all $n \in \mathbb{N} \cup \infty$, the group structure of PH_n and H_n induces a group structure on $\pi_0 PH_n$ and $\pi_0 H_n$, respectively. In fact, these groups are called the **mapping class groups** of H_n and PH_n, resectively. Moreover, recall that the concatenation of paths "$\star$" induces an $H-$space structure on the loop spaces ΩF_n and ΩC_n.

Proposition 1.15. *For all* $n \in \mathbb{N} \cup \infty$*, the maps*

$$\varphi_n : PH_n \to \Omega F_n \quad \text{and} \quad \overline{\varphi}_n : H_n \to \Omega C_n$$

are maps of H-spaces, and thus induce homomorphisms

$$\pi_0\varphi_n : \pi_0 PH_n \to \pi_1 F_n \quad \text{and} \quad \pi_0\overline{\varphi}_n : \pi_0 H_n \to \pi_1 C_n,$$

respectively.

Proof. We only prove the case $n = \infty$, whereas the case $n \in \mathbb{N}$ is proved in [1, 2]. Pick any elements $g, h \in PH_\infty$, and observe that

$$\begin{array}{rcl} \varphi_\infty\big(g \circ h\big) & = & \big(K(g\circ h,\cdot)(\tau_i)\big)_{i\in\mathbb{N}} \\ & \overset{*}{\simeq} & \Big(\big(K(g,\cdot)\circ h\big) \star K(h,\cdot)(\tau_i)\Big)_{i\in\mathbb{N}} \\ & = & \Big(\big(K(g,\cdot)\circ h\big)(\tau_i)\Big)_{i\in\mathbb{N}} \star \Big(K(h,\cdot)(\tau_i)\Big)_{i\in\mathbb{N}} \\ & \overset{**}{=} & \big(K(g,\cdot)(\tau_i)\big)_{i\in\mathbb{N}} \star \big(K(h,\cdot)(\tau_i)\big)_{i\in\mathbb{N}} \\ & = & \varphi_\infty g \star \varphi_\infty h, \end{array}$$

where $(*)$ is given by Lemma A.3, because the paths $K(g\circ h,\cdot)$ and $\big(K(g,\cdot)\circ h\big) \star K(g,\cdot)$ both have the same initial- and endpoint. The equality $(**)$ comes

from the fact that $h(\tau_i) = \tau_i$ for all $i \in \mathbb{N}$. Applying π_0 to the resulting equation shows that $\pi_0\varphi_\infty$ is a homomorphism.
On the other hand, pick elements $g, h \in H_\infty$, and, recalling the natural projection $p_\infty : F_\infty \to C_\infty$, verify that

$$\begin{aligned}
\overline{\varphi}_\infty(g \circ h) &= p_\infty \circ \big(K(g \circ h, \cdot)(\tau_i)\big)_{i\in\mathbb{N}} \\
&\overset{*}{\simeq} p_\infty \circ \Big(\big(K(g, \cdot) \circ h\big) \star K(h, \cdot)(\tau_i)\Big)_{i\in\mathbb{N}} \\
&= p_\infty \circ \Big(\big(K(g, \cdot) \circ h\big)(\tau_i)\Big)_{i\in\mathbb{N}} \star p_\infty \circ \Big(K(h, \cdot)(\tau_i)\Big)_{i\in\mathbb{N}} \\
&\overset{**}{=} \Big(p_\infty \circ \big(K(g, \cdot)(\tau_i)\big)_{i\in\mathbb{N}}\Big) \star \Big(p_\infty \circ \big(K(h, \cdot)(\tau_i)\big)_{i\in\mathbb{N}}\Big) \\
&= \overline{\varphi}_\infty g \star \overline{\varphi}_\infty h,
\end{aligned}$$

where $(*)$ is given again by Lemma A.3, and $(**)$ comes from the fact that, as sets, $\big\{h(\tau_i)\big\}_{i\in\mathbb{N}} = \{\tau_i\}_{i\in\mathbb{N}}$. Thus, $\pi_0\overline{\varphi}_\infty$ is a homomorphism, as required. □

Theorem 1.16. *(E. Artin [1, 2]) For all $n \in \mathbb{N}$, the maps φ_n and $\overline{\varphi}_n$ are weak equivalences, and therefore induce isomorphisms*

$$\pi_0\varphi_n : \pi_0 PH_n \xrightarrow{\cong} \pi_1 F_n; \quad \pi_0\overline{\varphi}_n : \pi_0 H_n \xrightarrow{\cong} \pi_1 C_n$$

Moreover, as

$$\pi_k F_n = \pi_k C_n = 1 \qquad \forall k \geq 2, \forall n \in \mathbb{N}$$

it follows that

$$\pi_k H_n = \pi_k PH_n = 1 \qquad \forall k \geq 1, \forall n \in \mathbb{N}.$$

1.1.2 The direct system of braid groups

A presentation of the groups B_n for finite n was first found by E. Artin in 1925. For each $n \in \mathbb{N}$, it is given by generators

$$\sigma_1, \ldots, \sigma_{n-1}$$

and relations

$$\sigma_i\sigma_j \sim \sigma_j\sigma_i \qquad \text{if } |i-j| \geq 2,\ 1 \leq i, j \leq n-1 \tag{1.2}$$

$$\sigma_i\sigma_{i+1}\sigma_i \sim \sigma_{i+1}\sigma_i\sigma_{i+1}, \qquad 1 \leq i \leq n-2. \tag{1.3}$$

The particular notation for the generators comes from the fact that, if some given element $b \in B_n$ is represented by a word $\sigma_{i_1} \cdots \sigma_{i_k}$, then, the representative loops $\beta \in \Omega C_n$, satisfying

$$b = [\beta] \quad \text{in } \pi_1 C_n = B_n,$$

have associated permutation $\sigma_{i_1} \cdots \sigma_{i_k}$ in Σ_n, where, for each $j \in [1, n-1]$, $\sigma_j \in \Sigma_n$ is given by

$$\sigma_j = \left(\begin{array}{c} 1, \ldots, j, j+1, \ldots, n \\ 1, \ldots, j+1, j, \ldots, n \end{array}\right).$$

Artin's group presentation of the braid group B_n allows us to consider B_m as a subgroup of B_n for all $m < n$, with inclusion map

$$i_{m,n} : B_m \hookrightarrow B_n.$$

Moreover, through the isomorphisms $\{\pi_0\overline{\varphi}_n\}_{n\in\mathbb{N}}$, we can define injective maps $\{j_n\}_{n\in\mathbb{N}}$ such that the following diagram commutes for all $n \in \mathbb{N}$

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \pi_0 H_n & \overset{j_n}{\dashrightarrow} & \pi_0 H_{n+1} & \longrightarrow & \cdots \\ & & {\scriptstyle \pi_0\overline{\varphi}_n}\downarrow\cong & & \cong\downarrow{\scriptstyle \pi_0\overline{\varphi}_{n+1}} & & \\ \cdots & \longrightarrow & B_n & \xrightarrow{i_{n,n+1}} & B_{n+1} & \longrightarrow & \cdots . \end{array} \tag{1.4}$$

This corresponds to an isomorphism of direct systems, yieding an isomorhpism of colimits

$$\mathrm{colim}_n\big\{\pi_0 H_n, j_n\big\} \xrightarrow{\cong} \mathrm{colim}_n\big\{B_n, i_{n,n+1}\big\}.$$

As the maps$(i_{n,n+1})_{n\in\mathbb{N}}$ are group inclusions, the colimit of the braid groups is actually the union of the finite braid groups, which is called the **finitary infinite braid group**.

$$B_\infty^f := \bigcup_{n\in\mathbb{N}} B_n = \mathrm{colim}_n\big\{B_n, i_{n,n+1}\big\}.$$

As, in particular,

$$B_\infty^f \neq B_\infty, \quad \text{and} \quad B_\infty^f \neq \pi_1 C_\infty,$$

the approach of direct systems is not useful in the context of our work.

1.1.3 The inverse system of pure braid groups

Recall Artin's presentation of the finite braid groups, and consider the projection map of free groups

$$\begin{aligned} \widehat{r}_n : \mathcal{F}\big(\{\sigma_1,\ldots,\sigma_{n-1}\}\big) &\to \mathcal{F}\big(\{\sigma_1,\ldots,\sigma_{n-2}\}\big) \\ \sigma_i &\mapsto \begin{cases} \sigma_i\ , & i < n-1 \\ 1\ , & i = n-1. \end{cases} \end{aligned}$$

Factoring out Artin's relations, given by Eqs. 1.2 and 1.3, in $\mathcal{F}\big(\{\sigma_1,\ldots,\sigma_{n-1}\}\big)$ and $\mathcal{F}\big(\{\sigma_1,\ldots,\sigma_{n-2}\}\big)$, the map $\widehat{r}_n$ induces a map $r_n : B_n \to B_{n-1}$. Note that this map is not a homomorphism, as the following example shows.

$$\begin{array}{ccccc} [\sigma_{n-1}\sigma_n\sigma_{n-1}] & = & [\sigma_n\sigma_{n-1}\sigma_n] & & \in B_n \\ {\scriptstyle r_n}\downarrow & & {\scriptstyle r_n}\downarrow & & \\ [\sigma_{n-1}^2] & \neq & [\sigma_{n-1}] & & \in B_{n-1} \end{array}$$

In fact, within Artin's presentation of the finite braid groups, there does not seem to exist a natural way to define homomorphisms $B_n \to B_{n-1}$. Also, there doesn't seem to be a straightforward way to define a continuous underlying map

$$C_n \to C_{n-1} \quad \text{or} \quad \Omega C_n \to \Omega C_{n-1}.$$

Thus, an inverse system of braid groups does not seem to exist.
Considering the pure braid groups $PB_n \equiv \pi_1 F_n$, things work better, as we show next.

Definition 1.17. *For all $n, n' \in \mathbb{N} \cup \infty$ with $n' > n$, introduce projection maps*

$$\begin{array}{rcl} s_{n',n} : F_{n'} & \to & F_n \\ (x_i)_{i \in [1,n']} & \mapsto & (x_i)_{i \in [1,n]}. \end{array}$$

Observe that, for all $n > 1$, the inclusion

$$\iota_{n,n-1} : PH_n \hookrightarrow PH_{n-1}$$

makes the diagram

$$\begin{array}{ccc} PH_n & \xhookrightarrow{\iota_{n,n-1}} & PH_{n-1} \\ {\scriptstyle \varphi_n}\downarrow{\scriptstyle \sim} & & {\scriptstyle \sim}\downarrow{\scriptstyle \varphi_{n-1}} \\ \Omega F_n & \xrightarrow{\Omega s_{n,n-1}} & \Omega F_{n-1} \end{array}$$

commute for all $n \in \mathbb{N}$. All maps in this diagram are maps of H-spaces (ι_n is a map of topological groups, φ_n and φ_{n-1} are maps of H-spaces by Proposition 1.15, and $\Omega s_{n,n-1}$ is a map of H-spaces). This allows us to conclude as follows.

Proposition 1.18. *There is an isomorphism of inverse systems*

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \pi_0 PH_n & \xrightarrow{\pi_0 \iota_{n,n-1}} & \pi_0 PH_{n-1} & \longrightarrow & \cdots \\ & & {\scriptstyle \pi_0\varphi_n}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \pi_0\varphi_{n-1}} & & \\ \cdots & \longrightarrow & \pi_1 F_n & \xrightarrow{\pi_0 s_{n,n-1}} & \pi_1 F_{n-1} & \longrightarrow & \cdots. \end{array}$$

Proposition 1.19. *The inverse system of inclusions $\{PH_n, \iota_{n,n-1}\}_{n \in \mathbb{N}}$ has the limit*

$$PH_\infty = \lim_n \{PH_n, \iota_{n,n-1}\}_{n \in \mathbb{N}}.$$

Proof. As the limit of an inverse system of group inclusions is just the intersection of the groups, the result is given by observing that

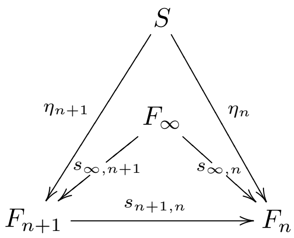
$$PH_\infty = \bigcap_{n \in \mathbb{N}} PH_n.$$

□

Proposition 1.20. *The inverse system* $\{F_n, s_{n,n-1}\}_{n\in\mathbb{N}}$ *has the limit*

$$F_\infty = \lim_n \{F_n, s_{n,n-1}\}_{n\in\mathbb{N}}.$$

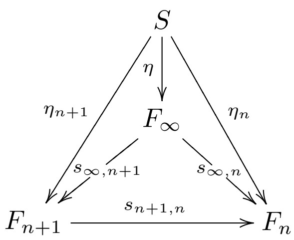
Proof. Assume there is a topological space S, and maps $\eta_n : S \to F_n$, such that the following diagram commutes for all $n \in \mathbb{N}$.



Define a map

$$\begin{array}{rcl} \eta : S & \to & F_\infty \\ s & \mapsto & \big((\eta_i(s))_i\big)_{i\in\mathbb{N}}, \end{array}$$

and observe that the diagram



commutes for all $n \in \mathbb{N}$. □

Corollary 1.21. *The inverse system of loop spaces* $\{\Omega F_n, \Omega s_{n,n-1}\}_{n\in\mathbb{N}}$ *has the limit*

$$\Omega F_\infty = \lim_n \{\Omega F_n, \Omega s_{n,n-1}\}_{n\in\mathbb{N}}.$$

Proof. In the category $\mathbf{Top}_*$, the functor Ω has a left adjoint, and thus, preserves limits. □

Theorem 1.22. *(Fadell, Neuwirth [5], also proved in [3, p. 12]) For each* $n \in \mathbb{N}$*, there is a fiber bundle*

$$\mathring{D}^2 \setminus \{\tau_i\}_{i\in[1,n-1]} \hookrightarrow F_n \overset{s_{n,n-1}}{\longrightarrow} F_{n-1},$$

where the fiber inclusion is given by

$$\begin{array}{rcl} \mathring{D}^2 \setminus \{\tau_i\}_{i\in[1,n-1]} & \hookrightarrow & F_n \\ x & \mapsto & (\tau_1, \dots, \tau_{n-1}, x). \end{array}$$

Corollary 1.23. *The following equations hold.*

$$\pi_1 F_\infty = \lim_n \left\{ \pi_1 F_n, \pi_1 s_{n,n-1} \right\},$$

$$\pi_1 \text{holim}_n PH_n = \lim_n \left\{ \pi_1 PH_n, \pi_1 \iota_{n,n-1} \right\}.$$

Moreover, for all $k \in \mathbb{N}$,

$$\pi_k F_\infty \cong \pi_{k-1} \text{holim}_n PH_n \quad (= 1 \quad \forall k \geq 2).$$

Proof. Consider the following isomorphism of exact sequences, which follows by Proposition 1.18.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \lim^1_n \pi_k PH_n & \longrightarrow & \pi_{k-1}\text{holim}_n PH_n & \longrightarrow & \lim_n \pi_{k-1} PH_n & \longrightarrow & 1 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ 1 & \longrightarrow & \lim^1_n \pi_{k+1} F_n & \longrightarrow & \pi_k \text{holim}_n F_n & \longrightarrow & \lim_n \pi_k F_n & \longrightarrow & 1 \end{array}$$

As, by Theorem 1.22, the maps $s_{n,n-1} : F_n \to F_{n-1}$ are, in particular, fibrations, it follows that, by Proposition 1.20,

$$\text{holim}_n F_n = \lim_n F_n = F_\infty, \qquad (A)$$

Moreover, observe that, by Theorem 1.22, there is a long exact homotopy sequence

$$\cdots \to \pi_1 \big(\overset{\circ}{D}{}^2 \backslash \{\tau_i\}_{i \in [1,n-1]} \big) \to \pi_1 F_n \overset{\pi_1 s_{n,n-1}}{\longrightarrow} \pi_1 F_{n-1} \to \pi_0 \big(\overset{\circ}{D}{}^2 \backslash \{\tau_i\}_{i \in [1,n-1]} \big).$$

As

$$\pi_0 \big(\overset{\circ}{D}{}^2 \setminus \{\tau_i\}_{i \in [1,n-1]} \big) = 1,$$

it follows that the map

$$\pi_1 F_n \overset{\pi_1 s_{n,n-1}}{\longrightarrow} \pi_1 F_{n-1}$$

is surjective, whereas, for all $k \geq 2$, $\pi_k F_n = 1$, such that, according to [12, Prop. 1.67],

$$\lim^1_n \pi_k F_n = 1 \quad \forall k \geq 2.$$

The required results can now be directly read from the above diagram, by replacing $\text{holim}_n F_n$ with F_∞ according to (A). □

1.1.4 General remarks

In the proof of Theorem 1.16, we used the fact that there is a fiber bundle

$$H_n \hookrightarrow H_0 \overset{\overline{\text{ev}}_n}{\longrightarrow} C_n(\overset{\circ}{D}{}^2)$$

to prove that the map $\pi_0 \overline{\varphi}_n$ is an isomorphism. This proof method does not extend to $n \to \infty$, as the map $\overline{\text{ev}}_\infty$ is not a fiber bundle, and is not even surjective, and thus, in particular, does not have the path lifting property. This can be seen by the fact that, for each $h \in H_0$, the unordered point set

$$\overline{\mathrm{ev}}_\infty(h) = \left[\big(h(\tau_i)\big)_{i\in\mathbb{N}}\right]$$

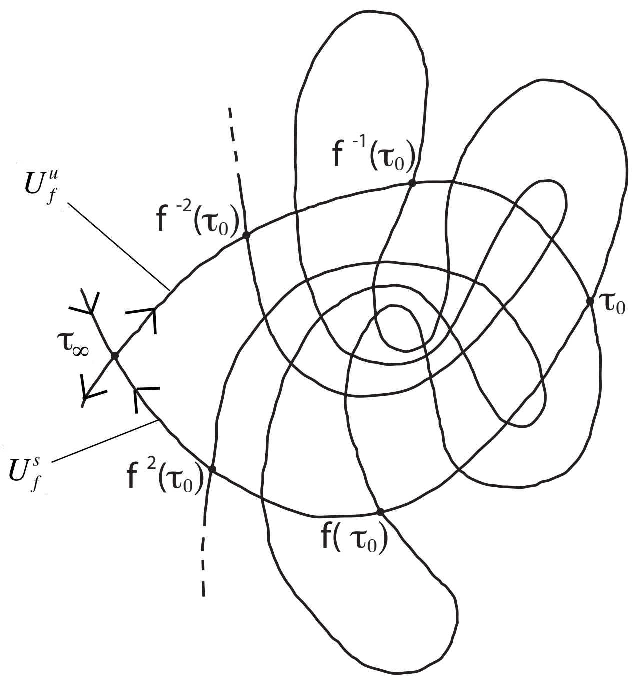
contains as many accumulation points as the set $\{\tau_i\}_{i\in\mathbb{N}}$ does, whereas the space C_∞ contains unordered point sets with any number of accumulation points.
This makes the maps φ_∞ and $\overline{\varphi}_\infty$ considerably more difficult to handle than the corresponding maps in the finite case. In particular, the map

$$\pi_0\overline{\varphi}_\infty : \pi_0 H_\infty \to \pi_1 C_\infty$$

turns out not to be an isomorphism, so that we are interested in finding its image and kernel, which is done in the subsequent sections.
The initial motivation for the study of the link between the infinite mapping class group $\pi_0 H_\infty$ and the infinite braid group $\pi_1 C_\infty(\overset{\circ}{D}{}^2)$ is its application to diffeomorphisms with a **hyperbolic fixed point**, a term that we briefly explain here. A fixed point x of a diffeomorphism $h \in Diff(D^2, D^2)$ is called **hyperbolic**, if the matrix of the linearization of h at x has eigenvalues λ_1, λ_2 with

$$|\lambda_1| > 1, \quad |\lambda_2| < 1.$$

Then, there are two smooth one-dimensional manifolds in D^2 that are invariant by the action of h, and that intersect at the fixed point x. This is given by the Invariant Manifold Theorem (See, e.g., [9]). On these manifolds, that are called the **stable** and the **unstable manifold**, the maps h and h^{-1}, respectively, move the points assymptotically towards the fixed point x. These manifolds cannot intersect themselves, but, in case they intersect each other transversely in some point other than in x, they necessarily meander in a complicated pattern, yielding an infinity of other intersection points that are called **homoclinic intersection points**, whereas the union of the stable and the unstable manifolds is called a **homoclinic tangle**. This subject was introduced by Poincaré, and is a field of current research, with many applications in physics and chemistry. In particular, the classification of homoclinic tangles is still an unsolved problem. The following drawing shows how a homoclinic tangle may look like.



Choose some $h \in H_0$, and let $x \in \overset{\circ}{D}{}^2$ be an arbitrary point. If the orbit

$$\left\{h^i(x)\right\}_{i\in\mathbb{N}}$$

is periodic, then, there is some finite set $\{\widehat{\tau}_i\}_{i\in[1,n]}$ of pairwise distinct points, such that

$$\left\{h^i(x)\right\}_{i\in\mathbb{N}} = \{\widehat{\tau}_i\}_{i\in[1,n]}.$$

Recalling the arbitrariness of $(\tau_i)_{i\in[1,n]}$, we may identify $\tau_i := \widehat{\tau}_i$ for all $i \in [1,n]$, which allows us, in particular, to consider h as an element of H_n. If the orbit of x is *not* periodic, then, similarly, h can be considered as an element of H_∞, and thus be evaluated by $\overline{\varphi}_\infty$. For technical reasons, the study of the maps $\overline{\varphi}_\infty$ and φ_∞ depends on the number of accumulation points of the set $\{\tau_i\}_{i\in\mathbb{N}}$, so that we prove most of the subsequent results using a particularly simple choice for $(\tau_i)_{i\in\mathbb{N}}$, that contains a single accumulation point in D^2. Thereafter, our main results are generalized to any choice of $(\tau_i)_{i\in\mathbb{N}} \in F_\infty$, such that, in D^2 the point set $\{\tau_i\}_{i\in\mathbb{N}}$ accumulates at a single point $\tau_\infty \in \overset{\circ}{D}{}^2$. As this is the case for any homoclinic orbit, the above described procedure allows us to study homeomorphisms in H_0 with a hyperbolic fixed point. In particular, a codification of the image of the map $\pi_0\overline{\varphi}_\infty$ can thus be used to codify the classes in $\pi_0 H_\infty$ of homeomorphisms with a hyperbolic fixed point, which might be useful for the investigation of such homeomorphisms, and for the study of homoclinic tangles themselves. Finally, note that, given any homeomorphism $h \in H_0$ with a hyperbolic fixed point τ_∞, h doesn't fix any of the points of the associated homoclinic orbit $\left\{h^i(\tau_0)\right\}_{i\in\mathbb{N}}$ (where τ_0 is any homoclinic intersection point). In other words, this permutation is not finitary. This motivates the fact that we consider Σ_∞ to be the group of *all* permutations of $\mathbb{N}$, and not only the union of all finite permutation groups.

Chapter 2

Comparision between $\pi_0\overline{\varphi}_\infty$ and $\pi_0\varphi_\infty$

As we observed in the preceeding chapter, the map $\pi_0\varphi_\infty : \pi_0 PH_\infty \to PB_\infty$ is easier to study than the map $\pi_0\overline{\varphi}_\infty : \pi_0 H_\infty \to B_\infty$, because both $\pi_0 PH_\infty$ and PB_∞ are limits of inverse systems. In the present chapter, we develop a result that allows us to study the map $\pi_0\overline{\varphi}_\infty$ in terms of the map $\pi_0\varphi_\infty$ (see Theorem 2.19). The proof of this result requires some knowledge of the infinite permutation group Σ_∞. In particular, we show in section 2.1 how to decompose the elements of Σ_∞ into infinite sequences of transpositions. Appart from the use in our particular context, these results are of interest themselves and might also be used for other purposes.

While in the preceding chapter, the basepoint $\mathcal{T}_\infty = (\tau_i)_{i\in\mathbb{N}} \in F_\infty$ was arbitrary, we restrict ourselves in the sequel to the case where the point set $\{\tau_i\}_{i\in\mathbb{N}}$ has a single accumulation point in D^2 that lies in $\mathring{D}^2$. Moreover, a particularly simple choice for $\mathcal{T}_\infty$ turns out to be useful in many proofs.

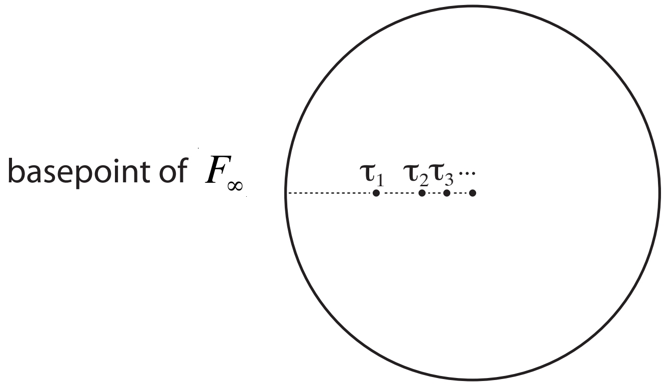
Definition 2.1. *For every $i \in \mathbb{N}$ write*

$$\tau_i = \left(-\frac{1}{i+1}, 0\right) \quad \in \quad \mathbb{R}^2,$$

and, for the remainder of this text, let the basepoint $\mathcal{T}_\infty \in F_\infty$ be

$$\mathcal{T}_\infty := (\tau_i)_{i\in\mathbb{N}},$$

unless specified otherwise.



We proceed by first proving our results using the chosen canonical basepoint $\mathcal{T}_\infty$, and thereafter generalizing the main results to an arbitrary choice of $\mathcal{T}_\infty := (\tau_i)_{i\in\mathbb{N}}$ where the point set $\{\tau_i\}_{i\in\mathbb{N}}$ contains a single accumulation point in D^2, that lies in $\overset{\circ}{D}{}^2$, as we pointed out above.
We first show a combinatorial result concerning the group Σ_∞, which allows us thereafter to define a continuous map $\pi_{\Sigma H} : \Sigma_\infty \to H_\infty$, that satisfies

$$\pi_{\Sigma H}(\sigma)(\tau_i) = \tau_{\sigma(i)} \quad \forall i \in \mathbb{N}, \ \ \forall \sigma \in \Sigma_\infty.$$

Using this map, we can then prove the main result of this chapter, given by Theorem 2.19.

2.1 On the infinite permutation group Σ_∞.

2.1.1 Decomposition of infinite permutations into sequences of transpositions.

Recall that the group structure of Σ_n, for all $n \in \mathbb{N} \cup \infty$ is given by

$$\big(\sigma\sigma'\big)(i) = \sigma' \circ \sigma(i) \quad \forall \sigma, \sigma' \in \Sigma_n, \quad \forall i \in [1, n].$$

Definition 2.2. *Given any $\sigma \in \Sigma_\infty$, define sets $\{\nu_{\sigma,i}\}_{i\in\mathbb{N}} \subseteq \mathbb{N}$ and $\{[\sigma]_i\}_{i\in\mathbb{N}} \subseteq \Sigma_\infty$ inductively by*

$$\begin{array}{rcrl} \nu_{\sigma,1} := & \sigma^{-1}(1), & [\sigma]_1 := & (1, \nu_{\sigma,1}), \\ \nu_{\sigma,n} := & [\sigma]_{n-1}\big(\sigma^{-1}(n)\big), & [\sigma]_n := & (n, \nu_{\sigma,n}) \circ [\sigma]_{n-1} \end{array}$$

for all $n \geq 2$, where (i, j) means the transposition of i and j.

Observe that these notations imply that, for all $\sigma \in \Sigma_\infty$,

$$[\sigma]_n = (n, \nu_{\sigma,n}) \circ \cdots \circ (1, \nu_{\sigma,1}) \quad \forall n \in \mathbb{N}.$$

Lemma 2.3. *Pick any $\sigma \in \Sigma_\infty$.*

$$\begin{array}{llll} (a) & \nu_{\sigma,n} \geq n & \forall n \in \mathbb{N}, & \\ (b) & [\sigma]_n^{-1}(i) = \sigma^{-1}(i) & \forall n \in \mathbb{N}, & \forall\, i \in [1,n], \\ (c) & [\sigma]_j(i) = \sigma(i) & \forall i \in \mathbb{N}, & \forall\, j \geq \sigma(i). \end{array}$$

Proof. Proceding by induction, observe that the case $n = 1$ is trivial, and assume that (a), (b) and (c) are satisfied for some $n \geq 2$. To prove the inductive step for (a), assume that, by contradiction, $\nu_{\sigma,n} < n$. Applying $[\sigma_{n-1}]^{-1}$ to the left of the equation $[\sigma]_{n-1}\sigma^{-1}(n) = \nu_{\sigma,n}$ gives

$$\sigma^{-1}(n) = [\sigma]_{n-1}^{-1}(\nu_{\sigma,n}) \overset{*}{=} \sigma^{-1}(\nu_{\sigma,n}),$$

where $(*)$ is given by the inductive hypothesis for (b). This means that $\nu_{\sigma,n} = n$, which contradicts our assumption.
To verify (b), observe that, for all $i \in [1, n-1]$,

$$\begin{aligned} [\sigma]_n^{-1}(i) &= \Big((1,\nu_{\sigma,1}) \circ \cdots \circ (n-1,\nu_{\sigma,n-1}) \circ (n,\nu_{\sigma,n})\Big)(i) \\ &\overset{*}{=} \Big((1,\nu_{\sigma,1}) \circ \cdots \circ (n-1,\nu_{\sigma,n-1})\Big)(i) \\ &= [\sigma]_{n-1}^{-1}(i) \\ &\overset{**}{=} \sigma^{-1}(i), \end{aligned}$$

where $(*)$ and $(**)$ are given by the induction hypothesis (a) and (b), respectively. It remains to show that this also holds for $i = n$.

$$\begin{aligned} [\sigma]_n^{-1}(n) &= \Big((1,\nu_{\sigma,1}) \circ \cdots \circ (n-1,\nu_{\sigma,n-1}) \circ (n,\nu_{\sigma,n})\Big)(n) \\ &= \Big((1,\nu_{\sigma,1}) \circ \cdots \circ (n-1,\nu_{\sigma,n-1}) \circ (n,[\sigma]_{n-1}\sigma^{-1}(n)\Big)(n) \\ &= \Big((1,\nu_{\sigma,1}) \circ \cdots \circ (n-1,\nu_{\sigma,n-1}))\Big)\big([\sigma]_{n-1}\sigma^{-1}(n)\big) \\ &= [\sigma]_{n-1}^{-1}[\sigma]_{n-1}\sigma^{-1}(n) \\ &= \sigma^{-1}(n) \end{aligned}$$

To prove (c), observe that, for all $k \geq \sigma(i)$,

$$\begin{aligned} [\sigma]_k(i) &= \big((k,\nu_k) \circ \cdots \circ (\sigma(i)+1,\nu_{\sigma(i)+1}) \circ [\sigma]_{\sigma(i)}\big)(i) \\ &\overset{*}{=} \Big((k,\nu_k) \circ \cdots \circ \big(\sigma(i)+1,\nu_{\sigma(i)+1}\big)\Big)\big(\sigma(i)\big) \\ &\overset{**}{=} \sigma(i), \end{aligned}$$

where $(*)$ holds because, writing $n := \sigma(i)$ in Eq. (b), it follows that

$$\begin{aligned} [\sigma]_{\sigma(i)}^{-1}(\sigma(i)) &= \sigma^{-1}\big(\sigma(i)\big) \\ &= i. \end{aligned}$$

On the other hand, Eq. $(**)$ is given by Eq. (a). $\square$

Recall that the space Σ_∞ has the topology of *pointwise convergence*.

Proposition 2.4. *For all $\sigma \in \Sigma_\infty$,*

$$\sigma = \lim_{n\to\infty} \big((n, \nu_{\sigma,n}) \circ \cdots \circ (1, \nu_{\sigma,1})\big),$$

where the integers $\{\nu_{\sigma,i}\}_{i\in\mathbb{N}}$ are defined as above.

Proof. By Eq. (c) of Lemma 2.3, we know that, for every $i \in \mathbb{N}$

$$\sigma(i) = \big((n, \nu_{\sigma,n}) \circ \cdots \circ (1, \nu_{\sigma,1})\big)(i) =: [\sigma]_n(i) \quad \forall\, n \geq \sigma(i).$$

Thus, it follows from [Munkres, Thm. 46.1], that the sequence $\big((n, \nu_{\sigma,n}) \circ \cdots \circ (1, \nu_{\sigma,1})\big)_{n\in\mathbb{N}}$ converges to σ. $\square$

2.1.2 Identification of Σ_∞ in $\mathbb{N}^\infty$

In the last subsection, we showed how to decompose an infinite permutation into a convergent sequence of products of transpositions. We now consider the inverse problem, which is to see under what conditions a sequence of natural numbers $(\nu_i)_{i\in\mathbb{N}}$ gives rise to a sequence of products of transpositions

$$\ldots (n, \nu_n) \circ \cdots \circ (1, \nu_1)$$

that converges in $\mathbb{N}^\infty$, i.e., yields a well defined element of Σ_∞. This allows us thereafter to find a criterion to identify the sequences $(\nu_{\sigma,i})_{i\in\mathbb{N}}$ for all $\sigma \in \Sigma_\infty$.

Definition 2.5. *Recalling Definition 2.2, define a map by*

$$\begin{array}{rcl} Seq : \Sigma_\infty & \to & \mathbb{N}^\infty \\ \sigma & \mapsto & (\nu_{\sigma,i})_{i\in\mathbb{N}}. \end{array}$$

Moreover, for any given $n \geq 1$ and $(\nu_i)_{i\in[1,n]} \in \mathbb{N}^n$, introduce the notation

$$\begin{array}{rcl} [\sigma_\nu]_0 & := & \mathrm{Id}_{\mathbb{N}}, \quad \text{and} \\ [\sigma_\nu]_n & := & (n, \nu_n) \circ \cdots \circ (1, \nu_1) \quad \forall n \geq 1, \end{array}$$

and notice that this notation is consistent with the expression for $[\sigma]_n$, for some given σ, that is given immediately after Definition 2.2.

Proposition 2.6. *Let $(\nu_i)_{i\in\mathbb{N}}$ be a sequence of integers satisfying*

$$\nu_i \geq i \quad \forall i \in \mathbb{N},$$

and define sequences $(\lambda_{i,n})_{n\in\mathbb{N}}$ inductively by

$$\lambda_{i,1} := i, \quad \lambda_{i,2} := \nu_i, \quad \lambda_{i,3} := \nu_{\nu_i}, \quad \lambda_{i,n} = \nu_{\lambda_{i,n-1}}$$

for all $i \in \mathbb{N}$.
The sequence $\big([\sigma_\nu]_k\big)_{k\in\mathbb{N}}$ converges in $\mathbb{N}^\infty$ if and only if, for each $i \in \mathbb{N}$, there is an $n_i \in \mathbb{N}$, such that either

$$\begin{array}{ll} (a) & \lambda_{i,n_i} = \lambda_{i,n_i+1}, \quad \textit{or} \\ (b) & \lambda_{i,n_i} = \nu_j, \ \textit{for some } j \in \ [\lambda_{i,n_i-1}+1, \lambda_{i,n_i}-1], \end{array}$$

Remark: If, for some $i \in \mathbb{N}$, the sequence $(\lambda_{i,n})_{n\in\mathbb{N}}$ satisfies $\lambda_{i,n} = \lambda_{i,n+1}$ for some $n \in \mathbb{N}$, then,

$$\lambda_{i,n} = \lambda_{i,n+k} \quad \forall k \geq 1.$$

Proof. Pick any sequence of integers $(\nu_i)_{i\in\mathbb{N}}$, satisfying

$$\nu_i \geq i \quad \forall i \in \mathbb{N},$$

fix some $i \in \mathbb{N}$ and define the sequence $(\lambda_{i,n})_{n\in\mathbb{N}}$ as required. Recall that, by [11, Thm. 46.1] the sequence $\big([\sigma_\nu]_n\big)_{n\in\mathbb{N}}$ converges if and only if the sequence $\big([\sigma_\nu]_n(i)\big)_{n\in\mathbb{N}}$ converges for all $i \in \mathbb{N}$.
Note that, as $\nu_i \geq i$ for all $i \in \mathbb{N}$,

$$\lambda_{i,n+1} \geq \lambda_{i,n} \quad \forall n \in \mathbb{N}.$$

Fix some $i \in \mathbb{N}$, and observe that

$$[\sigma_\nu]_{\lambda_{i,0}}(i) = i = \lambda_{i,1}.$$

Assuming that $[\sigma_\nu]_{\lambda_{i,n}}(i) = \lambda_{i,n+1}$ for some $n \geq 0$, write

$$\begin{aligned}
[\sigma_\nu]_{\lambda_{i,n+1}}(i) &= (\lambda_{i,n+1}, \nu_{\lambda_{i,n+2}}) \circ \cdots \circ (\lambda_{i,n}+1, \nu_{\lambda_{i,n}+1}) \circ [\sigma_\nu]_{\lambda_{i,n}}(i) \\
&= (\lambda_{i,n+1}, \nu_{\lambda_{i,n+1}}) \circ \cdots \circ (\lambda_{i,n}+1, \nu_{\lambda_{i,n}+1})(\lambda_{i,n+1}),
\end{aligned}$$

which shows that, by our assumption,

$$[\sigma_\nu]_{\lambda_{i,n+1}}(i) = \lambda_{i,n+2} \quad \Leftrightarrow \quad \lambda_{i,n+1} \neq \nu_j \quad \forall j \in [\lambda_{i,n}+1, \lambda_{i,n+1}-1]. \qquad (A)$$

Notice that, if $\lambda_{i,n} = \lambda_{i,n+1}$, the interval $[\lambda_{i,n}+1, \lambda_{i,n+1}-1]$ is empty. We continue our inductive procedure separately in two different cases.

First case: (b) **holds.** Let n be the least integer such that $\lambda_{i,n+1} = \nu_{\widehat{j}}$ for some $\widehat{j} \in [\lambda_{i,n}+1, \lambda_{i,n+1}-1]$, and let j be the least among these $\widehat{j}$. According to (A), we know that

$$[\sigma_\nu]_{\lambda_{i,n}}(i) = \lambda_{i,n+1},$$

such that

$$\begin{aligned}
[\sigma_\nu]_{\lambda_{i,n+1}}(i) &= (\lambda_{i,n+1}, \nu_{\lambda_{i,n+2}}) \circ \cdots \circ (\lambda_{i,n}+1, \nu_{\lambda_{i,n}+1}) \circ [\sigma_\nu]_{\lambda_{i,n}}(i) \\
&= (\lambda_{i,n+1}, \nu_{\lambda_{i,n+1}}) \circ \cdots \circ (\lambda_{i,n}+1, \nu_{\lambda_{i,n}+1})(\lambda_{i,n+1}) \\
&\overset{*}{=} (\lambda_{i,n+1}, \nu_{\lambda_{i,n+1}}) \circ \cdots \circ (j, \nu_j)(\lambda_{i,n+1}) \\
&= (\lambda_{i,n+1}, \nu_{\lambda_{i,n+1}}) \circ \cdots \circ (j, \nu_j)(\nu_j) \\
&\overset{**}{=} j,
\end{aligned}$$

where $(*)$ holds by our particular choice of j, and $(**)$ comes from the fact that $\nu_k \geq k$ for all $k \in \mathbb{N}$. A generalisation of the same argument shows that

$$[\sigma_\nu]_m(i) = j \quad \forall m \geq \lambda_{i,n+1},$$

i.e., the sequence $\big([\sigma_\nu]_n(i)\big)_{n\in\mathbb{N}}$ converges.

Second case: (b) **doesn't hold.** In this case, the statement (A) admits the induction step, which means that

$$[\sigma_\nu]_{\lambda_{i,n}}(i) = \lambda_{i,n+1} \quad \forall n \in \mathbb{N}.$$

Thus, the sequence $\big([\sigma_\nu]_{\lambda_{i,n}}(i)\big)_{n\in\mathbb{N}}$, converges if and only if (a) holds.
Thus, $\big([\sigma_\nu]_n\big)_{n\in\mathbb{N}}$ converges if and only if (a) or (b) holds for all $i \in \mathbb{N}$. □

With a little more effort, we can prove the following stronger result.

Theorem 2.7. *The map $Seq : \Sigma_\infty \to \mathbb{N}^\infty$ induces a bijection*

$$\begin{array}{rcl} Seq : \Sigma_\infty & \xrightarrow{\cong} & \Big\{(\nu_i)_{i\in\mathbb{N}} \in \mathbb{N}^\infty \quad | \quad \nu_i \geq i \quad \forall\, i \in \mathbb{N}, \\ && \quad \textit{and } \forall\, i \in \mathbb{N},\ \exists\, n_i \in \mathbb{N},\ \textit{such that} \\ && \qquad (a) \quad \lambda_{i,n_i} = \lambda_{i,n_i+1}, \quad \textit{or} \\ && \qquad (b) \quad \lambda_{i,n_i} = \nu_j,\ \textit{for some } j \in\ [\lambda_{i,n_i-1}+1, \lambda_{i,n_i}-1]\Big\}, \end{array}$$

where the integers $\lambda_{i,j}$ are defined as in the proposition above.

Proof. To shorten the notation, write

$$\begin{array}{rcl} \mathcal{S} & := & \Big\{(\nu_i)_{i\in\mathbb{N}} \in \mathbb{N}^\infty \quad | \quad \nu_i \geq i \quad \forall\, i \in \mathbb{N}, \\ && \quad \text{and } \forall\, i \in \mathbb{N},\ \exists\, n_i \in \mathbb{N},\ \text{such that} \\ && \qquad (a) \quad \lambda_{i,n_i} = \lambda_{i,n_i+1}, \quad \text{or} \\ && \qquad (b) \quad \lambda_{i,n_i} = \nu_j,\ \text{for some } j \in\ [\lambda_{i,n_i-1}+1, \lambda_{i,n_i}-1]\Big\}. \end{array}$$

We show that the inverse of Seq is given by

$$\begin{array}{rcl} \mathrm{Per} : \mathcal{S} & \to & \Sigma_\infty \\ (\nu_i)_{i\in\mathbb{N}} & \mapsto & \sigma_\nu, \end{array}$$

where σ_ν denotes the limit of the sequence $\big([\sigma_\nu]_n\big)_{n\in\mathbb{N}}$, which exists by Proposition 2.6.
Pick an element $(\nu_i)_{i\in\mathbb{N}} \in \mathcal{S}$. By Proposition 2.6, we know that σ_ν is in $\mathbb{N}^\infty$. To see that σ_ν is actually a *bijection*, i.e., an element of Σ_∞, pick some $i \in \mathbb{N}$, and observe that, for every $k > i$,

$$\begin{array}{rcl} [\sigma_\nu]_k\big([\sigma_\nu]_i^{-1}(i)\big) & = & \big((k,\nu_k)\circ\cdots\circ(1,\nu_1)\circ(1,\nu_1)\circ\cdots\circ(i,\nu_i)\big)(i) \\ & = & \big((k,\nu_k)\circ\cdots\circ(i+1,\nu_{i+1})\big)(i) \\ & = & i. \end{array}$$

Thus, σ_ν is surjective. The injectivity of σ_ν is shown by the fact that, if

$$\sigma_\nu(i) = \sigma_\nu(j),\ \text{i.e., } \lim_{k\to\infty}[\sigma_\nu]_k(i) = \lim_{k\to\infty}[\sigma_\nu]_k(j)$$

for some $i, j \in \mathbb{N}$, then,

$$[\sigma_\nu]_k(i) = [\sigma_\nu]_k(j),$$

for some $k \in \mathbb{N}$, which means that $i = j$, because $[\sigma_\nu]_k$ is a bijection. This shows that $\mathrm{Per} : \mathcal{S} \to \Sigma_\infty$ is well defined.
It remains to show that Seq and Per are mutually inverse. By Proposition 2.4, it follows directly that

$$\mathrm{Per} \circ \mathrm{Seq}(\sigma) = \sigma \quad \forall\, \sigma \in \Sigma_\infty.$$

To see that $\mathrm{Seq} \circ \mathrm{Per} = \mathrm{Id}_{\mathcal{S}}$, pick any $(\nu_i)_{i \in \mathbb{N}} \in \mathcal{S}$, and write $\sigma_\nu := \mathrm{Per}\big((\nu_i)_{i \in \mathbb{N}}\big)$. Proceeding by induction, observe that

$$\big(\mathrm{Seq}(\sigma_\nu)\big)_1 = \sigma_\nu^{-1}(1) \overset{*}{=} [\sigma_\nu]_1^{-1}(1) = (1, \nu_1)(1) = \nu_1,$$

where $(*)$ is given by Lemma 2.3. Now, assume that

$$\big(\mathrm{Seq}(\sigma_\nu)\big)_i = \nu_i \quad \forall i \in [1, n-1].$$

Then,

$$\begin{aligned}
\big(\mathrm{Seq}(\sigma_\nu)\big)_n &= [\sigma_\nu]_{n-1}\big(\sigma_\nu^{-1}(n)\big) \\
&\overset{*}{=} [\sigma_\nu]_{n-1}[\sigma_\nu]_n^{-1}(n) \\
&= (n-1, \nu_{n-1}) \circ \cdots \circ (1, \nu_1) \circ (1, \nu_1) \circ \cdots \circ (n-1, \nu_{n-1}) \circ (n, \nu_n)(n) \\
&= (n, \nu_n)(n) \\
&= \nu_n,
\end{aligned}$$

where $(*)$ is given by Lemma 2.3. Thus,

$$\mathrm{Seq} \circ \mathrm{Per}\big((\nu_i)_{i \in \mathbb{N}}\big) = \mathrm{Seq}(\sigma_\nu) = (\nu_i)_{i \in \mathbb{N}},$$

which finishes the proof. □

Definition 2.8. *For all $i \in \mathbb{N}$, write*

$$\varrho_i := \|\tau_i - \tau_\infty\|.$$

Proposition 2.9. *By our choice of the basepoint $\mathcal{T}_\infty = (\tau_i)_{i \in \mathbb{N}}$,*

$$\varrho_i \geq \varrho_{i+1} \ \ \forall\, i \in \mathbb{N}, \quad \lim_{i \to \infty} \varrho_i = 0, \quad \text{and} \quad \tau_j \in B(\tau_\infty, \varrho_i) \ \ \forall\, j \geq i,$$

where $B(x, r)$ is the open ball , centered at x, with radius r.

Definition 2.10. *Define a map*

$$\pi_{H\Sigma} : H_\infty \to \Sigma_\infty, \quad g \mapsto \sigma_g,$$

where σ_g is the unique element of Σ_∞, that satisfies

$$g(\tau_i) = \tau_{\sigma_g(i)} \quad \forall\, i \in \mathbb{N}.$$

Lemma 2.11. *For each pair of integers $i \leq j$, there is an element $\widehat{f}_{i,j} \in H_\infty$, satisfying the following conditions.*

$$\widehat{f}_{i,j}(\tau_k) = \begin{cases} \tau_j, & \text{if } k = i \\ \tau_i, & \text{if } k = j \\ \tau_k, & \text{else,} \end{cases}$$

i.e.,

$$\pi_{H\Sigma}(\widehat{f}_{i,j}) = (i,j),$$

and

$$\widehat{f}_{i,j}|_{D^2 \setminus B(\tau_\infty, 2\varrho_i)} = Id.$$

Proof. The proof of this result involves Dehn twists. For a detailed introduction to this construction, see [13]. Throughout the proof, we identify S^1 as follows

$$S^1 := \{z \in \mathcal{C}^2 \mid \|z\| = 1\}.$$

Pick any pair of integers (i,j) with $i \leq j$, let

$$a_{i,j} : S^1 \to B(\tau_\infty, 2\varrho_i) \setminus \{\tau_k\}_{k \geq N_i, k \neq i,j},$$

be a simple closed curve where $N_i := \min_{k \in \mathbb{N}} \{\tau_k \in B(\tau_\infty, \varrho_i)\}$. Knowing that neither τ_i nor τ_j is an accumulation point of the set $\{\tau_k\}_{k \in \mathbb{N}}$, we can assume that, moreover,

$$a_{i,j}(1) = \tau_i, \quad a_{i,j}(-1) = \tau_j.$$

Furthermore, let

$$v_{i,j} : S^1 \times I \to B(\tau_\infty, 2\varrho_i) \setminus \{\tau_k\}_{k \geq N_i}$$

be a tubular neighbourhood of $a_{i,j}$, i.e., an oriented embedding satisfying

$$v_{i,j}(z, 1/2) = a_{i,j}(z) \quad \forall z \in S^1.$$

Then, the map $\widehat{f}_{i,j} : D^2 \to D^2$ defined by

$$\big(\widehat{f}_{i,j} \circ v_{i,j}\big)(z,t) := v_{i,j}\left(e^{2i\pi t} z, t\right) \quad \forall (z,t) \in S^1 \times I,$$

and

$$\widehat{f}_{i,j}(x) = x \quad \forall x \in D^2 \setminus \mathrm{Im}\; v_{i,j}$$

satisfies the required properties. Note that, under this definition, $\widehat{f}_{i,j}$ is called a **Dehn twist** along $a_{i,j}$. □

Theorem 2.12. *The map $\pi_{H\Sigma} : H_\infty \to \Sigma_\infty$ has a right inverse*

$$\begin{aligned} \pi_{\Sigma H} : \Sigma_\infty &\to H_\infty \\ \sigma &\mapsto f_\sigma, \end{aligned}$$

i.e.,

$$\pi_{H\Sigma} \circ \pi_{\Sigma H} = \mathrm{Id}_{\Sigma_\infty}.$$

In other words, for every element $\sigma \in \Sigma_\infty$, there is a map $f_\sigma \in H_\infty$, such that

$$f_\sigma(\tau_i) = \tau_{\sigma(i)} \quad \forall\, i \in \mathbb{N}.$$

Proof. Pick any $\sigma \in \Sigma_\infty$. To define a homeomorphism $f_\sigma \in H_\infty$ with $\pi_{H\Sigma}(f_\sigma) = \sigma$, we make use of Proposition 2.4 that allows us to decompose σ into a sequence of transpositions, i.e.,

$$\sigma = \lim_{n\to\infty} [\sigma]_n,$$

where

$$[\sigma]_n := (n, \nu_{\sigma,n}) \circ \cdots \circ (2, \nu_{\sigma,2}) \circ (1, \nu_{\sigma,1}),$$

and $(\nu_{\sigma,n})_{n\in\mathbb{N}} = \mathrm{Seq}(\sigma)$. Recall that, by Lemma 2.3 (a),

$$\nu_{\sigma,n} \geq n \qquad \forall n \in \mathbb{N},$$

and let $(\widehat{f}_{n,\nu_{\sigma,n}})_{n\in\mathbb{N}}$ be elements of PH_∞ as given by Lemma 2.11. Thus, writing

$$f_{[\sigma]_n} := \widehat{f}_{n,\nu_{\sigma,n}} \circ \cdots \circ \widehat{f}_{2,\nu_{\sigma,2}} \circ \widehat{f}_{1,\nu_{\sigma,1}}$$

for all $n \in \mathbb{N}$, it follows in particular that

$$f_{[\sigma]_n}(\tau_i) = \tau_{[\sigma]_n(i)} \quad \forall i \in \mathbb{N}. \qquad (A)$$

We show that the limit

$$f_\sigma := \lim_{n\to\infty} f_{[\sigma]_n}$$

exists, that it is in H_∞, and that $\pi_{H\Sigma}(f_\sigma) = \sigma$, which finishes the proof.
To prove the existence of the limit $f_\sigma := \lim_{n\to\infty} f_{[\sigma]_n}$, pick any $x \in D^2$, and observe that, for every $n \in \mathbb{N}$,

$$\begin{aligned} \|f_{[\sigma]_{n+1}}(x) - f_{[\sigma]_n}(x)\| &= \left\|\widehat{f}_{n+1,\nu_{\sigma,n+1}}\big(f_{[\sigma]_n}(x)\big) - f_{[\sigma]_n}(x)\right\| \\ &\leq 4\varrho_{n+1}. \end{aligned}$$

As $\lim_{i\to\infty} \varrho_i = 0$, it follows that $\big(f_{[\sigma]_n}(x)\big)_{n\in\mathbb{N}}$ is a Cauchy sequence. By the completeness of the unit disk, the pointwise convergence of the sequence $\big(f_{[\sigma]_n}(x)\big)_{n\in\mathbb{N}}$ follows, which allows us to define a map

$$\begin{aligned} f_\sigma : D^2 &\to D^2 \\ x &\mapsto \lim_{n\to\infty} f_{[\sigma]_n}(x). \end{aligned}$$

We show that, moreover, the sequence $\big(f_{[\sigma]_n}(x)\big)_{n\in\mathbb{N}}$ converges *uniformly.*
Observe that, for all $x \in D^2$, and for all integers n, n' with $n' \geq n$,

$$\begin{aligned} \left\|f_{[\sigma]_{n'}}(x) - f_{[\sigma]_n}(x)\right\| &= \left\|\big(\widehat{f}_{n',\nu_{\sigma,n'}} \circ \cdots \circ \widehat{f}_{n,\nu_{\sigma,n}} \circ f_{[\sigma]_n}(x) - f_{[\sigma]_n}(x)\right\| \\ &\overset{*}{<} \sup_{k\in[n+1,n']} \big\{4\varrho_k\big\} \\ &= 4\varrho_{n+1}, \end{aligned}$$

where $(*)$ is given by Lemma 2.11. Thus, for any $n \in \mathbb{N}$,

$$\left\| f_\sigma(x) - f_{[\sigma]_n}(x) \right\| \leq 4\varrho_{n+1} \quad \forall\, x \in D^2.$$

As $\lim_{n\to\infty} \varrho_n = 0$, this means that, for every $\varepsilon > 0$, there is a $N_\varepsilon \in \mathbb{N}$, such that

$$\left\| f_\sigma(x) - f_{[\sigma]_n}(x) \right\| < \varepsilon \quad \forall\, x \in D^2, \;\; \forall\, n \geq N_\varepsilon,$$

which means that the sequence $\left(f_{[\sigma]_n}\right)_{n\in\mathbb{N}}$ converges *uniformly* to f_σ. Thus, by [11, Thms 46.5, 46.7, 46.8], it follows that f_σ is continuous. Similarly, one can show that the sequence $\left(f^{-1}_{[\sigma]_n}\right)_{n\in\mathbb{N}}$ converges uniformly to f_σ^{-1}, which thus is continuous. Thus f_σ is a homeomorphism. Finally, observing that

$$f_{[\sigma]_n} \in H_\infty \quad \forall n \in \mathbb{N}$$

shows that $f_\sigma \in H_\infty$.
To show that $\pi_{H\Sigma}(f_\sigma) = \sigma$, observe that, by Lemma 2.3,

$$[\sigma]_n(i) = \sigma(i) \quad \forall\, n \geq \sigma(i) \quad \forall\, i \in \mathbb{N},$$

such that, by (A),

$$f_{[\sigma]_n}(\tau_i) = \tau_{\sigma(i)} \quad \forall\, n \geq \sigma(i) \quad \forall\, i \in \mathbb{N}.$$

Thus,

$$f_\sigma(\tau_i) = \lim_{n\to\infty} \left(f_{[\sigma]_n}(\tau_i) \right) = \tau_{\sigma(i)} \quad \forall\, i \in \mathbb{N},$$

which means that

$$\pi_{H\Sigma}(f_\sigma) = \sigma.$$

□

Theorem 2.13. *The maps $\pi_{\Sigma H}$, $\pi_{H\Sigma}$ are continuous.*

Proof. As $\mathcal{H}(D^2, D^2)$ has the uniform topology, which is metric, its subspace H_∞ is metric too. Thus, by [Munkres, Thm. 21.3], the map $\pi_{H\Sigma}$ is continuous if (and only if) it maps convergent sequences to convergent sequences. Pick any convergent sequence $(g_i)_{i\in\mathbb{N}}$ in H_∞, such that

$$g := \lim_{i\to\infty} g_i,$$

and write

$$\sigma := \pi_{H\Sigma}(g), \quad \sigma_i := \pi_{H\Sigma}(g_i) \quad \forall i \in \mathbb{N}.$$

Observe that, for all $n \in \mathbb{N}$,

$$\lim_{i\to\infty} \tau_{\sigma_i(n)} = \lim_{i\to\infty} g_i(\tau_n) = g(\tau_n) = \tau_{\sigma(n)}. \qquad (A)$$

By assumption, the sequence $(\tau_i)_{i\in\mathbb{N}}$ converges to τ_∞ in such a way that for all $i \in \mathbb{N}$, τ_i is not an accumulation point of the set $\{\tau_i\}_{i\in\mathbb{N}}$. Thus, it follows from (A) that for every $n \in \mathbb{N}$, there is a $N_n \in \mathbb{N}$, such that

$$\tau_{\sigma_i(n)} = \tau_{\sigma(n)} \quad \forall\, i \geq N_n,$$

i.e.,

$$\sigma_i(n) = \sigma(n) \quad \forall\, i \geq N_n.$$

This proves that $\pi_{H\Sigma}$ maps convergent sequences to convergent sequences, because Σ_∞ has the topology of pointwise convergence.
Note that in Appendix, Lemma A.4, we prove that Σ_∞ is metric. Thus, again, the map $\pi_{\Sigma H}$ is continuous if it maps convergent sequences to convergent sequences.
Let $(\sigma_i)_{i\in\mathbb{N}}$ be an arbitrary sequence in Σ_∞ that converges to an element $\sigma \in \Sigma_\infty$. We need to show that

$$\lim_{i\to\infty} \pi_{\Sigma H}(\sigma_i) = \pi_{\Sigma H}(\sigma).$$

As, given any $n \in \mathbb{N}$,

$$\lim_{i\to\infty} \sigma_i(n) = \sigma(n),$$

there is an integer N_n^+, such that

$$\sigma_i(n) = \sigma(n) \quad \forall\, i \geq N_n^+.$$

Also, for every $n \in \mathbb{N}$, there is a $N_n^- \in \mathbb{N}$, such that

$$\sigma_i^{-1}(n) = \sigma^{-1}(n) \quad \forall\, i \geq N_n^-.$$

By definition,

$$\nu_{\sigma_i,n} := [\sigma_i]_{n-1}\sigma_i^{-1}(n) \quad \forall\, n \in \mathbb{N}.$$

It follows that, for every $n \in \mathbb{N}$,

$$\nu_{\sigma_i,n} = \nu_{\sigma,n} \quad \forall\, i \geq \max\{N^+_{\sigma^{-1}(n)}, N_n^-\}. \qquad (B)$$

Recall the definition

$$\pi_{\Sigma H}(\sigma) = f_\sigma := \lim_{n\to\infty} f_{[\sigma]_n}$$

in the proof of Theorem 2.12, where

$$f_{[\sigma]_n} := \widehat{f}_{n,\nu_{\sigma,n}} \circ \cdots \circ \widehat{f}_{2,\nu_{\sigma,2}} \circ \widehat{f}_{1,\nu_{\sigma,1}}$$

for all $\sigma \in \Sigma_\infty$ and all $n \in \mathbb{N}$. We need to show that

$$\lim_{i\to\infty} f_{\sigma_i} = f_\sigma.$$

Recall that, by [11, Thms 46.7, 46.8], the compact-open topology on H_0 coincides with the topology of uniform convergence. Thus, for every $\varepsilon > 0$, we need to find an integer N_ε, such that

$$\left\| f_{\sigma_i}(x) - f_\sigma(x) \right\| < \varepsilon \quad \forall\, x \in D^2 \;\; \forall\, i \geq N_\varepsilon.$$

Pick some $\varepsilon > 0$, and fix an integer n_0, such that

$$8\varrho_{n_0+1} < \varepsilon. \qquad (C)$$

Write

$$N_\varepsilon := \max_{n \le n_0} \big\{ \max\{N^+_{\sigma^{-1}(n)}, N^-_n\} \big\},$$

and, by (B), observe that

$$[\sigma_i]_{n_0} = [\sigma]_{n_0} \quad \forall\, i \ge N_\varepsilon,$$

and therefore

$$f_{[\sigma_i]_{n_0}} = f_{[\sigma]_{n_0}} \quad \forall\, i \ge N_\varepsilon. \qquad (D)$$

Also, we showed in the proof of Theorem 2.12, that, for any $\widetilde{\sigma} \in \Sigma_\infty$,

$$\big\| f_{\widetilde{\sigma}}(x) - f_{[\widetilde{\sigma}]_n}(x) \big\| \le 4\varrho_{n+1} \quad \forall x \in D^2,\ \ \forall n \in \mathbb{N}.$$

Thus, in particular,

$$\big\| f_\sigma(x) - f_{[\sigma]_n}(x) \big\| \le 4\varrho_{n+1}, \quad \forall x \in D^2\ \ \forall n \in \mathbb{N}, \qquad (E)$$

and

$$\big\| f_{\sigma_i}(x) - f_{[\sigma_i]_n}(x) \big\| \le 4\varrho_{n+1}, \quad \forall x \in D^2\ \ \forall n \in \mathbb{N}. \qquad (F)$$

Finally, for all $i \ge N_\varepsilon$,

$$\begin{aligned} \big\| f_{\sigma_i}(x) - f_\sigma(x) \big\| &\le \big\| f_{\sigma_i}(x) - f_{[\sigma_i]_{n_0}}(x) \big\| + \big\| f_{[\sigma_i]_{n_0}}(x) - f_\sigma(x) \big\| \\ &\overset{D}{=} \big\| f_{\sigma_i}(x) - f_{[\sigma_i]_{n_0}}(x) \big\| + \big\| f_{[\sigma]_{n_0}}(x) - f_\sigma(x) \big\| \\ &\overset{E,F}{\le} 8\varrho_{n_0+1} \\ &\overset{C}{<} \varepsilon, \quad \forall\, x \in D^2, \end{aligned}$$

which finishes the proof. □

Finally, there is another map that will be useful below.

Definition 2.14. *Consider the set $\mathcal{T}_\infty\Sigma_\infty := \{\mathcal{T}_\infty\sigma \mid \sigma \in \Sigma_\infty\}$ as a subspace of F_∞, and define a map*

$$\begin{aligned} \pi_{T\Sigma} : \mathcal{T}_\infty\Sigma_\infty &\to \Sigma_\infty \\ \mathcal{T}_\infty\sigma &\mapsto \sigma. \end{aligned}$$

Proposition 2.15. *The map $\pi_{T\Sigma}$ is continuous.*

Proof. By [11, p. 280, Exercise 1], the product topology on $\prod_{i\in\mathbb{N}} \overset{\circ}{D}{}^2$ is metric. As the space F_∞ is topologized as a subspace of $\prod_{i\in\mathbb{N}} \overset{\circ}{D}{}^2$, it is metric too. As, thus, $\mathcal{T}_\infty\Sigma_\infty \subset F_\infty$ is itself metric, the proposition follows from the fact that $\pi_{T\Sigma}$ maps convergent sequences to convergent sequences. (see [11, Thm. 21.3]). □

2.2 Reducing the map $\pi_0\varphi_\infty$ to $\pi_0\overline{\varphi}_\infty$

In this section, we develop a commutative diagram, given in Theorem 2.19, that shows how the maps $\pi_0\overline{\varphi}_\infty$ and $\pi_0\varphi_\infty$ are related to each other.

Recall the space

$$\mathcal{O}C_\infty := \mathcal{C}\big((I,\dot{I}),(F_\infty,\mathcal{T}\Sigma_\infty)\big)\Big/\Sigma_\infty,$$

and observe that

$$\operatorname{Im}\left(\overline{\varphi}_\infty\right) \subseteq \mathcal{O}C_\infty.$$

Thus, we use the same notation for $\overline{\varphi}_\infty : H_\infty \to \Omega C_\infty$ and its corestriction $\overline{\varphi}_\infty : H_\infty \to \mathcal{O}C_\infty$.

Proposition 2.16.

$$\pi_0\Sigma_\infty = \Sigma_\infty$$

Proof. Pick a path $\gamma : I \to \Sigma_\infty$, and define, for all $i \in \mathbb{N}$,

$$\begin{aligned} \gamma_i : I &\to \mathbb{N} \\ t &\mapsto \gamma(t)(i). \end{aligned}$$

By [11, Thm. 21.3], we know that γ maps convergent sequences to convergent sequences, which thus is also the case for the maps γ_i. As I is metric, it follows by the same theorem that γ_i is continuous for all $i \in \mathbb{N}$. As $\mathbb{N}$ is discrete, this shows that γ_i is constant for all $i \in \mathbb{N}$, which implies that γ is also constant. □

Proposition 2.17. *There is a continuous bijection*

$$\begin{aligned} \xi : \Sigma_\infty \times PH_\infty &\xrightarrow{\cong} H_\infty \\ (\sigma, h) &\mapsto \pi_{\Sigma H}(\sigma) \circ h. \end{aligned}$$

Proof. The continuity is given by Theorem 2.13. Moreover, we know by Proposition 2.12 that there is a split short exact sequence

$$1 \longrightarrow PH_\infty \hookrightarrow H_\infty \underset{\pi_{H\Sigma}}{\overset{\pi_{\Sigma H}}{\rightleftarrows}} \Sigma_\infty \longrightarrow 1\ ,$$

which proves the result (see [14, Prop 10.5]). □

Proposition 2.18. *The map*

$$\begin{aligned} \zeta : \Sigma_\infty \times \Omega F_\infty &\longrightarrow \mathcal{O}C_\infty \\ (\sigma, \beta) &\mapsto p_\infty \circ \big(K\big(\pi_{\Sigma H}(\sigma), \cdot\big)(\tau_i)\big)_{i\in\mathbb{N}} \star \beta(\cdot) \end{aligned}$$

induces a bijection

$$\pi_0\zeta : \Sigma_\infty \times \pi_1 F_\infty \xrightarrow{\cong} \pi_0\mathcal{O}C_\infty.$$

Proof. Notice that Proposition 1.7 allows us to define a map

$$\begin{array}{rcl} l : \mathcal{O}C_\infty & \to & \mathcal{C}\big((I,0,1),(F_\infty,\mathcal{T}_\infty\Sigma_\infty,\mathcal{T}_\infty)\big) \\ \overline{\beta} & \mapsto & \beta, \end{array}$$

where β is the unique lifting of $\overline{\beta}$ into $\mathcal{C}\big((I,0,1),(F_\infty,\mathcal{T}_\infty\Sigma_\infty,\mathcal{T}_\infty)\big)$ that satisfies $p_\infty \circ \beta = \overline{\beta}$. To see that this map is continuous, pick an open subset $U \subset \mathcal{C}\big((I,0,1),(F_\infty,\mathcal{T}_\infty\Sigma_\infty,\mathcal{T}_\infty)\big)$, and observe that $l^{-1}(U)$ is open in $\mathcal{O}C_\infty$, because, writing

$$q : \mathcal{C}\big((I,\dot{I}),(F_\infty,\mathcal{T}_\infty\Sigma_\infty)\big) \to \mathcal{O}C_\infty$$

for the quotient map, the preimage

$$q^{-1}\big(l^{-1}(U)\big) = \bigcup_{\sigma\in\Sigma_\infty} \big(\pi_{\Sigma H}(\sigma)\big)(U)$$

is open in $\mathcal{C}\big((I,\dot{I}),(F_\infty,\mathcal{T}_\infty\Sigma_\infty)\big)$.
Introduce maps

$$\begin{array}{rcl} \phi_p : \mathcal{O}C_\infty & \to & \Sigma_\infty \\ \gamma & \mapsto & \pi_{T\Sigma}\Big(l\big([\gamma]\big)(0)\Big) \end{array}$$

and

$$\begin{array}{rcl} \phi_l : \Sigma_\infty & \to & \mathcal{O}C_\infty \\ \sigma & \mapsto & p_\infty \circ \Big(K\big(\pi_{\Sigma H}(\sigma),\cdot\big)(\tau_i)\Big)_{i\in\mathbb{N}}, \end{array}$$

and observe that they are continuous, because all maps of which they are composed are continuous. (The continuity of the maps $\pi_{T\Sigma}$ and $\pi_{\Sigma H}$ is given by Theorem 2.13 and Proposition 2.15, respectively).
To verify that $\pi_0\zeta$ is a bijection, observe first that, for any $\sigma \in \Sigma_\infty$

$$\begin{array}{rcl} \phi_p \circ \phi_l(\sigma) & = & \pi_{T\Sigma}\Big(l \circ p_\infty\Big(K\big(\pi_{\Sigma H}(\sigma),\cdot\big)(\tau_i)\Big)(0)\Big) \\ & = & \pi_{T\Sigma}\Big(K\big(\pi_{\Sigma H}(\sigma),0\big)(\tau_i)\Big)_{i\in\mathbb{N}} \\ & = & \pi_{T\Sigma}\big(\pi_{\Sigma H}(\sigma)\big) \\ & = & \sigma. \end{array}$$

In particular, there is thus a split exact sequence of sets

$$1 \longrightarrow \Omega F_\infty \xrightarrow{\Omega p_\infty} \mathcal{O}C_\infty \underset{\phi_p}{\overset{\phi_l}{\rightleftarrows}} \Sigma_\infty \longrightarrow 1.$$

Recalling that $\pi_0\Sigma_\infty = \Sigma_\infty$, the result follows directly by considering the induced split exact sequence

$$1 \longrightarrow \pi_1 F_\infty \xrightarrow{\pi_1 p_\infty} \pi_0\mathcal{O}C_\infty \underset{\pi_0\phi_p}{\overset{\pi_0\phi_l}{\rightleftarrows}} \Sigma_\infty \longrightarrow 1,$$

because

$$\pi_0\zeta(\sigma, b) = \pi_0\phi_l(\sigma) \cdot b$$

(see [14, Prop 10.5]). $\square$

Theorem 2.19. *There is a commutative diagram of sets*

$$\begin{array}{ccc} \pi_0 H_\infty & \xrightarrow{\pi_0\overline{\varphi}_\infty} & \pi_0\mathcal{O}C_\infty \\ {\scriptstyle \pi_0\xi}\uparrow{\scriptstyle\cong} & & {\scriptstyle\cong}\uparrow{\scriptstyle \pi_0\zeta} \\ \Sigma_\infty \times \pi_0 PH_\infty & \xrightarrow{Id\times\pi_0\varphi_\infty} & \Sigma_\infty \times \pi_1 F_\infty.. \end{array}$$

Proof. First, we show that the following diagram of topological spaces commutes up to homotopy.

$$\begin{array}{ccc} H_\infty & \xrightarrow{\overline{\varphi}_\infty} & \mathcal{O}C_\infty \\ {\scriptstyle \xi}\uparrow & & \uparrow{\scriptstyle \zeta} \\ \Sigma_\infty \times PH_\infty & \xrightarrow{\mathrm{Id}\times\varphi_\infty} & \Sigma_\infty \times \Omega F_\infty \end{array}$$

Pick any $(\sigma, h) \in \Sigma_\infty \times PH_\infty$ and verify.

$$\begin{aligned} \overline{\varphi}_\infty \circ \xi(\sigma, h) &= \overline{\varphi}_\infty(f_\sigma \circ h) \\ &\overset{*}{\simeq} \overline{\varphi}_\infty(f_\sigma) \star \overline{\varphi}_\infty(h) \\ &= p_\infty \circ \big(K(f_\sigma, \cdot)(\tau_i)\big)_{i\in\mathbb{N}} \star p_\infty \circ \big(K(h, \cdot)(\tau_i)\big)_{i\in\mathbb{N}} \\ &= p_\infty \circ \Big(K(f_\sigma, \cdot)(\tau_i)_{i\in\mathbb{N}} \star K(h, \cdot)(\tau_i)_{i\in\mathbb{N}}\Big) \\ &= \zeta\big(\sigma, K(h, \cdot)(\tau_i)_{i\in\mathbb{N}}\big) \\ &= \zeta \circ (\mathrm{Id} \times \varphi_\infty)(\sigma, h), \end{aligned}$$

where $(*)$ is given by the fact that $\pi_0\overline{\varphi}_\infty$ is a homomorphism by Proposition 1.15. Thus, applying π_0 yields the required commutative diagram, because $\pi_0\Sigma_\infty = \Sigma_\infty$ by Proposition 2.16, and because the vertical maps are isomorphisms by Propositions 2.17 and 2.18. $\square$

This is a very useful result, because it reduces the question of the image and the kernel of $\pi_0\overline{\varphi}_\infty : \pi_0 H_\infty \to \pi_0\mathcal{O}C_\infty$ to the analogous question for the map $\pi_0\varphi_\infty : \pi_0 PH_\infty \to \pi_1 F_\infty$.

2.2.1 Conclusions

For finite $n \in \mathbb{N}$, the pure braid group PB_n is just a subgroup of the (full) braid group B_n, and in our context, the properties of the PB_n are analogous to those of the B_n. In the infinite case, the full braid group $\pi_1 C_\infty$ is more difficult to handle than the group of pure braids $\pi_1 F_\infty$, because there is an inverse system of pure braid groups PB_n with $\pi_1 F_\infty$ as its (category theoretic) limit, whereas the full braid groups B_n do not fit together as an inverse system. To solve the question of the image and the kernel of the homomorphism $\pi_0\overline{\varphi}_\infty : \pi_0 H_\infty \to \pi_1 C_\infty$, Theorem 2.19 allows us to bypass this difficulty, however, because the image and the kernel of $\pi_0\overline{\varphi}_\infty : \pi_0 H_\infty \to \pi_1 C_\infty$ are given directly in terms of the image and the kernel of $\pi_0\varphi_\infty : \pi_0 PH_\infty \to \pi_1 F_\infty$.

Chapter 3

The injectivity of the maps $\pi_0\varphi_\infty$ and $\pi_0\overline{\varphi}_\infty$

3.1 Definition of a suitable contracting homotopy of the space H_0

Definition 3.1. *Write $t_1 := 0$, and, for every $i \geq 2$,*

$$t_i := \sum_{k=1}^{i-1} \frac{1}{2^k}.$$

In particular, observe that $\lim_{i\to\infty} t_i = 1$. Recall that, by Definition 2.8,

$$\varrho_i := \|\tau_i - \tau_\infty\|.$$

Lemma 3.2. *There is a continuous map*

$$\kappa : [0,1) \to \mathcal{C}\Big(D^2, D^2\Big)$$

with the following properties.

$$\begin{array}{lll}
(i) & \kappa(t) : D^2 \to D^2 \text{ is a homeomorphism onto its image} & \forall t \in [0,1). \\
(ii) & Im\, \kappa(t_i) = \overline{B(\tau_\infty, \varrho_{i-1})}, & \forall i \in \mathbb{N}, \\
(iii) & Im\, \kappa(t) \subset B(\tau_\infty, \varrho_{i-1}) & \forall\ t > t_i,\ \forall i \in \mathbb{N}, \\
(iv) & \kappa(t)(x) = x & \forall x \in \overline{B(\tau_\infty, \varrho_i)},\ \forall t \in [0, t_i],\ \forall i \in \mathbb{N}. \\
(v) & \kappa(0) = Id_{D^2} &
\end{array}$$

Moreover, κ contracts D^2 along radii, so that, for each $t \in I$, there is an $r \in [0,1]$ with

$$Im\ \kappa(t) = \overline{B(\tau_\infty, r)}, \quad \text{and} \quad \partial Im\ \kappa(t) = \partial\overline{B(\tau_\infty, r)}.$$

Proof. For each $i \in \mathbb{N}$, define a map

$$R_i : I \to \mathcal{C}\big([0,1],[0,\varrho_{i-1}]\big)$$

by

$$R_1(t)(r) = \begin{cases} \varrho_2 + \big(r - \varrho_2\big)\left(1 - t\frac{1-\varrho_1}{1-\varrho_2}\right) & \text{, if } r \in [\varrho_2, 1] \\ r & \text{, if } r \in [0, \varrho_2], \end{cases}$$

for $i = 1$, and

$$R_i(t)(r) = \begin{cases} \varrho_{i+1} + \big(R_{i-1}(1)(r) - \varrho_{i+1}\big)\left(1 - t\frac{\varrho_{i-1}-\varrho_i}{\varrho_{i-1}-\varrho_{i+1}}\right) & \text{, if } r \in [\varrho_{i+1}, 1] \\ r & \text{, if } r \in [0, \varrho_{i+1}] \end{cases}$$

for $i \geq 2$, and for all $t \in I$. Note that for each $t \in I$, $R_1(t)$ is well defined at ϱ_2, because

$$R_1(t)(\varrho_2) = \varrho_2 + \big(\varrho_2 - \varrho_2\big)\left(1 - t\frac{1-\varrho_1}{1-\varrho_2}\right) = \varrho_2 \qquad \forall t \in I.$$

Also, for all $i \geq 2, t \in I$, $R_i(t)$ is well defined at ϱ_{i+1}, because, as $R_{i-1}(1)(\varrho_{i+1}) = \varrho_{i+1}$,

$$R_i(t)(\varrho_{i+1}) = \varrho_{i+1} + \big(\varrho_{i+1} - \varrho_{i+1}\big)\left(1 - t\frac{\varrho_{i-1}-\varrho_i}{\varrho_{i-1}-\varrho_{i+1}}\right) = \varrho_{i+1}.$$

For each $i \in \mathbb{N}$, the map R_i has the following properties:

$$\begin{array}{ll} (A) & \text{Im } R_i(1) = [0, \varrho_i], \\ (B) & \text{Im } R_i(t) \subset [0, \varrho_{i-1}) \qquad \forall t \in (0,1], \\ (C) & R_i(t)(r) = r \quad \forall r \in [0, \varrho_{i+1}], \ \ \forall t \in I. \\ (D) & R_i(1) = R_{i+1}(0) \end{array}$$

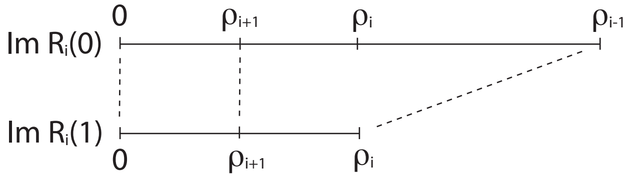
The properties (C) and (D) follow directly from the definition of R_i. To verify the properties (A) and (B), observe that

$$R_i(0)(0) = 0,$$

and, by induction,

$$\begin{array}{rcl} R_i(1)(1) & = & \varrho_i \quad \text{and} \\ R_i(0)(1) & = & \varrho_{i-1}, \end{array}$$

and that, moreover, $R_i(t)(r)$ is strictly increasing in r and strictly decreasing in t.



Now, define a continuous map

$$\widetilde{\kappa} : [0,1) \to \mathcal{C}\Big(I \times \mathbb{R}_{mod\ 2\pi}, I \times \mathbb{R}_{mod\ 2\pi}\Big)$$

piecewise by

$$\begin{array}{rcl} \widetilde{\kappa}|_{[t_i,t_{i+1}]} : [t_i, t_{i+1}] & \to & \mathcal{C}\Big(I \times \mathbb{R}_{mod\ 2\pi}, I \times \mathbb{R}_{mod\ 2\pi}\Big) \\ t & \mapsto & \Big((r,\phi) \mapsto \Big(R_i\left(\frac{t-t_i}{t_{i+1}-t_i}\right)(r), \phi\Big)\Big) \end{array}$$

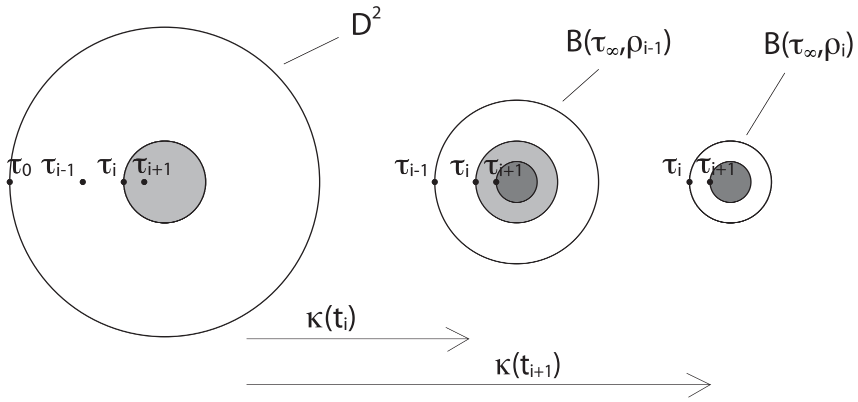
for all $i \in \mathbb{N}$. This map is well defined, because, at each t_i,

$$\begin{array}{rcl} \widetilde{\kappa}|_{[t_{i-1},t_i]}(t_i)(r,\phi) & = & \Big(R_{i-1}\left(\frac{t_i - t_{i-1}}{t_i - t_{i-1}}\right)(r), \phi\Big) \\ & = & \Big(R_{i-1}(1)(r), \phi\Big) \\ & \overset{(D)}{=} & \Big(R_i(0)(r), \phi\Big) \\ & = & \widetilde{\kappa}|_{[t_i,t_{i+1}]}(t_i)(r,\phi) \end{array}$$

for all $(r,\phi) \in I \times \mathbb{R}_{mod\ 2\pi}$. Identifying $I \times \mathbb{R}_{mod\ 2\pi}$ with the polar coordinates of D^2 turns $\widetilde{\kappa}$ into a map

$$\kappa : [0,1) \to \mathcal{C}\big(D^2, D^2\big).$$

For each $i \in \mathbb{N}$, the restricted map $\kappa|_{[t_i,t_{i+1}]}$ is represented as follows, where the grey zones are mapped by the identity.



As, for each $i \in \mathbb{N}$, the map $R_i(t)$ is open and injective for all $t \in I$, it follows that $\kappa(t)$ too is open and injective for all $t \in [0,1)$. Moreover, by the definition of κ, the properties (ii), (iii) and (iv) follow from (A), (B) and (C), respectively, and (v) follows from the fact that, at $t_1 \equiv 0$, $\widetilde{\kappa}|_{[t_1,t_2]}(0) = \mathrm{Id}_{I\times\mathbb{R}_{mod\ 2\pi}}$ □

Theorem 3.3. *There is a contracting homotopy $K : H_0 \times I \to H_0$, i.e., for all $f \in H_0$,*

$$K(f,0) = f, \quad K(f,1) = Id_{D^2}$$

with the following properties. For all $h \in PH_\infty$ and $i \in \mathbb{N}$,

$$\begin{array}{lll} (a) & K(h,t_i) \in PH_\infty, & \\ (b) & K(h,t)(\tau_i) = \tau_i & \forall t \in [0,t_i] \cup [t_{i+1},1], \end{array}$$

For each $h \in H_0$ that satisfies $h(\tau_\infty) = \tau_\infty$, in particular for all $h \in H_\infty$,

$$(c) \quad K(h,t)(\tau_\infty) = \tau_\infty \qquad \forall t \in I.$$

Furthermore, for all $f \in H_0$ and $i \in \mathbb{N}$,

$$\begin{array}{lll} (d) & K(f,t)(x) = x & \forall x \in D^2 \setminus B(\tau_\infty, \varrho_i), \ \forall t \geq t_{i+1} \\ (e) & K(f,t)(x) \in B(\tau_\infty, \varrho_i) & \forall x \in B(\tau_\infty, \varrho_i), \ \forall t \geq t_{i+1}. \end{array}$$

Finally,

$$(f) \ \ K(Id_{D^2}, t) = Id_{D^2} \quad \forall t \in I.$$

Proof. Recalling the injectivity of $\kappa(t) : D^2 \to D^2$ for all $t \in I$, define a map

$$\begin{array}{rcl} K : H_0 \times I & \to & H_0 \\ (h,t) & \mapsto & \left(x \mapsto \begin{cases} \big(\kappa(t) \circ h \circ \kappa^{-1}(t)\big)(x) & , \forall x \in \ \operatorname{Im} \kappa(t), \quad \forall t \in [0,1) \\ x & , \text{else} \end{cases} \right). \end{array}$$

We show that K is a contraction of H_0 with the required properties. First, we verify that K is well defined.
Observe that, for any $(h,t) \in H_0 \times [0,1)$, and $x \in \partial \operatorname{Im} \kappa(t)$,

$$\begin{array}{rcl} K(h,t)(x) & = & \big(\kappa(t) \circ h \circ \kappa^{-1}(t)\big)(x) \\ & = & \kappa(t)\Big(h\big(\kappa^{-1}(t)(x)\big)\Big) \\ & \overset{*}{=} & \kappa(t)\big(\kappa^{-1}(t)\big)(x) \\ & = & x, \end{array}$$

where $(*)$ is given by the fact that $\kappa^{-1}(t)(x) \in \partial D^2$, and because h fixes ∂D^2 pointwise. To see that, moreover, $\operatorname{Im} K \subset H_0$, pick some arbitrary $(h,t) \in H_0 \times I$. As $\kappa(t) : D^2 \to D^2$ is open, $K(h,t) : D^2 \to D^2$ is continuous, and, observing that $K(h,t)$ is mutually inverse to $K(h^{-1},t)$, it follows that $K(h,t) \in \mathcal{H}(D^2, D^2)$. Moreover, by the definition of K, we know that

$$K(h,t)|_{\partial D^2} = \operatorname{Id}_{\partial D^2},$$

i.e.,

$$K(h,t) \in H_0.$$

To verify that $K : H_0 \times I \to H_0$ is a continuous map, notice that $H_0 \times I$ is metric, as, by [11, Thms. 46.7/46.8] H_0 has the uniform topology, which in particular is *metric*. Thus, by [11, Thm. 21.3], we can verify the continuity of K by showing that, for any convergent sequence $(h_i, s_i)_{i\in\mathbb{N}}$ with

$$\lim_{i\to\infty} (h_i, s_i) =: (h, s),$$

the sequence $\{K(h_i, s_i)\}_{i\in\mathbb{N}}$ converges to $K(h, s)$ for $i \to \infty$.
Observe that K is continuous at all $(h, s) \in H_0 \times [0, 1)$, and pick a sequence $(h_i, s_i)_{i\in\mathbb{N}}$ that converges to $(h, 1)$ for some $h \in H_0$. We need to prove that

$$\lim_{i\to\infty} K(h_i, s_i) = \mathrm{Id}_{D^2}.$$

Recalling Definition 3.1, note that, by Lemma 3.2 (*ii*),

$$\|x - K(h, t_i)(x)\| < 2\|\tau_\infty - \tau_{i-1}\| \quad \forall x \in D^2 \quad \forall h \in H_0.$$

Write

$$k_i := \max\{j \in \mathbb{N} \mid t_j < s_i\},$$

and observe that

$$\lim_{i\to\infty} k_i = \infty,$$

because $\lim_{i\to\infty} s_i = 1$. Thus, it follows by Lemma 3.2 (*iii*) that

$$\|x - K(h_i, s_i)(x)\| < 2\|\tau_\infty - \tau_{k_i}\| \quad \forall x \in D^2 \quad \forall i \in \mathbb{N}.$$

In particular, as $\lim_{i\to\infty} \tau_i = \tau_\infty$, there is, for every $\varepsilon > 0$, an integer N_ε such that

$$\|x - K(h_i, s_i)(x)\| < \varepsilon \quad \forall x \in D^2 \quad \forall i \geq N_\varepsilon.$$

This proves that the sequence $\big(K(h_i, s_i)\big)_{i\in\mathbb{N}}$ converges uniformly to Id_{D^2} which means that, as H_0 has the uniform topology,

$$\lim_{i\to\infty} K(h_i, s_i) = \mathrm{Id}_{D^2}.$$

Thus,

$$K \in \mathcal{C}(H_0 \times I, H_0).$$

Moreover, it is easy to see by the definition of K that, for every $h \in H_0$,

$$K(h, 0) = h, \qquad K(h, 1) = \mathrm{Id}_{D^2}.$$

Fix some $h \in PH_\infty$ and $i \in \mathbb{N}$, and observe as follows that the homotopy K has the required properties.

Verification of (a). It follows from part (*ii*) of Lemma 3.2 that, for all $j \in [1, i-2]$, $\tau_j \notin \mathrm{Im}\ \kappa(t_i)$, and thus, by the definition of K,

$$K(h, t_i)(\tau_j) = \tau_j \qquad \forall j \in [1, i-2].$$

Moreover, $\tau_{i-1} \in \partial\text{Im } \kappa(t_i)$, which, by the definition of K, means that

$$K(h, t_i)(\tau_{i-1}) = \tau_{i-1}$$

Finally, for every $j \geq i$, $\tau_j \in \text{Im } \kappa(t_i)$, which means that

$$\begin{aligned} K(h, t_i)(\tau_j) &= \big(\kappa(t_i) \circ h \circ \kappa(t_i)^{-1}\big)(\tau_j) \\ &\overset{*}{=} \kappa(t_i)\big(h(\tau_j)\big) \\ &\overset{**}{=} \kappa(t_i)(\tau_j) \\ &\overset{*}{=} \tau_j, \end{aligned}$$

where $(*)$ follows from part (iv) of Lemma 3.2, and $(**)$ is given by the fact that $h \in PH_\infty$.

Verification of (b). By part (iii) of Lemma 3.2, we know that, for each $j \in [1, i-2]$,

$$\tau_j \notin \text{Im } \kappa(t) \qquad \forall t \in [t_i, t_{i+1}],$$

and so, by the definition of K,

$$K(h, t)(\tau_j) = \tau_j \qquad \forall j \in [1, i-2], \quad \forall t \in [t_i, t_{i+1}].$$

Again, $\tau_{i-1} \in \partial\text{Im } \kappa(t_i)$, which means that, by the definition of K,

$$K(h, t)(\tau_{i-1}) = \tau_{i-1} \quad \forall t \in [t_i, t_{i+1}].$$

Furthermore, for each $j \geq i+1$, $\tau_j \in \overline{B(\tau_\infty, \varrho_{i+1})}$, such that, by part (iv) of Lemma 3.2,

$$\kappa(t)(\tau_j) = \tau_j \quad \forall t \in [0, t_{i+1}].$$

Thus, by the definition of K, for all $t \in [t_i, t_{i+1}]$,

$$\begin{aligned} K(h, t)(\tau_j) &= \big(\kappa(t) \circ h \circ \kappa(t)^{-1}\big)(\tau_j) \\ &= \kappa(t_i)\big(h(\tau_j)\big) \\ &\overset{*}{=} \kappa(t_i)(\tau_j) \\ &= \tau_j, \end{aligned}$$

where $(*)$ is given by the fact that $h \in PH_\infty$.

Verification of (c). Observe that, by part (iv) of Lemma 3.2,

$$\kappa(t)(\tau_\infty) = \tau_\infty \qquad \forall t \in I.$$

As, moreover, $h(\tau_\infty) = \tau_\infty$, it follows by the definition of $K : H_0 \times I \to H_0$ that

$$K(h, t)(\tau_\infty) = \tau_\infty \qquad \forall t \in I.$$

Finally, we check the properties (d) and (e) for a given $f \in H_0$.
Verification of (d). Pick some $\widehat{t} \geq t_{i+1}$ and $\widehat{x} \in D^2 \setminus B(\tau_\infty, \varrho_i)$. If $\widehat{t} = t_{i+1}$, then, by part (ii) of Lemma 3.2,

$$\widehat{x} \notin \text{Int Im } \kappa(\widehat{t}),$$

and, if $\widehat{t} > t_{i+1}$, then, this same fact holds by part (iii) of Lemma 3.2. Thus, by the definition of K,

$$K(f, \widehat{t})(\widehat{x}) = \widehat{x}.$$

Verification of (e). Pick any $f \in H_0$, $\widehat{t} \geq t_{i+1}$ and $\widehat{x} \in B(\tau_\infty, \varrho_i)$. If $\widehat{x} \notin \text{Im } \kappa(\widehat{t})$, then, by the definition of K,

$$K(f, \widehat{t})(\widehat{x}) = \widehat{x} \quad \in B(\tau_\infty, \varrho_i).$$

On the other hand, if $\widehat{x} \in \text{Im } \kappa(\widehat{t})$, then, as

$$K(f, \widehat{t})(\widehat{x}) = \big(\kappa(\widehat{t}) \circ f \circ \kappa(\widehat{t})^{-1}\big)(\widehat{x}) \quad \in \text{Im } \kappa(\widehat{t}),$$

it follows that

$$K(f, \widehat{t})(\widehat{x}) \in B(\tau_\infty, \varrho_i),$$

which is given by part (ii) of Lemma 3.2 if $\widehat{t} = t_{i+1}$, and by part (ii) of Lemma 3.2 if $t > t_{i+1}$. Thus, resuming these facts,

$$K(f, t)(x) \in B(\tau_\infty, \varrho_i) \qquad \forall f \in H_0,\ \ \forall x \in B(\tau_\infty, \varrho_i),\ \ \forall t \geq t_{i+1}.$$

Finally, the property (f) follow directly from the definition of K. □

3.2 Proof of the injectivity of $\pi_0\varphi_\infty$

Given two elements $f, g \in PH_\infty$ such that

$$\pi_0\varphi_\infty[f] = \pi_0\varphi_\infty[g],$$

we need to prove that there is a path in PH_∞ from f to g. Our construction of such a path, which is given in the proof of Theorem 3.7, requires some preliminary work.
Henceforth, the map

$$\begin{array}{rcl} \varphi_\infty : PH_\infty & \rightarrow & \Omega F_\infty \\ h & \mapsto & \big(K(h, \cdot)(\tau_i)\big)_{i \in \mathbb{N}}, \end{array}$$

is assumed to be given in terms of the homotopy K defined in Theorem 3.3. For all $x \in D^2$, let $p_x : I \rightarrow D^2$ be the constant path at x.

Lemma 3.4. *Assume that*

$$\pi_0\varphi_\infty[h] = 1$$

for some $h \in PH_\infty$*. Then,*

$$\big[\big(K(h,\cdot)(\tau_i)\big)_{i\in[1,n]\cup\infty}\big] = \big[(p_{\tau_i})_{i\in[1,n]\cup\infty}\big],$$

for all $n \in \mathbb{N}$*.*

Proof. Fix some $n \in \mathbb{N}$, and write

$$(\beta_i)_{i\in\mathbb{N}} := \big(K(h,\cdot)(\tau_i)\big)_{i\in\mathbb{N}}, \quad \text{and} \quad \beta_\infty := K(h,\cdot)(\tau_\infty).$$

By point (c) of Theorem 3.3, $\beta_\infty = p_{\tau_\infty}$, which implies that $\beta_i(t) \neq \tau_\infty$ for all $t \in I$ and all $i \in \mathbb{N}$. Thus, by the tube lemma, there is an $\varepsilon > 0$ that satisfies

$$\beta_i(t) \notin B(\tau_\infty, 4\varepsilon) \qquad \forall t \in I,\ \forall i \in [1,n]. \qquad (A)$$

Notice that, by the continuity of the map $K(h,\cdot) : I \times D^2 \to D^2$, the subset

$$K(h,\cdot)^{-1}\big(B(\tau_\infty,\varepsilon)\big) \subset I \times D^2$$

is an open neighbourhood of $I \times \{\tau_\infty\}$, such that, by the tube lemma, there is an $r > 0$ that satisfies

$$I \times \overline{B(\tau_\infty, r)} \subset K(h,\cdot)^{-1}\big(B(\tau_\infty,\varepsilon)\big),$$

i.e.,

$$K(h,t)\big(\overline{B(\tau_\infty, r)}\big) \subset B(\tau_\infty,\varepsilon) \quad \forall t \in I. \qquad (B)$$

Pick some $N \in \mathbb{N}$ with

$$\tau_N \in B(\tau_\infty, r),$$

and observe that, by (B),

$$\beta_N(t) \in B(\tau_\infty,\varepsilon) \quad \forall t \in I.$$

Thus, in particular,

$$B(\beta_N(t), 2\varepsilon) \subset B(\tau_\infty, 3\varepsilon) \qquad \forall t \in I,$$

which means that, by (A),

$$\|\beta_i(t) - \beta_N(t)\| \geq 2\varepsilon \quad \forall i \in [1,n],\ \forall t \in I. \qquad (C)$$

As $\pi_0\varphi_\infty[h] = 1$, there is a path $\widehat{\Lambda} := (\widehat{\Lambda}_i)_{i\in\mathbb{N}} : I \to \Omega\big(F_\infty, \mathcal{T}_\infty\big)$ with

$$\widehat{\Lambda}_i(0) = \beta_i, \quad \widehat{\Lambda}_i(1) = p_{\tau_i} \qquad \forall i \in \mathbb{N}. \qquad (D)$$

Using this path, we construct a path Λ in the following way. Let

$$\xi : D^2 \setminus 0 \xrightarrow{\cong} D^2 \setminus B(0,\varepsilon) \qquad (E)$$

be a homeomorphism that satisfies

$$\xi|_{D^2\setminus B(0,2\varepsilon)} = \mathrm{Id}, \qquad (F)$$

Clearly, such a homeomorphism exists. For the sequel of this proof, keep in mind that

$$\tau_\infty = 0 \quad \big(= (0,0) \in \mathbb{R}^2 \big).$$

Define a path $\Lambda := (\Lambda_i)_{i\in[1,n]\cup\infty} : I \to \Omega\big(F_{n+1}\big(\overline{B(\tau_\infty, 2)}\big), (\tau_1, \dots, \tau_n, \tau_\infty)\big)$ by

$$\begin{aligned} \Lambda_i(s)(t) &:= \widehat{\Lambda}_N(s)(t) + \xi\big(\widehat{\Lambda}_i(s)(t) - \widehat{\Lambda}_N(s)(t)\big) \qquad \forall i \in [1,n], \\ \Lambda_\infty(s)(t) &:= \widehat{\Lambda}_N(s)(t) + (1-s)\big(\tau_\infty - \beta_N(t)\big) + s\big(\tau_\infty - \tau_N\big) \end{aligned}$$

for all $s, t \in I$. First, we verify that $\Lambda(s)$ is a well defined loop in $\big(F_{n+1}\big(\overline{B(\tau_\infty, 2)}\big), (\tau_1, \dots, \tau_n, \tau_\infty)\big)$ for all $s \in I$. Clearly,

$$\Lambda_i(s)(t) \in \overline{B(\tau_\infty, 2)} \quad \forall t \in I, \forall s \in I.$$

Fix some $s \in I$, and observe that, for all $i \in [1,n]$,

$$\Lambda_i(s)(0) = \Lambda_i(s)(1) = \tau_N + \xi\big(\tau_i - \tau_N\big) \overset{*}{=} \tau_N + \tau_i - \tau_N = \tau_i \quad \forall s \in I,\ \forall i \in [1,n],$$

where $(*)$ is given by (F), because, by (C),

$$\|\tau_i - \tau_N\| > 2\varepsilon \qquad \forall i \in [1,n].$$

Also,

$$\Lambda_\infty(s)(0) = \Lambda_\infty(s)(1) = \tau_N + (1-s)\big(\tau_\infty - \tau_N\big) + s\big(\tau_\infty - \tau_N\big) = \tau_\infty \qquad \forall s \in I.$$

Thus, $\Lambda(s)(0) = \Lambda(s)(1) = (\tau_1, \dots, \tau_n, \tau_N, \tau_\infty)$. Furthermore, it is easy to see that, for every $s, t \in I$, the points $\big(\Lambda_i(s)(t)\big)_{i\in[1,n]}$ are pairwise distinct, i.e.,

$$\big(\Lambda_i(s)(t)\big)_{i\in[1,n]} \quad \in \quad F_n\big(\overline{B(\tau_\infty, 2)}\big) \qquad \forall s, t \in I.$$

By (E), we know that $\mathrm{Im}\,\xi \cap B(0,\varepsilon) = \varnothing$, which shows that, in particular,

$$\|\xi\big(\widehat{\Lambda}_i(s)(t) - \widehat{\Lambda}_N(s)(t)\big)\| > \varepsilon \qquad \forall s, t \in I,\ \forall i \in [1,n].$$

Thus, by the definition of the Λ_i's,

$$\Lambda_i(s)(t) \notin B(\widehat{\Lambda}_N(s)(t), \varepsilon) \qquad \forall s, t \in I,\ \forall i \in [1,n]. \qquad (G)$$

Now, observe that, by (B), and by the choice of τ_N,

$$\|\tau_\infty - \tau_N\| < \varepsilon, \text{ and } \|\tau_\infty - \beta_N(t)\| < \varepsilon \quad \forall t \in I,$$

which means that

$$\Lambda_\infty(s)(t) := \widehat{\Lambda}_N(s)(t) + (1-s)\big(\tau_\infty - \beta_N(t)\big) + s\big(\tau_\infty - \tau_N\big) \in B\big(\widehat{\Lambda}_N(s)(t), \varepsilon\big) \qquad \forall s, t \in I.$$

Thus, by (G),

$$\Lambda_\infty(s)(t) \cap \big\{\Lambda_i(s)(t)\big\}_{i\in[1,n]} = \varnothing \qquad \forall s,t \in I,$$

i.e., $\Lambda(s)$ is a well defined loop in $\big(F_{n+1}\big(\overline{B(\tau_\infty,2)}\big),(\tau_1,\dots,\tau_n,\tau_\infty)\big)$.
Notice that, as I is compact, there is, for all $i \in [1,n]$, a $\widehat{t_i} \in I$, such that

$$\|\beta_i(t) - \tau_\infty\| \le \|\beta_i(\widehat{t_i}) - \tau_\infty\| \quad \forall i \in [1,n], \forall t \in I$$

by the extreme value theorem [11, Thm. 27.4]. As

$$\beta_i(t) \in \overset{\circ}{D}{}^2 \quad \forall t \in I, \forall i \in [1,n],$$

there is some $\widehat{r}$ with

$$\max_{i\in[1,n]} \|\beta_i(\widehat{t_i}) - \tau_\infty\| < \widehat{r} < 1,$$

such that, in particular,

$$\sup_{t\in I, i\in[1,n]} \|\beta_i(t) - \tau_\infty\| < \widetilde{r} < 1.$$

Let $\chi : \overline{B(\tau_\infty,2)} \to D^2$ be a homeomorphism such that

$$\chi|_{\overline{B(\tau_\infty,\widetilde{r})}} = \mathrm{Id}. \qquad (H)$$

Clearly, $\chi \circ \Lambda$ is a well defined path in $\big(F_{n+1},(\tau_1,\dots,\tau_n,\tau_\infty)\big)$. We show that, furthermore, $\chi\circ\Lambda(0) = (\beta_i)_{i\in[1,n]\cup\infty}$ and $\chi\circ\Lambda(1) = (p_{\tau_i})_{i\in[1,n]\cup\infty}$, which proves the lemma.
For all $i \in [1,n]$,

$$\begin{aligned}
\chi\circ\Lambda_i(0)(t) &= \chi\circ\Big(\widehat{\Lambda}_N(0)(t) + \xi\big(\widehat{\Lambda}_i(0)(t) - \widehat{\Lambda}_N(0)(t)\big)\Big)\\
&= \chi\circ\Big(\beta_N(t) + \xi\big(\beta_i(t) - \beta_N(t)\big)\Big)\\
&\overset{C,F}{=} \chi\circ\big(\beta_N(t) + \beta_i(t) - \beta_N(t)\big)\\
&\overset{H}{=} \beta_i(t).
\end{aligned}$$

On the other hand, again for all $i \in [1,n]$,

$$\begin{aligned}
\chi\circ\Lambda_i(1)(t) &= \chi\circ\Big(\widehat{\Lambda}_N(1)(t) + \xi\big(\widehat{\Lambda}_i(1)(t) - \widehat{\Lambda}_N(1)(t)\big)\Big)\\
&= \chi\circ\Big(\tau_N + \xi\big(\tau_i - \tau_N\big)\Big)\\
&\overset{C,F}{=} \chi\circ\big(\tau_N + \tau_i - \tau_N\big)\\
&\overset{H}{=} \tau_i.
\end{aligned}$$

Finally,

$$\begin{aligned}\chi\circ\Lambda_\infty(0)(t) &= \chi\circ\big(\widehat{\Lambda}_N(0)(t)+\tau_\infty-\beta_N(t)\big)\\ &= \chi\circ\big(\beta_N(t)+\tau_\infty-\beta_N(t)\big)\\ &\overset{H}{=} \tau_\infty,\end{aligned}$$

and

$$\begin{aligned}\chi\circ\Lambda_\infty(1)(t) &= \chi\circ\big(\widehat{\Lambda}_N(1)(t)+\tau_\infty-\tau_N\big)\\ &= \chi\circ\big(\tau_N+\tau_\infty-\tau_N\big)\\ &\overset{H}{=} \tau_\infty.\end{aligned}$$

□

Corollary 3.5. *Let h be an element of PH_∞ that satisfies*

$$\pi_0\varphi_\infty\big[h\big]=1.$$

Then, for every $n\in\mathbb{N}$, there is a path $\Gamma_n : I\to H_0$ such that

$$\Gamma_n(0)=Id,\quad \Gamma_n(1)=h^{-1},\quad \Gamma_n(t)(\tau_i)=\tau_i\quad \forall i\in[1,n]\cup\infty.$$

Proof. Fix some $n\in\mathbb{N}$, and define a point in F_{n+1} by

$$\widetilde{\mathcal{T}}_{n+1}:=\big(\tau_i\big)_{i\in[1,n]\cup\infty}.$$

Also, define a space $\widetilde{F}_{n+1}$ similarly to F_{n+1}, by replacing $\mathcal{T}_{n+1}$ by $\widetilde{\mathcal{T}}_{n+1}$. Also, define a map

$$\widetilde{\varphi}_{n+1}: PH_{n+1}\to\Omega\widetilde{F}_{n+1}$$

by

$$\widetilde{\varphi}_{n+1}\big([h]\big)=\big(K(h,\cdot)(\tau_i)\big)_{i\in[1,n]\cup\infty}.$$

As we pointed out earlier, the map $\pi_0\varphi_i:\pi_0PH_i\to PB_i$ is an isomorphism for all $i\in\mathbb{N}$, independently of the choice of the basepoint $\mathcal{T}_i$ of F_i. In particular, the map

$$\pi_0\widetilde{\varphi}_{n+1}:\pi_0PH_{n+1}\to\pi_1\widetilde{F}_{n+1}$$

is an isomorphism. Furthermore, observe that, for all $h\in PH_\infty$

$$\pi_0\widetilde{\varphi}_{n+1}[h]=\Big[\big(K(h,\cdot)(\tau_i)\big)_{i\in[1,n]\cup\infty}\Big]\overset{*}{=}\Big[\big(p_{\tau_i}\big)_{i\in[1,n]\cup\infty}\Big]=\pi_0\widetilde{\varphi}_{n+1}[\mathrm{Id}_{D^2}],$$

where $(*)$ follows from Lemma 3.4, because $\pi_0\varphi_\infty[h]=1$. As the map $\pi_0\widetilde{\varphi}_{n+1}:\pi_0\widetilde{PH}_{n+1}\to\pi_1\widetilde{F}_{n+1}$ is injective, it thus follows that

$$\big[h\big]=\big[\mathrm{Id}\big]\qquad\text{in}\quad \pi_0PH_{n+1},$$

i.e., there is a path $\widehat{\Gamma}_n : I \to H_0$ such that

$$\widehat{\Gamma}_n(0) = h, \quad \widehat{\Gamma}_n(1) = \mathrm{Id}, \quad \widehat{\Gamma}_n(t)(\tau_i) = \tau_i \quad \forall i \in [1,n] \cup \infty.$$

Thus, the path

$$\Gamma_n := h^{-1} \circ \widetilde{\Gamma}_n$$

satisfies the required properties, because h is in PH_∞. □

Lemma 3.6. *Let h be an element of PH_∞ that satisfies*

$$\pi_0 \varphi_\infty \big[h\big] = 1.$$

Then, for every $i \in \mathbb{N}$, there is a path $\Gamma_i : I \to PH_\infty$, such that

$$\begin{aligned} &(i) \quad \Gamma_i(0) = h, \\ &(ii) \quad K\big(\Gamma_i(1), t\big) \in PH_\infty \qquad \forall t \in [t_i, t_{i+1}]. \end{aligned}$$

Proof. We show the existence of such a path Γ_i for some arbitrary, fixed $i \in \mathbb{N}$. Recall that, by part (iv) of Lemma 3.2, the map

$$\kappa : [0,1) \to \mathcal{C}\big(D^2, D^2\big)$$

satisfies

$$\kappa(t)(\tau_\infty) = \tau_\infty \qquad \forall t \in [0,1), \forall i \in \mathbb{N}.$$

Thus, in particular

$$\kappa(t)^{-1}(\tau_i) \neq \tau_\infty, \quad h^{-1}\big(\kappa(t)^{-1}\big)(\tau_i) \neq \tau_\infty \qquad \forall t \in [0,1).$$

By the continuity of both $\kappa(\cdot)^{-1}(\tau_i) : I \to D^2$ and $h^{-1}\big(\kappa(\cdot)^{-1}\big)(\tau_i) : I \to D^2$, and by the fact that D^2 is normal, it follows that there is a large enough $N \in \mathbb{N}$ that satisfies

$$\begin{aligned} &(A) \quad \kappa(t)^{-1}(\tau_i) \qquad\quad \in D^2 \setminus B(\tau_\infty, \varrho_N) \quad \forall t \in [t_i, t_{i+1}], \\ &(B) \quad h^{-1}\big(\kappa(t)^{-1}\big)(\tau_i) \in D^2 \setminus B(\tau_\infty, \varrho_N) \quad \forall t \in [t_i, t_{i+1}], \end{aligned}$$

because $\lim_{j \to \infty} \varrho_j = 0$. Moreover, according Corollary 3.5, there is a path

$$\widehat{\Gamma}_i : I \to PH_N, \quad \text{s.t.} \quad \widehat{\Gamma}_i(0) = \mathrm{Id}, \quad \widehat{\Gamma}_i(1) = h^{-1}, \quad \widehat{\Gamma}_i(t)(\tau_\infty) = \tau_\infty \quad \forall t \in I.$$

Choose reals $\widetilde{\varrho}_o, \widetilde{\varrho}_m, \widetilde{\varrho}_i$ with

$$\varrho_N > \widetilde{\varrho}_o > \widetilde{\varrho}_m > \widetilde{\varrho}_i > \varrho_{N+1},$$

and write

$$D_o := B(\tau_\infty, \widetilde{\varrho}_o), \;\; D_m := B(\tau_\infty, \widetilde{\varrho}_m), \;\; D_i := B(\tau_\infty, \widetilde{\varrho}_i).$$

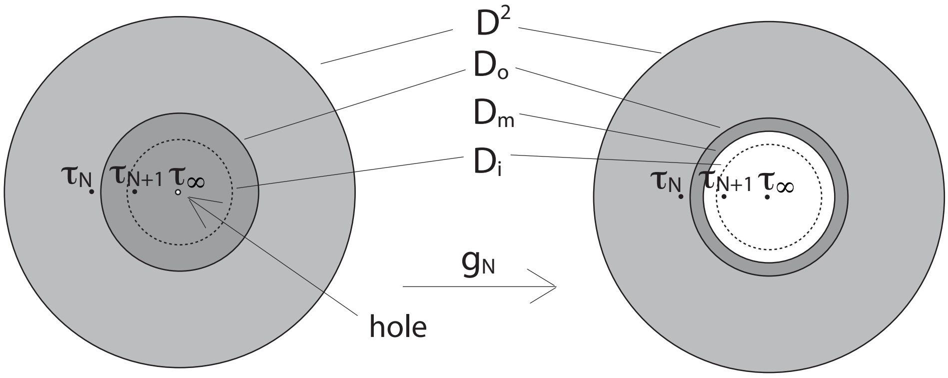
Let g_N be a homeomorphism

$$g_N : D^2 \setminus \tau_\infty \xrightarrow{\cong} D^2 \setminus D_m,$$

that contracts $D^2 \setminus \tau_\infty$ along radii, and that satisfies

$$g_N|_{D^2 \setminus D_o} = \mathrm{Id}|_{D^2 \setminus D_o}.$$

Thus, in the following drawing, g_n maps the outer grey area by the identity.



Consider the path

$$g_N \circ \widehat{\Gamma}_i(\cdot) \circ g_N^{-1} : I \to \mathcal{H}(D^2 \setminus D_m),$$

and extend it to a path $\widetilde{\Gamma}_i : I \to \mathcal{H}(D^2)$ defined by

$$\widetilde{\Gamma}_i(t)(x) = \begin{cases} \Big(g_N \circ \widehat{\Gamma}_i(t) \circ g_N^{-1}\Big)(x) & , x \in D^2 \setminus D_m \\ \mathrm{ext}(t)(x) & , x \in D_m \setminus D_i \\ x & , x \in D_i, \end{cases}$$

for all $t \in I, x \in D^2$, where

$$\mathrm{ext} : I \to \mathcal{H}\big(D_m \setminus D_i\big)$$

can be chosen as follows. Endowing D^2 with polar coordinates, define a homeomorphism

$$\arg := g_N \circ \widehat{\Gamma}_i(t) \circ g_N^{-1}|_{\partial D_m} : \mathbb{R}_{mod\ 2\pi} \xrightarrow{\cong} \mathbb{R}_{mod\ 2\pi},$$

and write

$$\begin{array}{rcl} \mathrm{ext} : [\varrho_i, \varrho_m] \times \mathbb{R}_{mod\ 2\pi} & \xrightarrow{\cong} & [\varrho_i, \varrho_m] \times \mathbb{R}_{mod\ 2\pi} \\ (r, \varphi) & \mapsto & \left(r, \varphi + \big(\arg(\varphi) - \varphi\big)\dfrac{r - \varrho_i}{\varrho_m - \varrho_i}\right). \end{array}$$

Observing that

$$\mathrm{ext}(\varrho_i, \varphi) = (\varrho_i, \varphi) \qquad \forall \varphi \in \mathbb{R}_{mod\ 2\pi},$$

and

$$\mathrm{ext}(\varrho_m, \varphi) \equiv \big(\varrho_m, \arg(\varphi)\big) = \big(g_N \circ \widehat{\Gamma}_i(t) \circ g_N^{-1}\big)(\varrho_m, \varphi) \qquad \forall \varphi \in \mathbb{R}_{mod\ 2\pi},$$

it follows that ext suitably extends $\widetilde{\Gamma}_i$. Moreover, as

$$g_N \circ \widehat{\Gamma}_i(t) \circ g_N^{-1}|_{\partial D^2} = \mathrm{Id}|_{\partial D^2},$$

it follows that $\widetilde{\Gamma}_i$ is actually a path in H_0.
We show that the map

$$\Gamma_i(\cdot) := h \circ \widetilde{\Gamma}_i(\cdot)$$

satisfies the required properties. First, we verify that

$$\Gamma_i(t) \in PH_\infty \qquad \forall t \in I.$$

Observe that, for each $j \geq N+1$, τ_j lies in D_i, such that

$$\Gamma_i(t)(\tau_j) = h(\tau_j) = \tau_j \qquad \forall t \in I,$$

because $h \in PH_\infty$. On the other hand, for each $j \leq N$, τ_j lies in $D^2 \setminus D_o$, such that

$$\begin{aligned}
\Gamma_i(t)(\tau_j) &= \big(h \circ g_N \circ \widehat{\Gamma}_i(t) \circ g_N^{-1}\big)(\tau_j) \\
&\overset{*}{=} \big(h \circ g_N \circ \widehat{\Gamma}_i(t)\big)(\tau_j) \\
&\overset{**}{=} \big(h \circ g_N\big)(\tau_j) \\
&\overset{*}{=} h(\tau_j) \\
&\overset{***}{=} \tau_j,
\end{aligned}$$

where $(*)$ is given by the fact that, by the definition of g_N, $g_N(x) = x$ for all $x \in D^2 \setminus D_o$, $(**)$ follows from the fact that $\widehat{\Gamma}_i(t) \in PH_N$ for all $t \in I$, and $(***)$ holds, because $h \in PH_\infty$.

Furthermore, as $\widetilde{\Gamma}_i(0) = \mathrm{id}_{D^2}$, it follows that

$$\Gamma_i(0) = h.$$

It remains to verify the condition (ii);

$$K\big(\Gamma_i(1), t\big) \in PH_\infty \qquad \forall t \in [t_i, t_{i+1}].$$

As $\Gamma_i(1) \in PH_\infty$, it follows from the statement (b) of Theorem 3.3 that

$$K\big(\Gamma_i(1), t\big)(\tau_j) = \tau_j \qquad \forall j \neq i \quad \forall t \in [t_i, t_{i+1}].$$

Finally, to prove that $K\big(\Gamma_i(1), t\big)(\tau_i) = \tau_i$ $\forall t \in [t_i, t_{i+1}]$, observe that, for all $x \in D^2 \setminus D_o$ with $h^{-1}(x) \in D^2 \setminus D_o$,

$$\begin{aligned}
\Gamma_i(1)(x) &= \big(h \circ g_N \circ \widehat{\Gamma}_i(1) \circ g_N^{-1}\big)(x) \\
&\overset{*}{=} \big(h \circ g_N \circ h^{-1} \circ g_N^{-1}\big)(x) \\
&\overset{**}{=} \big(h \circ g_N\big)\big(h^{-1}(x)\big) \\
&\overset{**}{=} \big(h \circ h^{-1}\big)(x) \\
&= x, \qquad\qquad (C)
\end{aligned}$$

where $(*)$ is given by the definition of $\widehat{\Gamma}_i$, and $(**)$ holds by the fact that $g_N|_{D^2\setminus D_o} = \mathrm{Id}|_{D^2\setminus D_o}$. But, by our choice of N,

$$\kappa(t)^{-1}(\tau_i) \in D^2 \setminus D_o, \quad \text{and} \quad h^{-1}\big(\kappa(t)^{-1}\big)(\tau_i) \in D^2 \setminus D_o \qquad \forall t \in [t_i, t_{i+1}],$$

because $D_o \subset B(\tau_\infty, \varrho_N)$. Thus, by (C),

$$\begin{aligned} K\big(\Gamma_i(1), t\big)(\tau_i) &= \big(\kappa(t) \circ \Gamma_i(1) \circ \kappa(t)^{-1}\big)(\tau_i) \\ &= \big(\kappa(t) \circ \Gamma_i(1)\big)\big(\kappa(t)^{-1}\big)(\tau_i)\big) \\ &= \big(\kappa(t) \circ \kappa(t)^{-1}\big)(\tau_i) \\ &= \tau_i \end{aligned}$$

for all $t \in [t_i, t_{i+1}]$. □

Theorem 3.7. *The map*

$$\pi_0\varphi_\infty : \pi_0 PH_\infty \to PB_\infty$$

is injective.

Proof. Pick any $h \in PH_\infty$, with

$$\pi_0\varphi_\infty\big([h]\big) = \pi_0\varphi_\infty\big([\mathrm{Id}]\big).$$

We need to show that

$$[h] = [\mathrm{Id}] \qquad \text{in } \pi_0 PH_\infty.$$

In other words, we are looking for a path $\mathcal{G} : I \to PH_\infty$, that satisfies

$$\mathcal{G}(0) = h, \quad \mathcal{G}(1) = \mathrm{Id}_{D^2}.$$

According to Lemma 3.6, there is a path $\Gamma_1 : I \to PH_\infty$, such that

$$\Gamma_1(0) = h, \quad K\big(\Gamma_1(1), t\big) \in PH_\infty \qquad \forall t \in [t_1 \equiv 0, t_2].$$

Moreover, by induction, there is a set of paths $\{\Gamma_i : I \to PH_\infty\}_{i\in\mathbb{N}}$, such that, for each $i > 1$,

$$\Gamma_i(0) = \Gamma_{i-1}(1), \quad K\big(\Gamma_i(1), t\big) \in PH_\infty \qquad \forall t \in [t_i, t_{i+1}].$$

Recall that, for every $t \in [0, 1)$, there is an $i \in \mathbb{N}$, such that $t \in [t_i, t_{i+1}]$. This allows us to define, piecewise for all $i \in \mathbb{N}$, a path

$$\begin{aligned} \mathcal{G} : I &\to PH_\infty \\ t &\mapsto \begin{cases} K\left(\Gamma_i\left(2\frac{t-t_i}{t_{i+1}-t_i}\right), t_i\right) & , t \in \left[t_i, \frac{t_i+t_{i+1}}{2}\right] \\ K\left(\Gamma_i(1), 2t - t_{i+1}\right) & , t \in \left[\frac{t_i+t_{i+1}}{2}, t_{i+1}\right] \\ \mathrm{Id}_{D^2} & , t = 1. \end{cases} \end{aligned}$$

We show that this path is well defined, and that it satisfies the required properties. First, observe that, at $t := t_1 \equiv 0$,

$$\mathcal{G}(0) = K\left(\Gamma_1(0), 0\right) = \Gamma_1(0) = h.$$

Fix some $i \geq 2$. At $t := t_i$,

$$K\left(\Gamma_i\Big(2\frac{t_i - t_i}{t_{i+1} - t_i}\Big), t_i\right) = K\big(\Gamma_i(0), t_i\big) = K\big(\Gamma_{i-1}(1), 2t_i - t_i\big),$$

whereas, at $t := \frac{t_i + t_{i+1}}{2}$,

$$K\left(\Gamma_i\Big(2\frac{\frac{t_i+t_{i+1}}{2} - t_i}{t_{i+1} - t_i}\Big), t_i\right) = K\big(\Gamma_i(1), t_i\big) = K\left(\Gamma_i(1), 2\frac{t_i + t_{i+1}}{2} - t_{i+1}\right).$$

Thus, $\mathcal{G}$ is well defined and continuous at all $t \in [0, 1)$. Now, we show the continuity at $t = 1$. We need to show that $\mathcal{G}(t)$ converges (uniformly) to Id_{D^2} for $t \to 1$. Let $\{\hat{t}_i\}_{i \in \mathbb{N}}$ be any sequence in $[0, 1]$ with

$$\lim_{i \to \infty} \hat{t}_i = 1.$$

Pick any $\varepsilon > 0$, and choose an integer n with

$$2\varrho_n < \varepsilon,$$

and an integer N such that

$$\hat{t}_i > t_n \qquad \forall i \geq N.$$

According to part (e) of Theorem 3.3,

$$K(f, \hat{t}_i)(x) \in B(\tau_\infty, \varrho_n) \qquad \forall f \in H_0,\ \ \forall x \in B(\tau_\infty, \varrho_n), ;\ \forall i \geq N.$$

Thus,

$$\big\|K(f, \hat{t}_i)(x) - x\big\| \leq 2\varrho_n < \varepsilon \qquad \forall f \in H_0,\ \ \forall x \in B(\tau_\infty, \varrho_n), ;\ \forall i \geq N.$$

Moreover, by part (d) of Theorem 3.3,

$$K(f, \widehat{t_i})(x) = x \qquad \forall x \in D^2 \setminus B(\tau_\infty, \varrho_N) \qquad \forall f \in H_0,\ \ \forall i \geq N.$$

Resuming these facts, we know that

$$\big\|K(f, \hat{t}_i)(x) - x\big\| < \varepsilon \qquad \forall f \in H_0,\ \ \forall x \in D^2, ;\ \forall i \geq N.$$

As this holds for all $f \in H_0$, it follows from the definition of $\mathcal{G}$, that

$$\big\|\mathcal{G}(\hat{t}_i)(x) - x\big\| < \varepsilon \qquad \forall x \in D^2, \quad \forall i \geq N,$$

which shows the uniform convergence. Thus, $\mathcal{G}$ is a well defined path in H_0 with $\mathcal{G}(0) = h$ and $\mathcal{G}(1) = \mathrm{Id}_{D^2}$. It remains to show that $\mathcal{G}$ is actually a path in PH_∞, i.e., as we already know that $\mathcal{G}(1) \in PH_\infty$, we need to prove that, for each $i \in \mathbb{N}$,

$$\mathcal{G}(t) \in PH_\infty \qquad \forall t \in [t_i, t_{i+1}].$$

Pick some $i \in \mathbb{N}$. By part (a) of Theorem 3.3,

$$\mathcal{G}(t) = K\left(\Gamma_i\Big(2\frac{t-t_i}{t_{i+1}-t_i}\Big), t_i\right) \in PH_\infty \quad \forall t \in \left[t_i, \frac{t_i + t_{i+1}}{2}\right], \qquad (A)$$

because $\Gamma_i(t) \in PH_\infty$ for all $t \in I$ by Lemma 3.6. Moreover,

$$\mathcal{G}(t) = K\big(\Gamma_i(1), 2t - t_{i+1}\big) \quad \forall t \in \left[\frac{t_i + t_{i+1}}{2}, t_{i+1}\right].$$

But for all $t \in \left[\frac{t_i+t_{i+1}}{2}, t_{i+1}\right]$,

$$2t - t_{i+1} \in [t_i, t_{i+1}],$$

such that, by part (ii) of Lemma 3.6,

$$\mathcal{G}(t) \in PH_\infty \qquad \forall\ t \in \left[\frac{t_i + t_{i+1}}{2}, t_{i+1}\right]. \qquad (B)$$

Putting together (A) and (B), we finally obtain that

$$\mathcal{G}(t) \in PH_\infty \qquad \forall t \in [t_i, t_{i+1}].$$

□

Corollary 3.8. *The map*

$$\pi_0\overline{\varphi}_\infty : \pi_0 H_\infty \to \pi_0 \mathcal{O}C_\infty$$

is injective.

Proof. This follows directly from Theorems 3.7 and 2.19. □

Chapter 4

The image of the maps $\pi_0\varphi_\infty$ and $\pi_0\overline{\varphi}_\infty$

After proving the injectivity of the maps $\pi_0\varphi_\infty$ and $\pi_0\overline{\varphi}_\infty$, we are interested in identifying their image in PB_∞ and B_∞, respectively. Again, we can restrict our attention to the map $\pi_0\varphi_\infty$, as Theorem 2.19 directly yields the image of $\pi_0\overline{\varphi}_\infty$, once the image of $\pi_0\varphi_\infty$ is known. In order to identify the image of $\pi_0\varphi_\infty$, we introduce a map $\pi_0\varphi'_\infty$ that is closely related to the map $\pi_0\varphi'_\infty$, which admits an easier identification of its image than the map $\pi_0\varphi_\infty$ itself.
First, in section 4.1, we identify Im $\pi_0\varphi_\infty \subset PB_\infty$ in terms of representatives in ΩF_∞. Thereafter, in section 4.2, we introduce a suitable algebraic description of PB_∞ as an infinite semidirect product of free groups, and state a result concerning the image of $\pi_0\varphi_\infty$ within this semidirect product decomposition of PB_∞.
In this chapter, we often work with the configuration spaces of $\mathring{D}^2 \setminus \tau_\infty$. We thus introduce the following notation.

Definition 4.1. *For all $n \in \mathbb{N} \cup \infty$, define*

$$F'_n := F_n(\mathring{D}^2 \setminus \tau_\infty),$$

and let $PB'_n := \pi_1 F'_n$ be the corresponding pure braid groups. Moreover, for all integers $m > n$, write

$$s'_{m,n} : F'_m \to F'_n$$

for the corresponding (co)- restriction of the projection map $s_{m,n} : F_m \to F_n$. Similarly, we note

$$s'_{\infty,n} : F'_\infty \to F'_n$$

for all $n \in \mathbb{N}$.

In this chapter, let $K : H_0 \times I \to H_0$ be a contracting homotopy of H_0 with the properties given in Theorem 3.3. As, in particular,

$$K(h, \cdot)(\tau_\infty) = p_{\tau_\infty} \quad \forall h \in PH_\infty,$$

the map

$$\begin{array}{rcl} \varphi_\infty : PH_\infty & \to & \Omega F_\infty \\ h & \mapsto & \big(K(h,\cdot)(\tau_i)\big)_{i\in\mathbb{N}} \end{array}$$

corestricts to a well defined map

$$\varphi_\infty|_{\Omega F'_\infty} : PH_\infty \to \Omega F'_\infty.$$

to simplify the notation, we write

$$\varphi'_\infty := \varphi_\infty|_{\Omega F'_\infty}.$$

The inclusion map $\iota : F'_\infty \hookrightarrow F_\infty$ thus induces a commutative diagram

$$\begin{array}{ccc} & \pi_0 PH_\infty & \\ {\scriptstyle \pi_0\varphi'_\infty} \swarrow & & \searrow {\scriptstyle \pi_0\varphi_\infty} \\ PB'_\infty & \xrightarrow{\pi_1\iota} & PB_\infty, \end{array}$$

which in particular shows that the map $\pi_0\varphi'_\infty$ is injective, because the map $\pi_0\varphi_\infty$ is injective by Theorem 3.7.

Proposition 4.2. *The map $\pi_0\varphi'_\infty : \pi_0 PH_\infty \to PB'_\infty$ is injective.*

In fact, the above diagram allows us to characterize the image of $\pi_0\varphi_\infty$ in terms of the image of $\pi_0\varphi'_\infty$, which is easier to identify than the image of $\pi_0\varphi_\infty$. Also, we show that there is an isomorphism

$$\Psi_\infty : PB'_\infty \xrightarrow{\cong} PB_\infty,$$

which does not correspond to $\pi_1\iota$, however. Clearly, it would be interesting to know whether $\pi_1\iota$ is an isomorphism. We didn't solve this question.

4.1 Description of the image of $\pi_0\varphi_\infty$ in terms of representatives

Recall the definition

$$t_1 := 0, \quad t_i := \sum_{k=1}^{i-1} \frac{1}{2^k} \quad \forall i \geq 2.$$

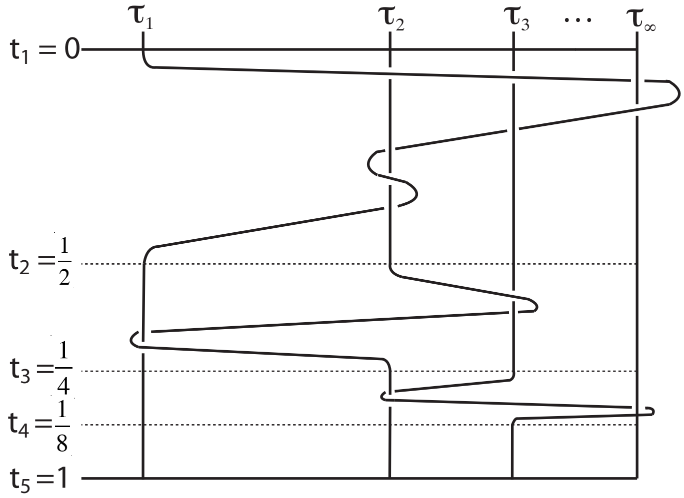
Definition 4.3. *Define the space of* ***infinite combed braids*** *in $\Omega F'_\infty$ as the subspace $\big(\Omega F'_\infty\big)_c \subset \Omega F'_\infty$, given by all braids $(\beta_i)_{i\in\mathbb{N}} \in \Omega F'_\infty$, such that, for each $i \in \mathbb{N}$,*

$$\beta_i(t) = \tau_i \quad \forall t \in \big[0, t_i\big] \cup \big[t_{i+1}, 1\big].$$

Moreover, for all $n \in \mathbb{N}$, define

$$\big(\Omega F'_n\big)_c := \Omega s_{\infty,n}\Big(\big(\Omega F'_\infty\big)_c\Big).$$

A typical element of $\left(\Omega F'_\infty\right)_c$ is drawn below.



The next theorem characterizes the image of the map $\pi_0\varphi_\infty$ in terms of the following subset of $\left(\Omega F'_\infty\right)_c$.

Definition 4.4. *Define a subspace* $\left(\Omega F'_\infty\right)_{cc} \subset \left(\Omega F'_\infty\right)_c$ *by*

$$\left(\Omega F'_\infty\right)_{cc} := \left\{(\beta_i)_{i\in\mathbb{N}} \in \left(\Omega F'_\infty\right)_c \mid \lim_{i\to\infty} \beta_i = p_{\tau_\infty}\right\},$$

where p_{τ_∞} *is the constant path at* τ_∞. *We call* $\left(\Omega F'_\infty\right)_{cc}$ *the space of* ***converging braids*** *in* $\left(\Omega F'_\infty\right)_c$ *(i.e.,* ***c****onverging* ***c****ombed infinite braids). Moreover, let* $\left(PB'_\infty\right)_{cc}$ *be the subset of* PB'_∞ *defined by*

$$\left(PB'_\infty\right)_{cc} := \left\{\left[(\beta_i)_{i\in\mathbb{N}}\right] \in PB'_\infty \mid (\beta_i)_{i\in\mathbb{N}} \in \left(\Omega F'_\infty\right)_{cc}\right\}.$$

Note that, by [11, Thms. 46.7, 46.8], the space $\mathcal{C}(I, D^2)$ has the topology of uniform convergence, so that the above given convergence condition on an element $(\beta_i)_{i\in\mathbb{N}} \in \left(\Omega F'_\infty\right)_{cc}$ is equivalent to the condition that, for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$\|\beta_i(t) - \tau_\infty\| < \varepsilon \quad \forall t \in I, \forall i \geq N.$$

To see that $\left(\Omega F'_\infty\right)_{cc}$ is a *strict* subspace of $\left(\Omega F'_\infty\right)_c$,

$$\left(\Omega F'_\infty\right)_{cc} \subsetneq \left(\Omega F'_\infty\right)_c,$$

we construct a combed braid that is *not* convergent. Consider the point $x := \left(0, \frac{1}{2}\right)$ in D^2, and choose, for each $i \in \mathbb{N}$, a continuous path

$$\widehat{\beta}_{x,i} \in \Omega\left(\overset{\circ}{D}{}^2 \setminus \{\tau_j\}_{j\in\mathbb{N}\cup\infty\setminus i}, \tau_i\right)$$

that loops around x. Define a braid $(\beta_i)_{i\in\mathbb{N}}$ by

$$\beta_i(t) := \begin{cases} \tau_i & \text{if } t \in [0, t_i] \\ \widehat{\beta}_i\left(2^i\left(t - t_i\right)\right) & \text{if } t \in [t_i, t_{i+1}] \\ \tau_n & \text{if } t \in [t_{i+1}, 1] \end{cases} \qquad \forall\ t \in I$$

for all $i \in \mathbb{N}$, and observe that $(\beta_i)_{i\in\mathbb{N}}$ is a well defined element of $\left(\Omega F'_\infty\right)_c$, because it satisfies the condition of Definition 4.3, and because, for each $i \in \mathbb{N}$, the map $\beta_i : S^1 \to D^2$ is continuous, which, by [11, Thm. 19.6], suffices for $(\beta_i)_{i\in\mathbb{N}} : S^1 \to F_\infty$ to be continuous. Notice that $(\beta_i)_{i\in\mathbb{N}}$ is not contained in $\left(\Omega F'_\infty\right)_{cc}$, however, because the condition

$$\lim_{i\to\infty} \beta_i = p_{\tau_\infty}$$

is not satisfied, i.e., the sequence $(\beta_i)_{i\in\mathbb{N}}$ does not converge uniformly to p_{τ_∞}. On the other hand, there is an interesting, unsolved question:

$$\pi_0\left(\Omega F'_\infty\right)_{cc} \stackrel{?}{=} \left(PB'_\infty\right)_{cc}$$

Theorem 4.5.

$$\text{Im } \pi_0\varphi'_\infty = \left(PB'_\infty\right)_{cc}.$$

Proof. To show that

$$\text{Im } \pi_0\varphi'_\infty \subseteq \left(PB'_\infty\right)_{cc},$$

pick an element $h \in PH_\infty$, and write

$$(\beta_i)_{i\in\mathbb{N}} := \varphi'_\infty(h) \equiv \left(K(h,\cdot)(\tau_i)\right)_{i\in\mathbb{N}}.$$

According to Theorem 3.3, item (b),

$$\beta_i(t) = \tau_i \quad \forall i \in \mathbb{N}, \quad \forall t \in [0, t_i] \cup [t_{i+1}, 1],$$

which means that $(\beta_i)_{i\in\mathbb{N}\cup\infty} \in \left(\Omega F'_\infty\right)_c$. To show that, moreover, $(\beta_i)_{i\in\mathbb{N}} \in \left(\Omega F'_\infty\right)_{cc}$, pick an $\varepsilon > 0$, and observe that, by the continuity of the map $K(h,\cdot) : I \times D^2 \to D^2$, the subset

$$K(h,\cdot)^{-1}\left(B(\tau_\infty, \varepsilon)\right) \subset I \times D^2$$

is an open neighbourhood of $I \times \{\tau_\infty\}$, which means that, by the tube lemma, there is an $r > 0$, such that

$$I \times \overline{B(\tau_\infty, r)} \subset K(h,\cdot)^{-1}\left(B(\tau_\infty, \varepsilon)\right),$$

i.e.,

$$K(h,t)\left(\overline{B(\tau_\infty, r)}\right) \subset B(\tau_\infty, \varepsilon) \quad \forall t \in I. \qquad (A)$$

Pick an $N \in \mathbb{N}$ such that

$$\tau_i \in B(\tau_\infty, r) \quad \forall i \geq N,$$

and notice that, by (A),

$$\|K(h,t)(\tau_i) - \tau_\infty\| < \varepsilon \quad \forall t \in I, \forall i \geq N,$$

which means that the sequence $(\beta_i)_{i\in\mathbb{N}} = \big(K(h,\cdot)(\tau_i)\big)_{i\in\mathbb{N}}$ converges uniformly to the constant path p_{τ_∞}. Thus

$$(\beta_i)_{i\in\mathbb{N}\cup\infty} \in \big(\Omega F'_\infty\big)_{cc},$$

i.e.,

$$\pi_0\varphi'_\infty\big([h]\big) = \big[(\beta_i)_{i\in\mathbb{N}}\big] \ \in \big(PB'_\infty\big)_{cc}.$$

It remains to prove that

$$\operatorname{Im} \pi_0\varphi'_\infty \supseteq \big(PB'_\infty\big)_{cc}.$$

Pick any element $b \in \big(PB'_\infty\big)_{cc}$, and let $(\beta_i)_{i\in\mathbb{N}} \in \big(\Omega F'_\infty\big)_{cc}$ be a representative of b. Also, write

$$r_i := 2\sup_{j\geq i}\left\{\max_{t\in I}\|\beta_j(t) - \tau_\infty\|\right\}$$

for all $i \in \mathbb{N}$. Note that

$$r_i \geq r_{i+1} \qquad \forall \ i \in \mathbb{N},$$

and, by the definition of $\big(\Omega F'_\infty\big)_{cc}$,

$$\lim_{i\to\infty} r_i = 0.$$

We show that, for every $i \in \mathbb{N}$, there is a path $g_i \in \mathcal{C}\big(I, H_0\big)$, such that, recalling the sequence $(t_i)_{i\in\mathbb{N}}$,

$$\begin{array}{rll} (i) & g_i(t)(\tau_j) = \begin{cases} \beta_i(t) & \text{if } j = i \quad \forall t \in I \\ \tau_j & j \in \mathbb{N}\cup\infty\setminus i \qquad \forall t \in I \end{cases} \\ (ii) & g_i(t)|_{D^2\setminus B(\tau_\infty, r_i)} = Id & \qquad \forall t \in I \\ (iii) & g_i(t) = \mathrm{Id} & \qquad \forall t \leq t_i \\ (iv) & g_i(t) = g_i(1) & \qquad \forall t \geq t_{i+1}, \end{array}$$

where $B(\tau_\infty, r_i)$ is the open ball in $\mathbb{R}^2$ with radius r_i, centered at τ_∞. For some fixed $i \in \mathbb{N}$, the existence can be shown as follows. By the definition of $\big(\Omega F'_\infty\big)_{cc}$, we know that, for every $t \in I$, there is at most one $i \in \mathbb{N}$, for which $\beta_i(t) \neq \tau_i$, i.e.,

$$\beta_i(t) \in \overset{\circ}{D}{}^2 \setminus \{\tau_j\}_{j\in\mathbb{N}\cup\infty\setminus i} \qquad \forall t \in I, \ \ \forall i \in \mathbb{N}.$$

Fix some $i \in \mathbb{N}$, and observe that, by the extreme value theorem [11, Thm. 27.4], there is an $\widetilde{r} > 0$, that satisfies

$$\beta_i(t) \notin \overline{B(\tau_\infty, \widetilde{r})} \quad \forall t \in I.$$

Thus, there is a finite $M \in \mathbb{N}$ such that $\tau_i \in \overline{B(\tau_\infty, \widetilde{r})}$ for all $i > M$, i.e.,

$$\beta_i(t) \in \overset{\circ}{D}{}^2 \setminus \left\{ \overline{B(\tau_\infty, \widetilde{r})} \cup \{\tau_j\}_{j \in [1,M] \setminus i} \right\} \qquad \forall t \in I.$$

Thus, we can again apply the extreme value theorem to conclude that there is a real $r > 0$ that satisfies

$$\bigcup_{t \in I} B(\beta_i(t), r) \subset \overset{\circ}{D}{}^2 \setminus \left\{ \overline{B(\tau_\infty, \widetilde{r})} \cup \{\tau_j\}_{j \in [1,M] \setminus i} \right\} \qquad \forall t \in I,$$

and thus, in particular,

$$\bigcup_{t \in I} B(\beta_i(t), r) \subset \overset{\circ}{D}{}^2 \setminus \{\tau_j\}_{j \in \mathbb{N} \cup \infty \setminus i}. \qquad (B)$$

Moreover, as, by the definition of r_i,

$$\beta_i(t) \in B(\tau_\infty, r_i) \qquad \forall\ t \in I,$$

we can choose r small enough, such that

$$\bigcup_{t \in I} B(\beta_i(t), r) \subset B(\tau_\infty, r_i). \qquad (C)$$

By the continuity of β_i, there is, for each $t \in [t_i, t_{i+1}]$, an open interval $]s_t^-, s_t^+[\subset [t_i, t_{i+1}]$ containing t, such that

$$\beta_i(\widehat{t}) \in B(\beta_i(t), r) \quad \forall \widehat{t} \in [s_t^-, s_t^+].$$

As I is compact, there is an $M \in \mathbb{N}$ and a point set $\{\hat{t}_j\}_{j \in [1,M]} \subset [t_i, t_{i+1}]$, such that $\bigcup_{j \in [1,M]} [s_{\hat{t}_j}^-, s_{\hat{t}_j}^+] = [t_i, t_{i+1}]$. In particular, $s_{\hat{t}_j}^+ > s_{\hat{t}_{j+1}}^-$ for all $j \in [1, M-1]$, such that, simplifying the notation by writing s_j instead of $s_{\hat{t}_j}^+$,

$$\bigcup_{j \in [1,M]} [s_{j-1}, s_j] = [t_i, t_{i+1}],$$

where $s_0 := t_i$. Also, notice that

$$\beta_i(\widehat{t}) \in B(\beta_i(t_j), r) \quad \forall \widehat{t} \in [s_{j-1}, s_j], \forall j \in [2, M].$$

Observe that that, by Theorem 1.12, the map

$$\begin{array}{rcl} \mathrm{ev}_{\tau_i} : H_0 & \to & D^2 \\ h & \mapsto & h(\tau_i) \end{array}$$

has the path lifting property. For each $j \in [1, M]$, choose a homeomorphism

$$f_j : \overset{\circ}{D}{}^2 \xrightarrow{\cong} B\big(\beta_i(t_j), r\big),$$

that satisfies

$$f_j(\tau_i) = \beta_i(s_{j-1}),$$

and consider the following commutative diagram

$$\begin{array}{ccc} H_0 & \xrightarrow[\cong]{f_j\circ(\cdot)\circ f_j^{-1}} & H_0^B \\ \Big\downarrow \mathrm{ev}_{\tau_i} & & \Big\downarrow \mathrm{ev}^B_{f_j(\tau_i)} \\ \overset{\circ}{D}{}^2 & \xrightarrow[\cong]{f} & B\big(\beta_i(t_j), r\big), \end{array}$$

where H_0^B is the space of homeomorphisms of $\overline{B\big(\beta_i(t_j), r\big)}$ that fix the boundary $\partial B\big(\beta_i(t_j), r\big)$ pointwise, and $\mathrm{ev}^B_{f_j(\tau_i)}$ is the evaluation at $f_j(\tau_i)$. Clearly, $\mathrm{ev}^B_{f_j(\tau_i)}$ has thus the path lifting property, which allows us to construct, for each $j \in [1, M]$, a path $g^{(j)} : [s_{j-1}, s_j] \to H_0$ satisfying

$$\begin{aligned} & g^{(j)}(s_{j-1}) = \mathrm{Id}_{D^2}, & & (D) \\ & g^{(j)}(t)\big(\beta_i(s_{j-1})\big) = \beta_i(t) & \forall t \in [s_{j-1}, s_j], \quad & (E) \\ & g^{(j)}(t)|_{D^2\setminus B(\beta_i(t_j),r)} = \mathrm{Id} & \forall t \in [s_{j-1}, s_j]. \quad & (F) \end{aligned}$$

Now, define a path $g_i : I \to H_0$ piecewise by

$$g_i(t) := \begin{cases} \mathrm{Id}_{D^2}, & t \in [0, t_i] \\ g^{(j)}(t) \circ g^{(j-1)}(s_{j-1}) \circ \cdots \circ g^{(1)}(s_1), & t \in [s_{j-1}, s_j] \quad \forall j \in [1, M] \\ g^{(M)}(s_M) \circ \cdots \circ g^{(1)}(s_1), & t \in [t_{i+1}, 1]\,, \end{cases}$$

and observe that, by (D), g_i is well defined. Moreover, g_i satisfies the condition (i), because, by (B) and by the properties (E) and (F) of the maps $g^{(j)}$. The condition (ii) is satisfied by (C) and (F), whereas the conditions (iii) and (iv) are given directly by the definition of g_i.
For all $n \in \mathbb{N}$, write

$$\mathcal{G}_n(-) = g_n(-) \circ \cdots \circ g_1(-).$$

Observe that, by the properties (i) and (iii) of the maps g_i, for every $n \in \mathbb{N}$,

$$\begin{aligned} \mathcal{G}_n(0) &= \mathrm{Id}_{D^2} \\ \mathcal{G}_n(t)(\tau_i) &= \beta_i(t), \qquad \forall\ i \le n, \forall\ t \in I, \qquad (G), \end{aligned}$$

and that, moreover,

$$\mathcal{G}_n(1)(\tau_i) = \tau_i \quad \forall i \in \mathbb{N}.$$

To finish the proof, we veriry the following fact.
Claim: There is a path $\mathcal{G} \in \mathcal{C}\big(I, H_0\big)$, such that

$$\lim_{n\to\infty} \mathcal{G}_n = \mathcal{G}. \qquad (H)$$

Then, by (G), it is clear that $\mathcal{G}$ satisfies

$$\mathrm{ev}_\infty \circ \big(\mathcal{G}(-)\big) = (\beta_i)_{i\in\mathbb{N}}. \qquad (I)$$

Write $\overline{\mathcal{G}}(t) := \mathcal{G}(1-t)$ for all $t \in I$, and observe that both

$$K(\mathcal{G}(1), -) \quad \text{and} \quad \overline{\mathcal{G}}$$

are paths in H_0 with startpoint $\mathcal{G}(1) \in PH_\infty$ and endpoint Id_{D^2} that, moreover, satisfy

$$K(\mathcal{G}(1), t)(\tau_\infty) = \overline{\mathcal{G}}(t)(\tau_\infty) = \tau_\infty \quad \forall t \in I$$

by the property (i) of the maps $\{g_i\}_{i\in\mathbb{N}}$ and by Theorem 3.3 (c). Thus, by Lemma A.3,

$$\Big[\mathrm{ev}_\infty \circ K\big(\mathcal{G}(1), -\big)\Big] = \Big[\mathrm{ev}_\infty \circ \overline{\mathcal{G}}(-)\Big] \qquad \text{in} \quad \pi_1 F'_\infty,$$

i.e., writing $(\overline{\beta}_i)_{i\in\mathbb{N}}$ for the inverse path of $(\beta_i)_{i\in\mathbb{N}}$,

$$\begin{aligned} \pi_0\varphi'_\infty\big([\mathcal{G}(1)]\big) &= \big[(\mathrm{ev}_\infty)_* K(\mathcal{G}(1), -)\big] \\ &= \big[(\mathrm{ev}_\infty)_* \overline{\mathcal{G}}(-)\big] \\ &\overset{I}{=} \big[(\overline{\beta}_i)_{i\in\mathbb{N}}\big]. \end{aligned}$$

Analoguous to the prove of Proposition 1.15, one can show that the map $\pi_0\varphi'_\infty$ is a homomorphism. Thus, defining $h_\beta := \mathcal{G}(1)^{-1}$, it follows that

$$\pi_0\varphi'_\infty\big([h_\beta]\big) = \big[(\beta_i)_{i\in\mathbb{N}}\big],$$

which proves the theorem.
To prove our claim (H), we proceed in two steps. First, we show that, for each $t \in I$, the sequence $(\mathcal{G}_n(t) \in H_0)_{n\in\mathbb{N}}$ converges in H_0. In a second step, we show that the set $\{\mathcal{G}(t)\}_{t\in I}$ depends continuously on t.

First step. Fix some $t \in I$, $x \in D^2$, and observe that, for every $n \in \mathbb{N}$,

$$\begin{aligned} \|\mathcal{G}_n(t)(x) - \mathcal{G}_{n-1}(t)(x)\| &= \|g_n \circ \mathcal{G}_{n-1}(t)(x) - \mathcal{G}_{n-1}(t)(x)\| \\ &\leq 2r_n \end{aligned}$$

by the property (ii) of the maps $\{g_i\}_{i\in[1,n]}$. Thus,

$$\lim_{n\to\infty} \|\mathcal{G}_n(t)(x) - \mathcal{G}_{n-1}(t)(x)\| = 0.$$

which means that, $\big(\mathcal{G}_n(t)(x)\big)_{n\in\mathbb{N}}$ is a Cauchy sequence. By the completeness of D^2, this sequence thus converges *pointwise* in D^2, which allows us to define a map

$$\begin{aligned} \mathcal{G}(t) : \mathring{D}^2 &\to \mathring{D}^2 \\ x &\mapsto \lim_{n\to\infty} \mathcal{G}_n(t)(x). \end{aligned}$$

To prove the uniform convergence of the sequence $\big(\mathcal{G}_n(t)\big)_{n\in\mathbb{N}}$, recall that, for all $n \in \mathbb{N}$,

$$g_n(t)(x) = x \qquad \forall\ x \in \overset{\circ}{D}{}^2 \setminus B(\tau_\infty, r_n),$$

by the property (ii) of g_n, and

$$g_n(t)(x) \in B(\tau_\infty, r_n) \qquad \forall\ x \in B(\tau_\infty, r_n)$$

by construction. As $r_{i+1} \leq r_i$ for all $i \in \mathbb{N}$, it follows that, for every $n' \geq n$,

$$\begin{aligned} \big(g_{n'}(t) \circ \cdots \circ g_n(t)\big)(x) &\in B(\tau_\infty, r_n) && \forall x \in B(\tau_\infty, r_n), \quad \text{and} \\ \big(g_{n'}(t) \circ \cdots \circ g_n(t)\big)(x) &= x && \forall x \in \overset{\circ}{D}{}^2 \setminus B(\tau_\infty, r_n). \end{aligned}$$

In other words, for all integers n, n' with $n' \geq n$,

$$\begin{aligned} \big\|\mathcal{G}_{n'}(t)(x) - \mathcal{G}_n(t)(x)\big\| &= \big\|\big(g_{n'}(t) \circ \cdots \circ g_{n+1}(t) \circ \mathcal{G}_n(t)\big)(x) - \mathcal{G}_n(t)(x)\big\| \\ &\leq 2r_{n+1} \end{aligned}$$

for all $x \in D^2$. This means that, for every $n \in \mathbb{N}$,

$$\big\|\mathcal{G}(t)(x) - \mathcal{G}_n(t)(x)\big\| \leq 2r_{n+1} \quad \forall\ x \in \overset{\circ}{D}{}^2,$$

which proves the uniform convergence of the sequence $\big(\mathcal{G}_n(t)\big)_{n\in\mathbb{N}}$, because $\lim_{n\to\infty} r_n = 0$. Thus, $\mathcal{G}(t)$ is a limit point of H_0, because H_0 is topologized as a subspace of ${D^2}^{D^2}$, endowed with the compact-open topology, that coincides with the uniform topology by [Munkres, Thms. 46.7/8].

On the other hand, H_0 is closed in ${D^2}^{D^2}$. For any sequence $(h_i)_{i\in\mathbb{N}}$ in H_0 that converges to an element $h \in {D^2}^{D^2}$, we know that

$$h \in \mathcal{C}(D^2, D^2),$$

as $\mathcal{C}(D^2, D^2)$ is closed in ${D^2}^{D^2}$ by [Munkres, Thm. 46,5]. By the same argument,

$$h^{-1} := \lim_{i\to\infty} h_i^{-1} \in \mathcal{C}(D^2, D^2),$$

i.e.,

$$h \in \mathcal{H}(D^2, D^2).$$

As every element of the sequence $(h_i)_{i\in\mathbb{N}}$ fixes the boundary ∂D^2 pointwise, so does its limit h, which means that

$$h \in H_0.$$

Thus, H_0 is closed in ${D^2}^{D^2}$, and thus, for every $t \in I$,

$$\mathcal{G}(t) \in H_0 \qquad \forall t \in I.$$

This finishes the first step of the proof.

Second step. Pick any $t \in [0,1)$, and an integer N, such that

$$t < t_N.$$

Then, by the property *(iii)* of the paths $\{g_i\}_{i\in\mathbb{N}}$,

$$\mathcal{G}_n(t) = \mathcal{G}_N(t) \qquad \forall \; n \geq N,$$

which means that

$$\mathcal{G}(t) = \mathcal{G}_N(t).$$

Thus,

$$\mathcal{G}|_{[0,1)} \in \mathcal{C}\Big([0,1), H_0\Big).$$

Let $(t_n)_{n\in\mathbb{N}}$ be a sequence in I that converges to 1. Given any $\epsilon > 0$, find a $k \in \mathbb{N}$, such that $4r_k \leq \epsilon$. Choose some $N \in \mathbb{N}$ that satisfies

$$t_n \in [t_{k+1}, 1] \qquad \forall \; n \geq N,$$

such that, by the property *(iv)* of the paths g_i,

$$g_i(t_n) = g_i(1) \quad \forall n \geq N, \;\; \forall i \leq k.$$

Consequently,

$$\mathcal{G}_k(t_n) = \mathcal{G}_k(1) \quad \forall n \geq N. \qquad (J)$$

Moreover, note that, by the property *(ii)* of the maps g_i,

$$\|\mathcal{G}(t) - \mathcal{G}_k(t)\| < 2r_k \quad \forall t \in I. \qquad (K)$$

Thus, for all $x \in D^2$,

$$\begin{aligned}
\big\|\mathcal{G}(t_n)(x) - \mathcal{G}(1)(x)\big\| &= \big\|\big(\mathcal{G}(t_n) - \mathcal{G}_k(t_n) + \mathcal{G}_k(t_n) - \mathcal{G}(1) + \mathcal{G}_k(1) - \mathcal{G}_k(1)\big)(x)\big\| \\
&\overset{J}{=} \big\|\big(\mathcal{G}(t_n) - \mathcal{G}_k(t_n) - \mathcal{G}(1) + \mathcal{G}_k(1)\big)(x)\big\| \\
&\leq \big\|\mathcal{G}(t_n)(x) - \mathcal{G}_k(t_n)(x)\big\| + \big\|\mathcal{G}(1)(x) - \mathcal{G}_k(1)(x)\big\| \\
&\overset{K}{\leq} 4r_k \quad \forall n \geq N \\
&\leq \varepsilon
\end{aligned}$$

i.e., the sequence $\big(\mathcal{G}(t_n)\big)_{n\in\mathbb{N}}$ converges uniformly to $\mathcal{G}(1)$. Thus, by [Munkres. Theorems 46.7/46.8],

$$\mathcal{G} \in \mathcal{C}(I, H_0),$$

which proves our claim (H). □

Corollary 4.6. *The maps $\varphi'_\infty, \varphi_\infty$ and $\overline{\varphi}_\infty$ induce isomorphisms*

$$\pi_0\varphi'_\infty : \pi_0 PH_\infty \xrightarrow{\cong} \big(PB'_\infty\big)_{cc},$$
$$\pi_0\varphi_\infty : \pi_0 PH_\infty \xrightarrow{\cong} \pi_1\iota\big(PB'_\infty\big)_{cc},$$

and a bijection

$$\pi_0\overline{\varphi}_\infty : \pi_0 H_\infty \xrightarrow{\cong} \pi_0\zeta\Big(\Sigma_\infty \ltimes \pi_1\iota\big((PB'_\infty)_{cc}\big)\Big)$$

where $\iota : F'_\infty \hookrightarrow F_\infty$ is the inclusion map.

Proof. The result follows directly from Theorems 4.5, 3.7 and Proposition 4.2. □

4.2 Algebraic description of PB_∞

To algebraically describe the image of $\pi_0\varphi_\infty$, we first need an algebraic description of PB_∞. The most straightforward way to do this is to use the inverse system

$$PB_\infty \to \cdots \to PB_{n+1} \xrightarrow{\pi_1 s_{n+1,n}} PB_n \to \cdots \to PB_1$$

to decompose PB_∞ into an infinite semidirect product

$$PB_\infty \cong \ltimes_{i\geq 2} U_i,$$

where, for all $i \geq 2$, $U_i = \text{Ker } \pi_1 s_{n+1,n}$. In subsection 4.2.5, this is explained in detail. However, it seems that, within $\ltimes_{i\geq 2}U_i$, the image of $\pi_0\varphi_\infty$ is complicated to describe. To avoid this difficulty, we introduce the braid groups of the punctured disk $PB'_n := \pi_1 F_n(\overset{\circ}{D}{}^2 \setminus \tau_\infty)$ for all $n \in \mathbb{N}$, and show that PB_∞ is the limit of the resulting inverse system

$$PB_\infty \to \cdots \to PB'_{n+1} \xrightarrow{\pi_1 s'_{n+1,n}} PB'_n \to \cdots \to PB'_1.$$

According to subsection 4.2.5, this allows us to write PB_∞ as the infinite semidirect product

$$PB_\infty \cong \ltimes_{i\in\mathbb{N}} U'_i,$$

where, for all $i \in \mathbb{N}$, $U'_i = \text{Ker } \pi_1 s'_{n+1,n}$. Within this semidirect product decomposition of PB_∞, the image of $\pi_0\varphi_\infty$ seems to be easier to identify (see section 4.3).

Observing that, for all $n \in \mathbb{N}$, PB'_n is isomorphic to the braid group of the bi-infinite cylinder $S^1 \times \mathbb{R}$, which, by an easy argument, is isomorphic to the braid group of the cylinder, we presume that the groups PB'_n and U'_n, are well known for all finite n. Nevertheless, the particular statements concerning these groups that we need for the identification of Im $\pi_0\varphi_\infty$ in section 4.3 seem hard to be found in literature. Therefore, we fully develop the introduction of these groups, and identify their presentation using the presentation of the standard pure braid groups PB_n. In particular, we suppose that the content of the present section is essentially known.

4.2.1 Introduction of the braid groups of the punctured disk PB'_n

In this paragraph, we introduce abstract groups PB'_n for all $n \in \mathbb{N}$, that we identify later with $\pi_1 F'_n$, the groups of n-strand braids in $\overset{\circ}{D}{}^2 \setminus \tau_\infty$.
For all $n \in \mathbb{N}$, define an isomorphism $\widehat{\Phi}_n$ of abstract free groups by

$$\begin{array}{rcll} \widehat{\Phi}_n : \left\langle \{A_{i,j}\}_{1\leq i<j\leq n-1}, \{\delta_i^{(n-1)}\}_{1\leq i\leq n-1}\} \right\rangle & \to & \left\langle \{A_{i,j}\}_{1\leq i<j\leq n} \right\rangle & \\ A_{i,j} & \mapsto & A_{i,j} & \forall\, 1 \leq i < j \leq n-1 \\ \delta_i^{(n-1)} & \mapsto & A_{i,i+1} \cdots A_{i,n} & \forall\, i \in [1, n-1]. \end{array}$$

where $\widehat{\Phi}_n^{-1}$ is given by

$$\begin{array}{rcll} \widehat{\Phi}_n^{-1} : \left\langle \{A_{i,j}\}_{1\leq i<j\leq n} \right\rangle & \to & \left\langle \{A_{i,j}\}_{1\leq i<j\leq n-1} \{\delta_i^{(n-1)}\}_{1\leq i\leq n-1}\} \right\rangle & \\ A_{i,j} & \mapsto & A_{i,j} & \forall\, 1 \leq i < j \leq n-1 \\ A_{i,n} & \mapsto & A_{i,n-1}^{-1} \cdots A_{i,i+1}^{-1} \delta_i^{(n-1)} & \forall i \in [1, n-2]. \\ A_{n-1,n} & \mapsto & \delta_{n-1}^{(n-1)}. & \end{array}$$

Identify the set $\{A_{i,j}\}_{1\leq i<j\leq n}$ with the identical set of generators of the group PB_n for all $n \in \mathbb{N}$ (cf. [Hansen, Lemma 4.2]), and define a projection map

$$q_n : \left\langle \{A_{i,j}\}_{1\leq i<j\leq n} \right\rangle \to \left| \{A_{i,j}\}_{1\leq i<j\leq n} : \mathbf{r}_n \right| \cong PB_n,$$

where $\mathbf{r}_n$ is the set of the relations in PB_n with respect to this presentation, which are given by

$$A_{r,s}^{-1} A_{i,j} A_{r,s} \sim \begin{cases} A_{i,j} & \text{if } i < r < s < j \quad \text{or } r < s < i < j \\ A_{r,j} A_{i,j} A_{r,j}^{-1} & \text{if } r < i = s < j \\ A_{r,j} A_{s,j} A_{i,j} A_{s,j}^{-1} A_{r,j}^{-1} & \text{if } i = r < s < j \\ A_{r,j} A_{s,j} A_{r,j}^{-1} A_{s,j}^{-1} A_{i,j} A_{s,j} A_{r,j} A_{s,j}^{-1} A_{r,j}^{-1} & \text{if } r < i < s < j \end{cases} \tag{4.1}$$

This presentation is related to Artin's by

$$A_{i,j} = \sigma_{j-1}\sigma_{j-2} \cdots \sigma_{i+1}\sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-2}^{-1}\sigma_{j-1}^{-1}$$

for all $1 \leq i < j$ (see [Birman, p. 20]). For all $n \in \mathbb{N}$, define

$$\mathbf{r}'_n := \widehat{\Phi}_n^{-1}(\mathbf{r}_n),$$

and, introducing the projection

$$q'_n : \left\langle \{A_{i,j}\}_{1\leq i<j\leq n-1} \cup \{\delta_i^{(n-1)}\}_{1\leq i\leq n-1}\} \right\rangle \to \left| \{A_{i,j}\}_{1\leq i<j\leq n-1}, \{\delta_i^{(n-1)}\}_{1\leq i\leq n-1}\} : \mathbf{r}'_n \right| := PB'_{n-1},$$

observe that there is a commutative diagram

$$\begin{array}{ccc} \left\langle \{A_{i,j}\}_{1\le i<j\le n-1} \cup \{\delta_i^{(n-1)}\}_{1\le i\le n-1}\right\rangle & \xrightarrow[\cong]{\widehat{\Phi}_n} & \left\langle \{A_{i,j}\}_{1\le i<j\le n}\right\rangle , \\ \downarrow q_n' & & \downarrow q_n \\ PB_{n-1}' & \xrightarrow[\cong]{\Phi_n} & PB_n \end{array}$$

where

$$\Phi_n : PB_{n-1}' \xrightarrow{\cong} PB_n$$

is the isomorphism of groups induced by $\widehat{\Phi}_n$.

4.2.2 Identification of $\pi_1 F_n'$ with PB_n'.

Recalling the definition $F_n' := F_n(\overset{\circ}{D}{}^2 \setminus \tau_\infty)$, we now identify the group PB_n' with the fundamental group $\pi_1 F_n'$ for all $n \in \mathbb{N}$, as shown below.

Proposition 4.7. *For each $n \in \mathbb{N}$, there is an element $\widehat{\phi}_n \in H_0$ that satisfies*

$$\begin{array}{rcl} \widehat{\phi}_n(\tau_\infty) & = & \tau_n, \quad \textit{and} \\ \widehat{\phi}_n(x) & = & x \quad \forall x \in D^2 \setminus B\big(\tau_\infty, \|\tau_{n-1} - \tau_\infty\|\big). \end{array}$$

In particular, this map induces a well defined map of pointed spaces

$$\begin{array}{rcl} \phi_n : \big(F_{n-1}', \mathcal{T}_{n-1}\big) & \to & \big(F_n, \mathcal{T}_n\big) \\ (x_1, \ldots, x_{n-1}) & \mapsto & \big(\widehat{\phi}_n(x_1), \ldots, \widehat{\phi}_n(x_{n-1}), \tau_n\big). \end{array}$$

Proof. Observe that, as Theorem 1.12 holds for any choice for $(\tau_i)_{i\in[1,n]}$, the map

$$\begin{array}{rcl} \widetilde{\mathrm{ev}}_n : H_0 & \to & F_n \\ h & \mapsto & \big(h(\tau_i)\big)_{i\in[1,n-1]\cup\infty} \end{array}$$

is a fiber bundle, and thus, in particular, is surjective. Choose any homeomorphism

$$f : D^2 \xrightarrow{\cong} \overline{B\big(\tau_\infty, \|\tau_{n-1} - \tau_\infty\|\big)},$$

and consider the following commutative diagram

$$\begin{array}{ccc} H_0 & \xrightarrow[\cong]{f\circ(\cdot)\circ f^{-1}} & H_0^B \\ \downarrow \widetilde{\mathrm{ev}}_n & & \downarrow \widetilde{\mathrm{ev}}_n^B \\ F_n & \xrightarrow[\cong]{f} & F_n B\big(\tau_\infty, \|\tau_{n-1} - \tau_\infty\|\big), \end{array}$$

where H_0^B is the space of homeomorphisms of $\overline{B\big(\beta_i(t_j), \|\tau_{n-1} - \tau_\infty\|\big)}$ that fix the boundary $\partial \overline{B\big(\beta_i(t_j), \|\tau_{n-1} - \tau_\infty\|\big)}$ pointwise, and $\widetilde{\mathrm{ev}}_n^B$ is the induced map. This shows that

$$\widetilde{\mathrm{ev}}_n^B : H_0^B \to F_n \overline{B\big(\tau_\infty, \|\tau_{n-1} - \tau_\infty\|\big)}$$

is surjective, which means that there is a map $\widehat{\phi}_n^B$ in H_0^B that satisfies

$$\begin{array}{rcl} \widehat{\phi}_n^B(\tau_\infty) & = & \tau_n, \\ \widehat{\phi}_n^B(\tau_i) & = & \tau_i \quad \forall \; i \in [1, n-1], \end{array}$$

which, when extended by the identity map on $D^2 \setminus B\big(\tau_\infty, \|\tau_{n-1} - \tau_\infty\|\big)$, yields the required map $\widehat{\phi}_n$. □

Proposition 4.8. *For each $n \in \mathbb{N}$, the map $\phi_n : F'_{n-1} \to F_n$ induces an isomorphism*

$$\pi_1 \phi_n : \pi_1\big(F'_{n-1}, \mathcal{T}_{n-1}\big) \xrightarrow{\cong} \pi_1\big(F_n, \mathcal{T}_n\big).$$

Proof. Fix some $n \in \mathbb{N}$. According to [Birman, Thm. 1.2], there is a fiber bundle

$$\begin{array}{ccccc} \big(F'_{n-1}, \mathcal{T}_{n-1}\big) & \overset{\nu_n}{\hookrightarrow} & \big(F_n, \mathcal{T}_n\big) & \longrightarrow & \big(\overset{\circ}{D}{}^2, \tau_n\big) \\ (x_1, \ldots, x_{n-1}) & \mapsto & (x_1, \ldots, x_{n-1}, \tau_n) & & \\ & & (x_1, \ldots, x_n) & \mapsto & x_n \end{array}$$

Moreover, recalling that

$$\pi_1(\overset{\circ}{D}{}^2, \tau_n) = \pi_2(\overset{\circ}{D}{}^2, \tau_n) = 1,$$

the corresponding long exact homotopy sequence yields an isomorphism

$$\pi_1 \nu_n : \pi_1\Big(F'_{n-1}, \mathcal{T}_{n-1}\Big) \xrightarrow{\cong} \pi_1\Big(F_n, \mathcal{T}_n\Big).$$

Considering the following diagram of pointed spaces commutes

$$\begin{array}{ccc} \big(F'_{n-1}, \mathcal{T}_{n-1}\big) & \xrightarrow{\nu_n} & \big(F_n, \mathcal{T}_n\big) \\ {\scriptstyle \phi_n^c} \uparrow {\scriptstyle \cong} & \nearrow_{\phi_n} & \\ \big(F'_{n-1}, \mathcal{T}_{n-1}\big), & & \end{array}$$

where the corestricted map

$$\phi_n^c := \phi_n|^{\left(F'_{n-1}, \mathcal{T}_{n-1}\right)}$$

is actually a homeomorphism, the induced diagram of fundamental groups yields the required isomorphism. □

Consider the diagram

$$\begin{array}{ccc} \pi_1 F'_{n-1} & \xrightarrow[\cong]{\pi_1\phi_n} & \pi_1 F_n \\ & & \Big| \equiv \\ PB'_{n-1} & \xrightarrow[\cong]{\Phi_n} & PB_n \end{array}$$

Definition 4.9. *By the fact that, for every $n \in \mathbb{N}$, both $\pi_1\phi_n : \pi_1 F'_{n-1} \to \pi_1 F_n$ and $\Phi_n : PB'_{n-1} \to PB_n$ are isomorphisms of groups, we can identify, for all $n \in \mathbb{N}$, the abstract group PB'_{n-1} with the concrete group $\pi_1 F'_{n-1}$ and the isomorphism Φ_n with $\pi_1\phi_n$, such that the above diagram completes as follows*

$$\begin{array}{ccc} \pi_1 F'_{n-1} & \xrightarrow[\cong]{\pi_1\phi_n} & \pi_1 F_n \\ \equiv \Big\downarrow & & \Big\downarrow \equiv \\ PB'_{n-1} & \xrightarrow[\cong]{\Phi_n} & PB_n. \end{array}$$

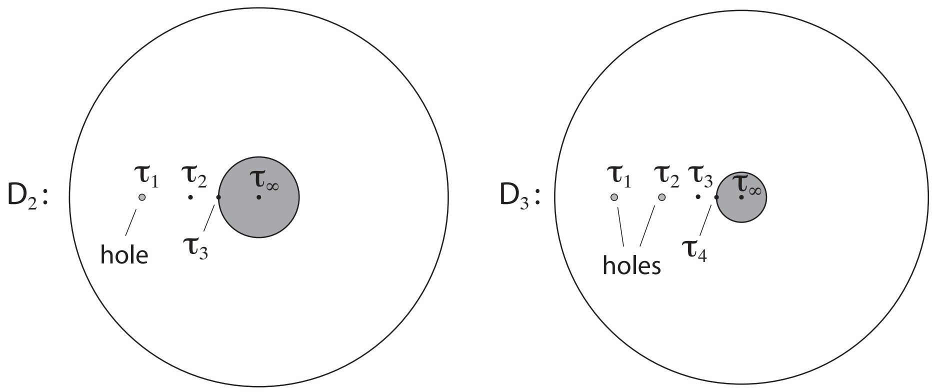
4.2.3 Canonical representatives of the generators of the groups PB'_n for finite n

The following notation is used repeatedly in the sequel.

Definition 4.10. *For all $n \in \mathbb{N}$ introduce a subset of D^2 by*

$$D_n := \overset{\circ}{D}{}^2 \setminus \Big\{ \{\tau_j\}_{j\in[1,n-1]} \cup \overline{B\big(\tau_\infty, \|\tau_{n+1} - \tau_\infty\|\big)} \Big\}.$$

Here are two examples.



Definition 4.11. *For all $i, n \in \mathbb{N}$ with $1 \le i \le n$, write*

$$\begin{array}{rcl} \kappa_i^{(n)} : \big(D_i, \tau_i\big) & \to & \big(F'_n, \mathcal{T}_n\big) \\ x & \mapsto & \big(\tau_1, \dots, \tau_{i-1}, x, \tau_{i+1}, \dots, \tau_n\big). \end{array}$$

Proposition 4.12. *For all integers i, j with $1 \leq i < j$, there is a loop*

$$\widehat{A}_{i,j} \in \Omega\big(D_j, \tau_j\big),$$

such that, for all $n \geq j$,

$$A_{i,j} = \big[\kappa_j^{(n)} \circ \widehat{A}_{i,j}\big] = \big[p_{\tau_1}, \ldots, p_{\tau_{j-1}}, \widehat{A}_{i,j}, p_{\tau_{j+1}}, \ldots, p_{\tau_n}\big]$$

in PB'_n. Also, for all $i \in \mathbb{N}$, there is a loop

$$\widehat{\delta}_i \in \Omega\big(D_i, \tau_i\big),$$

that satisfies, for all $n \geq i$,

$$\delta_i^{(n)} = \big[\kappa_i^{(n)} \circ \widehat{\delta}_j\big] = \big[p_{\tau_1}, \ldots, p_{\tau_{i-1}}, \widehat{\delta}_i, p_{\tau_{i+1}}, \ldots, p_{\tau_n}\big]$$

in PB'_n. Moreover, these loops can be chosen such that

$$\widehat{A}_{i,j}(t) \in D_j \cap B\left(\tau_\infty, \left\|\frac{1}{2}(\tau_{i-1} + \tau_i) - \tau_\infty\right\|\right) \quad \forall i, j \in \mathbb{N} \textit{ with } i < j, \ \ \forall t \in I,$$

and

$$\widehat{\delta}_i(t) \in D_i \cap B\left(\tau_\infty, \|\tau_{i-1} - \tau_\infty\|\right) \quad \forall i \in \mathbb{N}, \ \ \forall t \in I,$$

respectively.

Proof. Fix some $n \in \mathbb{N}$, and pick any $i, j \in \mathbb{N}$ with $1 \leq i < j \leq n$. According to basic braid theory, the generator $A_{i,j}$ of PB_{n+1} has a combed representative

$$\big(p_{\tau_1}, \ldots, p_{\tau_{j-1}}, \widehat{A}_{i,j}, p_{\tau_{j+1}}, \ldots, p_{\tau_{n+1}}\big) \in \Omega F_{n+1},$$

where $\widehat{A}_{i,j} \in \Omega(D^2, \tau_j)$ is a loop that winds around the i-th strand , and doesn't wind around any other strand. An example for $i = n-3, j = n$ is drawn below. Clearly, $\widehat{A}_{i,j}$ can be chosen such that

$$\widehat{A}_{i,j}(t) \in D_j \cap B\left(\tau_\infty, \left\|\frac{1}{2}(\tau_{i-1} + \tau_i) - \tau_\infty\right\|\right) \quad \forall i, j \in \mathbb{N} \text{ with } i < j, \ \ \forall t \in I, \qquad (A)$$

as required. Recalling Proposition 4.7 and Definition 4.9, notice that, in PB_{n+1},

$$\begin{aligned}
A_{i,j} &= \Big[\big(p_{\tau_1}, \ldots, p_{\tau_{j-1}}, \widehat{A}_{i,j}, p_{\tau_{j+1}}, \ldots, p_{\tau_{n+1}}\big)\Big] \\
&\overset{*}{=} \Big[\big(\widehat{\phi}_{n+1} \circ p_{\tau_1}, \ldots, \widehat{\phi}_{n+1} \circ p_{\tau_{j-1}}, \widehat{\phi}_{n+1} \circ \widehat{A}_{i,j}, \widehat{\phi}_{n+1} \circ p_{\tau_{j+1}}, \ldots, p_{\tau_{n+1}}\big)\Big] \\
&= \Big[\Omega\phi_{n+1}\big(p_{\tau_1}, \ldots, p_{\tau_{j-1}}, \widehat{A}_{i,j}, p_{\tau_{j+1}}, \ldots, p_{\tau_n}\big)\Big] \\
&= \Phi_{n+1}\Big[\big(p_{\tau_1}, \ldots, p_{\tau_{j-1}}, \widehat{A}_{i,j}, p_{\tau_{j+1}}, \ldots, p_{\tau_n}\big)\Big],
\end{aligned}$$

where $(*)$ follows from the properties of $\widehat{\phi}_n$, because, by (A),

$$\widehat{A}_{i,j}(t) \in D^2 \setminus B\left(\tau_\infty, \|\tau_{n+1} - \tau_\infty\|\right) \quad \forall t \in I.$$

Thus, in particular,

$$\Big[\big(p_{\tau_1},\ldots,p_{\tau_{j-1}},\widehat{A}_{i,j},p_{\tau_{j+1}},\ldots,p_{\tau_n}\big)\Big]=A_{i,j}\quad\text{ in } PB'_n.$$

On the other hand, for each $i\in[1,n]$, by Lemma A.9,

$$\Phi_{n+1}(\delta_i^{(n)})=A_{i,i+1}\cdots A_{i,n+1}\sim\sigma_i\cdots\sigma_{n-1}\sigma_n^2\sigma_{n-1}\cdots\sigma_i$$

in PB_{n+1}. Using standard representatives of the generators σ_i, one can show, by choosing an adequate homotopy in ΩF_{n+1}, that

$$\sigma_i\cdots\sigma_{n-1}\sigma_n^2\sigma_{n-1}\cdots\sigma_i=\Big[\big(p_{\tau_1},\ldots,p_{\tau_{i-1}},\widehat{\delta}_i,p_{\tau_{i+1}},\ldots,p_{\tau_{n+1}},\big)\Big],$$

where $\widehat{\delta}\in\Omega(D^2,\tau_i)$ is a loop that winds around all strands from the $i+1$-st to the $n+1$-st, and doesn't wind around the other strands. An example for $i=n$ is drawn below. Moreover, $\widehat{\delta}_i$ can be chosen such that

$$\widehat{\delta}_i(t)\in D_i\cap B\left(\tau_\infty,\|\tau_{i-1}-\tau_\infty\|\right)\quad\forall i\in\mathbb{N},\ \ \forall t\in I,\qquad (B)$$

as required. Thus, in PB_{n+1},

$$\begin{aligned}
\Phi_{n+1}\big(\delta_i^{(n)}\big) &= A_{i,i+1}\cdots A_{i,n+1}\\
&= \Big[\big(p_{\tau_1},\ldots,p_{\tau_{i-1}},\widehat{\delta}_i,p_{\tau_{i+1}},\ldots,p_{\tau_{n+1}},\big)\Big]\\
&\overset{*}{=} \Big[\big(\widehat{\phi}_{n+1}\circ p_{\tau_1},\ldots,\widehat{\phi}_{n+1}\circ p_{\tau_{i-1}},\widehat{\phi}_{n+1}\circ\widehat{\delta}_i,\widehat{\phi}_{n+1}\circ p_{\tau_{i+1}},\ldots,\widehat{\phi}_{n+1}\circ p_{\tau_n},p_{\tau_{n+1}}\big)\Big]\\
&= \Big[\Omega\phi_{n+1}\big(p_{\tau_1},\ldots,p_{\tau_{i-1}},\widehat{\delta}_i,p_{\tau_{i+1}},\ldots,p_{\tau_n},\big)\Big]\\
&= \Phi_{n+1}\Big[\big(p_{\tau_1},\ldots,p_{\tau_{i-1}},\widehat{\delta}_i,p_{\tau_{i+1}},\ldots,p_{\tau_n},\big)\Big],
\end{aligned}$$

where $(*)$ is given the above given properties of $\widehat{\phi}_n$, because, by (B),

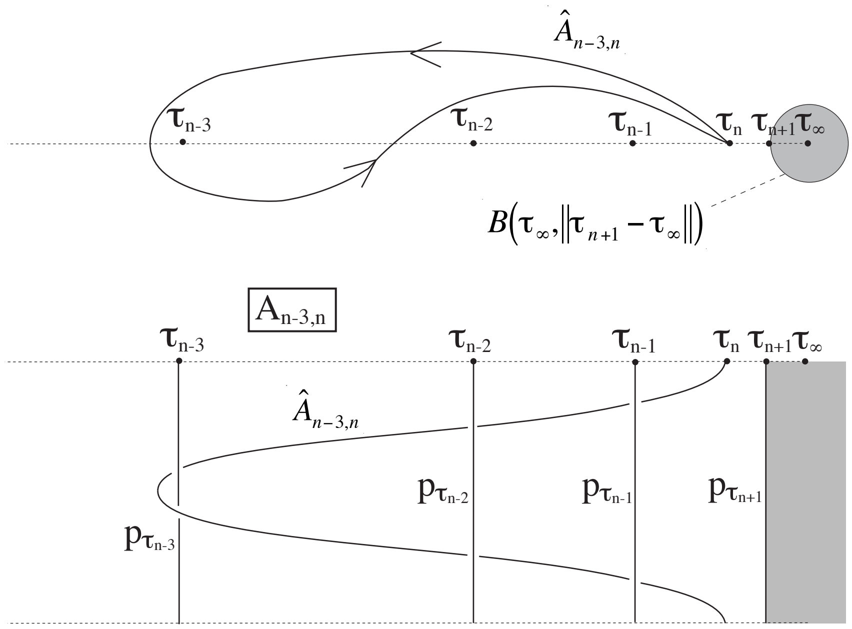
$$\widehat{\delta}_i(t)\in\overset{\circ}{D}{}^2\setminus B\left(\tau_\infty,\|\tau_i-\tau_\infty\|\right)\quad\forall t\in I.$$

Thus, as required,

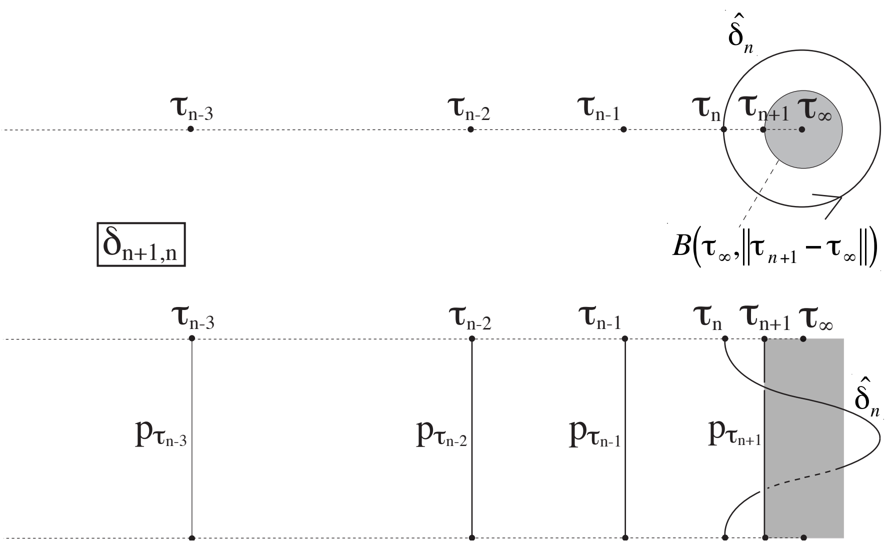
$$\Big[\big(p_{\tau_1},\ldots,p_{\tau_{i-1}},\widehat{\delta}_i,p_{\tau_{i+1}},\ldots,p_{\tau_n}\big)\Big]=\delta^{(n)}\quad\text{ in } PB'_n.$$

□

The loops $\widehat{A}_{n-3,n}$ and $\widehat{\delta}_n^{(n)}$, as well as the corresponding braids $\Omega\kappa_n^{(n+1)}(\widehat{A}_{n-3,n})$ and $\Omega\kappa_n^{(n+1)}(\widehat{\delta}_n)$ are shown in the following drawings, where the grey zones are to avoid by the given conditions.



$\hat{A}_{n-3,n}$
τ_{n-3} τ_{n-2} τ_{n-1} τ_n τ_{n+1} τ_∞
$B(\tau_\infty, \|\tau_{n+1} - \tau_\infty\|)$
$A_{n-3,n}$
τ_{n-3} τ_{n-2} τ_{n-1} τ_n τ_{n+1} τ_∞
$\hat{A}_{n-3,n}$
$p_{\tau_{n-3}}$ $p_{\tau_{n-2}}$ $p_{\tau_{n-1}}$ $p_{\tau_{n+1}}$



$\hat{\delta}_n$
τ_{n-3} τ_{n-2} τ_{n-1} τ_n τ_{n+1} τ_∞
$B(\tau_\infty, \|\tau_{n+1} - \tau_\infty\|)$
$\delta_{n+1,n}$
τ_{n-3} τ_{n-2} τ_{n-1} τ_n τ_{n+1} τ_∞
$\hat{\delta}_n$
$p_{\tau_{n-3}}$ $p_{\tau_{n-2}}$ $p_{\tau_{n-1}}$ $p_{\tau_{n+1}}$

4.2.4 Inverse system of pure braid groups revisited

In this subsection, we investigate the maps $\pi_1 s_{n,n-1} : PB_n \to PB_{n-1}$ and $\pi_1 s'_{n,n-1} : PB'_n \to PB'_{n-1}$ in algebraic terms, i.e., using the presentation of the groups PB_n and PB'_n, respectively. This allows us thereafter to construct an isomorphism between the inverse systems $\{PB'_n, \pi_1 s'_{n+1,n}\}_{n\in\mathbb{N}}$ and $\{PB_n, \pi_1 s_{n+1,n}\}_{n\in\mathbb{N}}$.

Proposition 4.13. *For each $n \geq 2$, the maps $\pi_1 s_{n,n-1} : PB_n \to PB_{n-1}$ and $\pi_1 s'_{n,n-1} : PB'_n \to PB'_{n-1}$ act as follows on the generators.*

$$\begin{array}{rcl} \pi_1 s_{n,n-1} : PB_n & \to & PB_{n-1} \\ A_{i,j} & \mapsto & A_{i,j} \qquad \forall\, 1 \leq i < j \leq n-1 \\ A_{i,n} & \mapsto & 1 \qquad \forall\, i \in [1, n-1] \end{array}$$

$$\begin{array}{rcl} \pi_1 s'_{n,n-1} : PB'_n & \to & PB'_{n-1} \\ A_{i,j} & \mapsto & A_{i,j} \qquad \forall\, 1 \leq i < j \leq n-1 \\ A_{i,n} & \mapsto & 1 \qquad \forall\, i \in [1, n-1] \\ \delta_i^{(n)} & \mapsto & \delta_i^{(n-1)} \qquad \forall\, i \in [1, n-1] \\ \delta_n^{(n)} & \mapsto & 1 \end{array}$$

Proof. Fix some $n \geq 2$. The statement concerning $\pi_1 s_{n,n-1}$ is proved in [Birman, p. 23], whereas the action of $\pi_1 s'_{n,n-1}$ follows directly from Proposition 4.12, by looking at representatives of the generators of PB'_n. □

For all integers $n' \geq n \geq 2$, introduce an isomorphism induced by conjugation

$$\begin{array}{rcl} c_n : PB_{n'} & \xrightarrow{\cong} & PB_{n'} \\ b & \mapsto & \sigma_{n-1} b \sigma_{n-1}^{-1}, \end{array}$$

where σ_{n-1} is the usual notation for a generator of Artin's presentation of the braid groups.

Lemma 4.14. *For every $n \geq 2$, the following diagram of homomorphisms of groups commutes.*

$$\begin{array}{ccc} PB'_{n-1} & \xrightarrow{\pi_1 s'_{n,n-1}} & PB'_{n-2} \\ {\scriptstyle \Phi_n}\downarrow{\scriptstyle \cong} & & \\ PB_n & & {\scriptstyle \cong}\downarrow{\scriptstyle \Phi_{n-1}} \\ {\scriptstyle c_n}\downarrow{\scriptstyle \cong} & & \\ PB_n & \xrightarrow{\pi_1 s_{n,n-1}} & PB_{n-1} \end{array}$$

Proof. Fix some $n \geq 2$. We prove that the diagram commutes by chasing each generator of PB'_{n-1} through it. Recall that the set of generators of PB'_{n-1} is given by $\{A_{i,j}\}_{1\leq i<j\leq n-1} \cup \{\delta_i^{(n-1)}\}_{i\in[1,n-1]}$. For the following calculations, we need Artin's relations of PB_n.

$$\begin{array}{rcll} \sigma_i\sigma_j & \sim & \sigma_j\sigma_i & \text{if } |i-j|\geq 2,\ 1\leq i,j\leq n-1 \qquad (A1)\\ \sigma_i\sigma_{i+1}\sigma_i & \sim & \sigma_{i+1}\sigma_i\sigma_{i+1}, & 1\leq i\leq n-2 \qquad (A2)\end{array}$$

At each stage, we underline the term to move, or to replace by an equivalent one. For all i,j with $1 \leq i < j \leq n-2$, the following holds in PB_{n-1}.

$$\begin{aligned} \pi_1 s_{n,n-1} \circ c_n \circ \Phi_n(A_{i,j}) &= \pi_1 s_{n,n-1} \circ c_n(A_{i,j}) \\ &= \pi_1 s_{n,n-1} \circ c_n\big(\sigma_{j-1}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\ldots\sigma_{j-1}\big) \\ &= \pi_1 s_{n,n-1}\big(\underline{\sigma_{n-1}}\sigma_{j-1}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\ldots\sigma_{j-1}\sigma_{n-1}^{-1}\big) \\ &\overset{A1}{=} \pi_1 s_{n,n-1}\big(\sigma_{j-1}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\ldots\sigma_{j-1}\sigma_{n-1}\sigma_{n-1}^{-1}\big) \\ &= \pi_1 s_{n,n-1}\big(\sigma_{j-1}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\ldots\sigma_{j-1}\big) \\ &= \pi_1 s_{n,n-1}\big(A_{i,j}\big) \overset{*}{=} A_{i,j} = \Phi_{n-1}(A_{i,j}) \\ &\overset{*}{=} \Phi_{n-1}\circ\pi_1 s'_{n,n-1}(A_{i,j}), \end{aligned}$$

where $(*)$ is given by Proposition 4.13. On the other hand, if $j = n-1$, then, for every $i \in [1, n-2]$,

$$\begin{aligned} \pi_1 s_{n,n-1} \circ c_n \circ \Phi_n(A_{i,n-1}) &= \pi_1 s_{n,n-1} \circ c_n(A_{i,n-1}) \\ &= \pi_1 s_{n,n-1} \circ c_n\big(\sigma_{n-2}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\ldots\sigma_{n-2}^{-1}\big) \\ &= \pi_1 s_{n,n-1}\big(\sigma_{n-1}\sigma_{n-2}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\ldots\sigma_{n-2}^{-1}\sigma_{n-1}^{-1}\big) \\ &= \pi_1 s_{n,n-1}\big(A_{i,n}\big) \overset{*}{=} 1 = \Phi_{n-1}(1) \\ &\overset{*}{=} \Phi_{n-1}\circ\pi_1 s'_{n,n-1}(A_{i,n-1}), \end{aligned}$$

where $(*)$ is given by Proposition 4.13. Furthermore, for every $i \in [1, n-2]$,

$$\begin{aligned}
&\pi_1 s_{n,n-1} \circ c_n \circ \Phi_n(\delta_i^{(n-1)}) \\
&\qquad = \pi_1 s_{n,n-1} \circ c_n(A_{i,i+1} \cdots A_{i,n}) \\
&\qquad = \pi_1 s_{n,n-1}(\sigma_{n-1} A_{i,i+1} \cdots A_{i,n} \sigma_{n-1}^{-1}) \\
&\qquad \stackrel{*}{=} \pi_1 s_{n,n-1}\big(\underline{\sigma_{n-1}}\sigma_i\sigma_{i+1}\cdots\sigma_{n-3}\sigma_{n-2}\sigma_{n-1}\sigma_{n-1}\sigma_{n-2}\sigma_{n-3}\ldots\sigma_i\underline{\sigma_{n-1}^{-1}}\big) \\
&\qquad \stackrel{A1}{=} \pi_1 s_{n,n-1}\big(\sigma_i\sigma_{i+1}\cdots\sigma_{n-3}\underline{\sigma_{n-1}\sigma_{n-2}\sigma_{n-1}}\sigma_{n-1}\sigma_{n-2}\sigma_{n-1}^{-1}\sigma_{n-3}\ldots\sigma_i\big) \\
&\qquad \stackrel{A2}{=} \pi_1 s_{n,n-1}\big(\sigma_i\sigma_{i+1}\cdots\sigma_{n-3}\sigma_{n-2}\sigma_{n-1}\underline{\sigma_{n-2}\sigma_{n-1}\sigma_{n-2}}\sigma_{n-1}^{-1}\sigma_{n-3}\ldots\sigma_i\big) \\
&\qquad \stackrel{A2}{=} \pi_1 s_{n,n-1}\big(\sigma_i\sigma_{i+1}\cdots\sigma_{n-3}\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\underline{\sigma_{n-1}\sigma_{n-1}^{-1}}\sigma_{n-3}\ldots\sigma_i\big) \\
&\qquad = \pi_1 s_{n,n-1}\big(\sigma_i\sigma_{i+1}\cdots\sigma_{n-3}\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\sigma_{n-3}\ldots\sigma_i\big) \\
&\qquad \stackrel{*}{=} \pi_1 s_{n,n-1}\big(A_{i,i+1}\cdots A_{i,n-1}A_{i,n}\big) \\
&\qquad \stackrel{*}{=} A_{i,i+1}\cdots A_{i,n-1} = \Phi_{n-1}(\delta_i^{(n-2)}) \\
&\qquad \stackrel{**}{=} \Phi_{n-1} \circ \pi_1 s'_{n,n-1}(\delta_i^{(n-1)}),
\end{aligned}$$

where $(*)$ and $(**)$ are given by Lemma A.9, and Proposition 4.13, respectively. Finally, if $i = n-1$, then,

$$\begin{aligned}
\pi_1 s_{n,n-1} \circ c_n \circ \Phi_n(\delta_{n-1}^{(n-1)}) &= \pi_1 s_{n,n-1}(\sigma_{n-1} A_{n-1,n} \sigma_{n-1}^{-1}) \\
&= \pi_1 s_{n,n-1}(\sigma_{n-1}\sigma_{n-1}^2\sigma_{n-1}^{-1}) \\
&= \pi_1 s_{n,n-1}(\sigma_{n-1}^2) = 1 = \Phi_{n-1}(1) \\
&= \Phi_{n-1} \circ \pi_1 s'_{n,n-1}(\delta_{n-1}^{(n-1)}).
\end{aligned}$$

□

Proposition 4.15. *The space F'_∞ is the inverse limit of pointed spaces*

$$F'_\infty = \lim \left\{F'_n, s'_{n,n-1}\right\}_{n\in\mathbb{N}}.$$

Proof. Similar to the proof of Proposition 1.19. □

Proposition 4.16.

$$PB'_\infty := \pi_1 F'_\infty = \lim \left\{PB'_n, \pi_1 s'_{n,n-1}\right\}_{n\in\mathbb{N}}.$$

Proof. Similar to the proof of Theorem 1.22 (see [5]), one can prove that the maps

$$s'_{n,n-1} : F'_n \to F'_{n-1}$$

are fiber bundles, such that, by Proposition 4.15, the result can be proved by [12], similarly to the proof of Corollary 1.23. □

For every $n \geq 2$, define an isomorphism $\Psi_n : PB'_{n-1} \xrightarrow{\cong} PB_n$ by iterated conjugation

$$\Psi_n := c_2 \circ \cdots \circ c_n \circ \Phi_n.$$

Theorem 4.17. *For each $n \in \mathbb{N}$, the isomorphism $\Psi_n : PB'_{n-1} \xrightarrow{\cong} PB_n$ induces an isomorphism of inverse systems*

$$\{\Psi_n\}_{n\geq 2} : \{PB'_{n-1}, \pi_1 s'_{n-1,n-2}\}_{n\in\mathbb{N}} \xrightarrow{\cong} \{PB_n, \pi_1 s_{n,n-1}\}_{n\in\mathbb{N}},$$

which itself induces an isomorphim

$$\Psi_\infty : PB'_\infty \xrightarrow{\cong} PB_\infty$$

on limits.

Proof. First, observe that the diagram

$$\begin{array}{ccc} PB_n & \xrightarrow{\pi_1 s_{n,n-1}} & PB_{n-1} \\ {\scriptstyle c_i}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle c_i} \\ PB_n & \xrightarrow{\pi_1 s_{n,n-1}} & PB_{n-1} \end{array}$$

commutes for every $n \in \mathbb{N}$ and $i \in [1, n-1]$, because, for each $b \in PB_n$,

$$\begin{aligned} c_i \circ \pi_1 s_{n,n-1}(b) &= \sigma_{i-1}\pi_1 s_{n,n-1}(b)\sigma_{i-1}^{-1} \\ &\overset{*}{=} \pi_1 s_{n,n-1}\big(\sigma_{i-1} b \sigma_{i-1}^{-1}\big) \\ &= \pi_1 s_{n,n-1} \circ c_i(b), \end{aligned}$$

where $(*)$ holds, because $\pi_1 s_{n,n-1}$ is a homomorphism, and

$$\pi_1 s_{n,n-1}(\sigma_i) = \sigma_i \quad \forall i \in [1, n-2].$$

By suitably putting together such diagrams for i varying from 1 to $n-1$, it follows that the diagram

$$\begin{array}{ccc} PB_n & \xrightarrow{\pi_1 s_{n,n-1}} & PB_{n-1} \\ {\scriptstyle c_2\cdots c_{n-1}}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle c_2\cdots c_{n-1}} \\ PB_n & \xrightarrow{\pi_1 s_{n,n-1}} & PB_{n-1} \end{array} \qquad (A)$$

also commutes for every $n \geq 2$. Using Lemma 4.14, it thus follows that the diagram

$$\begin{array}{ccccccc} \cdots \longrightarrow & PB'_{n-1} & \xrightarrow{\pi_1 s'_{n,n-1}} & PB'_{n-2} & \longrightarrow \cdots \longrightarrow & PB'_1 \\ & {\scriptstyle \Phi_n}\downarrow{\scriptstyle \cong} & & {\scriptstyle \Phi_{n-1}}\downarrow{\scriptstyle \cong} & & {\scriptstyle \Phi_2}\downarrow{\scriptstyle \cong} \\ & PB_n & & PB_{n-1} & & PB_2 \\ & {\scriptstyle c_n}\downarrow{\scriptstyle \cong} & \overset{\pi_1 s_{n,n-1}}{\nearrow} & {\scriptstyle c_{n-1}}\downarrow{\scriptstyle \cong} & & {\scriptstyle c_2}\downarrow{\scriptstyle \cong} \\ & PB_n & (A) & PB_{n-1} & & PB_2 \\ & {\scriptstyle c_2\circ\cdots\circ c_{n-1}}\downarrow{\scriptstyle \cong} & & {\scriptstyle c_2\circ\cdots\circ c_{n-2}}\downarrow{\scriptstyle \cong} & & {\scriptstyle =}\downarrow \\ \cdots \longrightarrow & PB_n & \xrightarrow{\pi_1 s_{n,n-1}} & PB_{n-1} & \longrightarrow \cdots \longrightarrow & PB_2 \end{array}$$

commutes, which shows that the maps $\{\Psi_n\}_{n\geq 2}$ yield the required isomorphism of inverse systems. Moreover, according to Corollary 1.23 and Proposition 4.16, the upper and lower inverse system have PB'_∞ and PB_∞ as limits, respectively. □

Recall that, by the comments on the beginning of the present chapter, there is a commutative diagram

$$\begin{array}{ccc} & PH_\infty & \\ {}^{\varphi'_\infty}\swarrow & & \searrow^{\varphi_\infty} \\ \Omega F'_\infty & \xhookrightarrow{\Omega\iota} & \Omega F_\infty, \end{array}$$

where ι is the inclusion map. However, it is important to keep in mind that the map $\Psi_\infty : PB'_\infty \to PB_\infty$ is different from the map $\pi_1\iota$. In particular, the diagram

$$\begin{array}{ccc} & \pi_0 PH_\infty & \\ {}^{\pi_0\varphi'_\infty}\swarrow & & \searrow^{\pi_0\varphi_\infty} \\ PB'_\infty & \xrightarrow[\cong]{\Psi_\infty} & PB_\infty \end{array}$$

does *not* commute.

Definition 4.18. *For every $n \in \mathbb{N}$, define subgroups $U_n \subset PB_n$ and $U'_n \subset PB'_{n+1}$ by*

$$U_n := Ker\,\big(\pi_1 s_{n,n-1}\big), \quad U'_n := Ker\,\big(\pi_1 s'_{n,n-1}\big).$$

Proposition 4.19. *For every $n \in \mathbb{N}$, the subgroups $U_n \lhd PB_n, U'_n \lhd PB'_{n+1}$ are presented as follows.*

$$U_n = \Big\langle \{A_{i,n}\}_{i\in[1,n-1]} \Big\rangle, \quad U'_n = \Big\langle \{A_{i,n}\}_{i\in[1,n-1]}, \delta_n^{(n)} \Big\rangle$$

In particular, these groups are free.

Proof. Fix some $n \in \mathbb{N}$. The presentation

$$U_n = \Big\langle \{A_{i,n}\}_{i\in[1,n-1]} \Big\rangle$$

is given in [Birman, p. 23]. Recalling that both $\pi_1 s_{n+1,n}$ and $\pi_1 s'_{n+1,n}$ are epimorphisms, the diagram of Lemma 4.14 can be extended to the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U'_n & \hookrightarrow & PB'_n & \xrightarrow{\pi_1 s'_{n,n-1}} & PB'_{n-1} & \longrightarrow & 1 \\ & & {\scriptstyle c_{n+1}\circ\Phi_{n+1}|^{U'_n}}\downarrow\cong & & {\scriptstyle c_{n+1}\circ\Phi_{n+1}}\downarrow\cong & & {\scriptstyle \Phi_n}\downarrow\cong & & \\ 1 & \longrightarrow & U_{n+1} & \hookrightarrow & PB_{n+1} & \xrightarrow{\pi_1 s_{n+1,n}} & PB_n & \longrightarrow & 1, \end{array}$$

where the corestricted map $c_{n+1} \circ \Phi_{n+1}|^{U'_n}$ is an isomorphism by the five lemma. Therefore, according to the presentation of U_{n+1}, U'_n must be the free group presented by

$$U'_n \cong \left\langle \left\{ \big(c_{n+1} \circ \Phi_{n+1}\big)^{-1}(A_{i,n+1}) \right\}_{i \in [1,n]} \right\rangle.$$

Recalling the identities

$$A_{i,n+1} := \sigma_n \sigma_{n-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{n-1}^{-1} \sigma_n^{-1}$$

for all $1 \leq i \leq n$, the generators are given by

$$\begin{aligned}
\big(c_{n+1} \circ \Phi_{n+1}\big)^{-1}(A_{i,n+1}) &= \Phi_{n+1}^{-1} \circ c_{n+1}^{-1}(A_{i,n+1}) \\
&= \Phi_{n+1}^{-1}(\sigma_n^{-1} \sigma_n \sigma_{n-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{n-1}^{-1} \sigma_n^{-1} \sigma_n) \\
&= \Phi_{n+1}^{-1}(\sigma_{n-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{n-1}^{-1}) \\
&= \Phi_{n+1}^{-1}(A_{i,n}) \\
&= A_{i,n}
\end{aligned}$$

for all $i \in [1, n-1]$. On the other hand,

$$\begin{aligned}
\big(c_{n+1} \circ \Phi_{n+1}\big)^{-1}(A_{n,n+1}) &= \Phi_{n+1}^{-1} \circ c_{n+1}^{-1}(A_{n,n+1}) \\
&= \Phi_{n+1}^{-1}(\sigma_n^{-1} \sigma_n^2 \sigma_n) \\
&= \Phi_{n+1}^{-1}(\sigma_n^2) \\
&= \Phi_{n+1}^{-1}(A_{n,n+1}) \\
&= \delta_n^{(n)}.
\end{aligned}$$

□

4.2.5 Semidirect product decomposition of the pure braid groups

Proposition 4.20. *For each $n \in \mathbb{N}$, there is an isomorphism*

$$\begin{aligned}
\mu_n : U'_1 \ltimes \cdots \ltimes U'_n &\to PB'_n \\
(u_1, \ldots, u_n) &\mapsto u_1 \cdots u_n.
\end{aligned}$$

Proof. Fix some $n \in \mathbb{N}$, and consider the split short exact sequence

$$1 \to U'_n \to PB'_n \xrightarrow{\pi_1 s'_{n,n-1}} PB'_{n-1} \to 1.$$

Thus,

$$PB'_n = PB'_{n-1} U'_n, \quad U'_n \lhd PB'_n, \quad PB'_{n-1} \cap U'_n = \{1\},$$

which means that there is an isomorphism

$$\begin{aligned}
PB'_{n-1} \ltimes U'_n &\xrightarrow{\cong} PB'_n \\
(b, u) &\mapsto b \cdot u.
\end{aligned}$$

Iterating this procedure yields the above defined isomorphism

$$\begin{array}{rcl} \mu_n : U'_1 \ltimes \cdots \ltimes U'_n & \to & PB'_n \\ (u_1, \ldots, u_n) & \mapsto & u_1 \cdots u_n. \end{array}$$

□

We now recall some basic facts concerning iterated products. First, note that, for any given $n \geq 2$,

$$U'_1 \ltimes \cdots \ltimes U'_n = \prod_{i=1}^{n} U'_i, \quad \text{as sets.}$$

Moreover, the group structure of $U'_1 \ltimes \cdots \ltimes U'_n$ is given as follows.

$$\begin{array}{rcl} \ltimes_{i=1}^{n} U_i \times \ltimes_{i=1}^{n} U'_i & \to & \ltimes_{i=1}^{n} U'_i \\ \big((u_i)_{i\in[1,n]}, (v_i)_{i\in[1,n]}\big) & \mapsto & \big(u_1v_1, v_1^{-1}u_2v_1v_2, \ldots, v_{n-1}^{-1} \cdots v_1^{-1}u_nv_1 \cdots v_n\big). \end{array}$$

That this structure is preserved by the map $\mu_n : \ltimes_{i=1}^{n-1} U'_i \to PB'_n$ can be illustrated as follows, for any given $(u_i)_{i\in[1,n]}$ and $(v_i)_{i\in[1,n]}$.

$$\begin{array}{rcl} \mu_n\big((u_i)_{i\in[1,n]} \cdot (v_i)_{i\in[1,n]}\big) & = & \mu_n\big(u_1v_1, v_1^{-1}u_2v_1v_2, \ldots, v_{n-1}^{-1} \cdots v_1^{-1}u_nv_1 \cdots v_n\big) \\ & = & u_1 \cdots u_n v_1 \cdots v_n \\ & = & \mu_n\big((u_i)_{i\in[1,n]}\big)\mu_n\big((v_i)_{i\in[1,n]}\big). \end{array}$$

Consider the following inverse system.

$$\cdots \to \ltimes_{i\in[1,n+1]} U'_i \xrightarrow{p_{n+1,n}} \ltimes_{i\in[1,n]} U'_i \to \cdots \to U'_1$$

As a set, its limit $\ltimes_{i\in\mathbb{N}} U'_i$ is given by

$$\ltimes_{i\in\mathbb{N}} U'_i = \prod_{i\in\mathbb{N}} U'_i.$$

Moreover, $\ltimes_{i\in\mathbb{N}} U'_i$ has a group structure induced by the group structure of the groups in the inverse system.

Proposition 4.21. *There is an isomorphism of inverse systems*

$$\begin{array}{ccccccc} \cdots \longrightarrow & \ltimes_{i\in[1,n+1]} U'_i & \xrightarrow{p_{n+1,n}} & \ltimes_{i\in[1,n]} U'_i & \longrightarrow \cdots \\ & \mu_{n+1} \downarrow \cong & & \mu_n \downarrow \cong & \\ \cdots \longrightarrow & PB'_{n+1} & \xrightarrow{\pi_1 s'_{n+1,n}} & PB'_n & \longrightarrow \cdots, \end{array}$$

where $p_{n,n-1}$ is the canonical projection. Thus, there is an induced isomorphism of limits

$$\mu_\infty : \ltimes_{i\in\mathbb{N}} U'_i \xrightarrow{\cong} PB'_\infty.$$

Proof. Recalling that $U'_n \equiv \ker \pi_1 s'_{n,n-1}$ for all n, it follows directly from the definition of the implied maps that each square commutes. Moreover,

$$\lim_n PB'_n = PB'_\infty$$

by Proposition 4.16. □

Similar to the maps μ_n, introduce maps of standard braid groups

$$\begin{aligned} \mu^s_n : U_2 \ltimes \cdots \ltimes U_n &\to PB_n \\ (u_2, \ldots, u_n) &\mapsto u_2 \cdots u_n \end{aligned}$$

for all $n \geq 2$.

Proposition 4.22. *For each $n \in \mathbb{N}$, the map μ^s_n is an isomorphism. Moreover, there is an isomorphism of inverse systems*

$$\begin{array}{ccccccc} \cdots \longrightarrow & \ltimes_{i\in[2,n+1]} U_i & \xrightarrow{p_{n+1,n}} & \ltimes_{i\in[2,n]} U_i & \longrightarrow \cdots \\ & \mu^s_{n+1} \downarrow \cong & & \mu^s_n \downarrow \cong & \\ \cdots \longrightarrow & PB_{n+1} & \xrightarrow{\pi_1 s_{n+1,n}} & PB_n & \longrightarrow \cdots, \end{array}$$

where $p_{n+1,n}$ is the canonical projection, which induces an isomorphism

$$\mu^s_\infty : \ltimes_{i\geq 2} U_i \xrightarrow{\cong} PB_\infty.$$

on limits.

Proof. Fix some $n \geq 2$. The proof of the fact that μ^s_n is an isomorphism is similar to the proof of Proposition 4.20. Moreover, recalling that $U_n \equiv \ker \pi_1 s_{n,n-1}$ for all n, it follows directly from the definition of the implied maps that each square commutes. Moreover, by Corollary 1.23,

$$\lim_n PB_n = PB_\infty.$$

□

4.2.6 Canonical representatives of the elements of the groups $\{PB'_n\}_{n\in\mathbb{N}}$

Definition 4.23. *Introduce a map*

$$\begin{aligned} \widehat{\mu}_\infty : \prod_{i\in\mathbb{N}} \Omega(D_i, \tau_i) &\to (\Omega F'_\infty)_c \\ (\beta_i)_{i\in\mathbb{N}} &\mapsto ((\beta'_i)_{i\in\mathbb{N}}), \end{aligned}$$

where, for each $i \in \mathbb{N}$, β_i' is given by

$$\beta_i'(t) := \begin{cases} \tau_i & \text{if } t \in [0, t_i] \\ \beta_i\Big(2^i\Big(t - t_i\Big)\Big) & \text{if } t \in [t_i, t_{i+1}] \\ \tau_i & \text{if } t \in [t_{i+1}, 1] \end{cases} \qquad \forall\ t \in I.$$

Also, write, for all $n \in \mathbb{N}$,

$$\begin{array}{rcl} \widehat{\mu}_n : \prod_{i\in[1,n-1]} \Omega\big(D_i, \tau_i\big) & \to & \Omega F_n' \\ (\beta_i)_{i\in[1,n-1]} & \mapsto & (\beta_i')_{i\in[1,n-1]}, \end{array}$$

where, for each $i \in \mathbb{N}$, the loop β_i' is defined as above.

Remark 4.24. *For any given $(\beta_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} \Omega(D_i, \tau_i)$, write*

$$\widehat{\mu}_\infty(\beta_i)_{i\in\mathbb{N}} := (\beta_i')_{i\in\mathbb{N}},$$

and observe that, for all $t \in I$,

$$\beta_i'(t) = \beta_j'(t) \quad \Leftrightarrow \quad i = j.$$

Thus, $(\beta_i')_{i\in\mathbb{N}} : I \to F_\infty'$ is a well defined map. Moreover, for each $i \in \mathbb{N}$, $\beta_i' : I \to D^2$ is continuous, which is a necessary and sufficient condition for $(\beta_i')_{i\in\mathbb{N}} : I \to F_\infty'$ to be a continuous map (see [11, Thm. 19.6]). This shows that $\widehat{\mu}_\infty : \times_{i\in\mathbb{N}}\Omega\big(D_i, \tau_i\big) \to \big(\Omega F_\infty'\big)_c$ is indeed well defined.
Finally, note that the map $\widehat{\mu}_n$ just concatenates braids:

$$\widehat{\mu}_\infty(\beta_i)_{i\in\mathbb{N}} = \big(\beta_1, (p_{\tau_i})_{i\geq 2}\big) \star \Big(\big(p_{\tau_i}, \beta_2, (p_{\tau_i})_{i\geq 3}\big) \star \dots\Big).$$

Recall that, for all $n \in \mathbb{N}$ the group U_n' is given by

$$U_n' = \big\langle \{A_{i,n}\}_{1\leq i<n}, \delta_n^{(n)} \big\rangle.$$

Recall the maps

$$\kappa_i^{(n)} : \big(D_i, \tau_i\big) \to (F_n', \mathcal{T}_n)$$

that we introduced in Definition 4.11 for all $n \in \mathbb{N}$ and $i \in [1, n-1]$. By Proposition 4.12, there are loops $\{\widehat{A}_{i,n}\}_{1\leq i<j\leq n}, \{\widehat{\delta}_i\}_{i\in[1,n]}$ in $\mathring{D}^2$ that satisfy

$$\begin{array}{rcl} A_{i,j} & = & \big[\kappa_j^{(n)} \circ \widehat{A}_{i,j}\big] \quad \forall i \in [1, n-1], \\ \delta_i^{(n)} & = & \big[\kappa_i^{(n)} \circ \widehat{\delta}_j\big] \quad \forall i \in [1, n]. \end{array}$$

Definition 4.25. *For all $n \in \mathbb{N}$, define a map $\mathrm{rep}_{U_n'} : U_n' \to \Omega\big(D_n, \tau_n\big)$ by*

$$\begin{array}{rcl} \mathrm{rep}_{U_n'}\big(A_{i,n}\big) & := & \widehat{A}_{i,n} \quad \forall i \in [1, n-1] \\ \mathrm{rep}_{U_n'}\big(\delta_n^{(n)}\big) & := & \widehat{\delta}_n, \end{array}$$

on the generators, and, for any $u \in U'_n$, by

$$\mathrm{rep}_{U'_n}(u) := \mathrm{rep}_{U'_n}(u_1) \star \Big(\mathrm{rep}_{U'_n}(u_2) \star (\dots)\Big)$$

where $u_1 \cdots u_k$ is the (unique) reduced word that represents u.

Proposition 4.26. *For all integers $i \leq n$,*

$$\big[\kappa_i^{(n)} \circ rep_{U'_i}\big] = Id_{U'_i}.$$

Proof. By Proposition 4.12, this follows directly from the definition of $\mathrm{rep}_{U'_i}$. □

Proposition 4.27. *For each $n \in \mathbb{N}$, the maps $\big[\mathrm{rep}_{U'_n}(\cdot)\big]$ and $\pi_1\kappa_n^{(n)}$ are mutually inverse isomorphisms.*

$$U'_n \underset{\pi_1\kappa_n^{(n)}}{\overset{[\mathrm{rep}_{U'_n}(\cdot)]}{\rightleftarrows}} \pi_1\big(D_n, \tau_n\big) \quad (\cong)$$

.

Proof. Fix some $n \in \mathbb{N}$, and extend $\kappa_n^{(n)}$ to a well defined map

$$\begin{aligned} \overline{\kappa}_n^{(n)} : \Big(D^2 \setminus \Big\{\{\tau_i\}_{i\in[1,n-1]\cup\infty}\Big\}, \tau_n\Big) &\to (F'_n, \mathcal{T}_n) \\ x &\mapsto \big(\tau_1, \dots, \tau_{n-1}, x\big). \end{aligned}$$

Observe that, by [Birman, Thm. 1.4], the sequence

$$1 \to \pi_1\Big(D^2 \setminus \Big\{\{\tau_i\}_{i\in[1,n-1]\cup\infty}\Big\}, \tau_n\Big) \overset{\pi_1\overline{\kappa}_n^{(n)}}{\longrightarrow} \pi_1 F'_n \overset{\pi_1 s'_{n,n-1}}{\longrightarrow} \pi_1 F'_{n-1} \to 1$$

is *exact*. Thus,

$$\pi_1\overline{\kappa}_n^{(n)} : \pi_1\Big(D^2 \setminus \Big\{\{\tau_i\}_{i\in[1,n-1]\cup\infty}\Big\}, \tau_n\Big) \overset{\cong}{\longrightarrow} \ker\, \pi_1 s'_{n,n-1} =: U'_n.$$

Observing that the injection

$$i : D_n := D^2 \setminus \Big\{\{\tau_i\}_{i\in[1,n-1]} \cup B\big(\tau_\infty, \|\tau_{n+1} - \tau_\infty\|\big)\Big\} \hookrightarrow D^2 \setminus \Big\{\{\tau_i\}_{i\in[1,n-1]\cup\infty}\Big\}$$

is a homotopy equivalence, and that, moreover,

$$\pi_1\kappa_n^{(n)} = \pi_1\overline{\kappa}_n^{(n)} \circ \pi_1 i,$$

it follows that $\pi_1\kappa_n^{(n)}$ is an isomorphism. Moreover, by Proposition 4.26

$$\pi_1\kappa_n^{(n)} \circ \big[\mathrm{rep}_{U'_n}\big] = \mathrm{Id}_{U'_n},$$

which finishes the proof. □

The following proposition gives a tool to find canonical representatives of finite, and also infinite braids in $\overset{\circ}{D}{}^2 \setminus \tau_\infty$ (see Corollary 4.29).

Proposition 4.28. *For each $n \in \mathbb{N}$, the following diagram of sets commutes.*

$$\begin{array}{ccc} \prod_{i\in[1,n]} U'_i & \xrightarrow[\cong]{\mu_n} & PB'_n \\ \Big\downarrow {\scriptstyle (\mathrm{rep}_{U'_i})_{i\in[1,n]}} & & \Big\uparrow {\scriptstyle [\cdot]} \\ \prod_{i\in[1,n]} \Omega(D_i,\tau_i) & \xrightarrow{\widehat{\mu}_n} & (\Omega F'_n)_c \end{array}$$

Moreover, these diagrams induce a commutative diagram of limits.

$$\begin{array}{ccc} \prod_{i\in\mathbb{N}} U'_i & \xrightarrow[\cong]{\mu_\infty} & PB'_\infty \\ \Big\downarrow {\scriptstyle (\mathrm{rep}_{U'_i})_{i\in\mathbb{N}}} & & \Big\uparrow {\scriptstyle [\cdot]} \\ \prod_{i\in\mathbb{N}} \Omega(D_i,\tau_i) & \xrightarrow{\widehat{\mu}_\infty} & (\Omega F'_\infty)_c \end{array}$$

Proof. Fix some $n \in \mathbb{N}$. To see that the diagram commutes, pick any $(u_i)_{i\in[1,n]} \in \prod_{i\in[1,n]} U'_i$, and verify that

$$\begin{aligned} &\Big[\widehat{\mu}_n\big(rep_{U'_1}(u_1),\dots,rep_{U'_n}(u_n)\big)\Big] \\ &\quad = \Big[\big(rep_{U'_1}(u_1), p_{\tau_2},\dots,p_{\tau_n}\big)\Big]\cdots\Big[\big(p_{\tau_1},p_{\tau_{n-1}}\dots,rep_{U'_n}(u_n)\big)\Big] \\ &\quad = \pi_1\kappa_1^{(n)}\big[rep_{U'_1}(u_1)\big]\cdots\pi_1\kappa_n^{(n)}\big[rep_{U'_n}(u_n)\big] \\ &\quad \overset{*}{=} u_1\cdots u_n, \end{aligned}$$

where $(*)$ is given by Proposition 4.26. This proves that the first diagram commutes for all $n \in \mathbb{N}$. Moreover, these diagrams induce a commutative diagram of inverse systems

$$\begin{array}{ccc} \Big\{\prod_{i\in[1,n]} U'_i, p_{n,n-1}\Big\}_{n\in\mathbb{N}} & \xrightarrow[\cong]{\{\mu_n\}_{n\in\mathbb{N}}} & \Big\{PB'_n, \pi_1 s'_{n,n-1}\Big\}_n \\ \Big\downarrow {\scriptstyle \{(rep_{U'_i})_{i\in[1,n]}\}_{n\in\mathbb{N}}} & & \Big\uparrow {\scriptstyle \{[\cdot]\}_{n\in\mathbb{N}}} \\ \Big\{\prod_{i\in[1,n]} \Omega(D_i,\tau_i), p_{n,n-1}\Big\}_n & \xrightarrow{\{\widehat{\mu}_n\}_{n\in\mathbb{N}}} & \Big\{\Omega F'_n, \Omega s'_{n,n-1}\Big\}_n, \end{array}$$

where we write $p_{n,n-1} : \prod_{i\in[1,n]} X_i \to \prod_{i\in[1,n-1]} X_i$ for the natural projection. Therefore, the induced diagram of limits also commutes, where

$$\lim_n PB_n = PB_\infty, \quad \lim_n PB'_n = PB'_\infty$$

by Corollary 1.23 and Proposition 4.16, respectively. □

The next corollary follows immediately from the proposition above.

Corollary 4.29. *For each $n \in \mathbb{N} \cup \infty$, the map*

$$\begin{array}{rcl} \mathrm{rep}_{PB'_n} : PB'_n & \to & \Omega F'_n \\ b & \mapsto & \widehat{\mu}_n\Big(\big(\mathrm{rep}_{U'_i}\big(\mu_n^{-1}(b)\big)_i\big)_{i\in[1,n-1]}\Big) \end{array}$$

satisfies

$$\big[\mathrm{rep}_{PB'_n}(b)\big] = b \quad \forall b \in PB'_n.$$

Moreover,

$$\mathrm{Im}\ \mathrm{rep}_{PB'_n} \subset \big(\Omega F'_n\big)_c.$$

In particular, this result allows us to attribute canonical representatives to the elements of PB'_n for any $n \in \mathbb{N} \cup \infty$.

4.3 Towards an identification of Im $\pi_0\varphi_\infty$ in $\ltimes_{i\in\mathbb{N}} U'_i$.

Recall that, in Theorem 4.5, we identified the image of $\pi_0\varphi'_\infty$ in terms of representatives in PB'_∞. Using the semidirect product decomposition

$$PB'_\infty \xleftarrow[\cong]{\mu_\infty} \ltimes_{i\in\mathbb{N}} U'_i.$$

of the preceding section, we now characterize a certain subset of Im $\pi_0\varphi_\infty \subset PB_\infty$ within $\ltimes_{i\in\mathbb{N}} U'_i$ (see Proposition 4.31).

Definition 4.30. *For each $i \in \mathbb{N}$, define a map $\theta_i : U'_i \to \mathbb{N}$ by*

$$\begin{aligned} \theta_i(b) \quad := \quad \min\Big\{ & i, j \in [1, i-1] \mid \textit{the reduced word that represents } b \\ & \textit{contains the letter } A_{j,i} \Big\} \end{aligned}$$

for all $b \in U'_i$.

Proposition 4.31.

$$\big(PB'_\infty\big)_{cc} \supset \Big\{\mu_\infty\big((b_i)_{i\in\mathbb{N}}\big) \mid (b_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} U'_i \ \textit{s.t.} \quad \lim_{i\to\infty} \theta_i(b_i) = \infty\Big\}.$$

The question whether the inverse inclusion also holds seems to depend on whether $\big(PB'_\infty\big)_{cc}$ is equal to $\pi_0\big(\Omega F'_\infty\big)_{cc}$. Unfortunately, we did not solve this problem.

Proof. Pick some $(b_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} U'_i$ with $\lim_{i\to\infty} \theta_i(b_i) = \infty$. Fix some $i \in \mathbb{N}$, and write b_i as a word

$$b_i = u_1 \cdots u_k$$

in the alphabet $\mathrm{Gen}(U'_i)$. As, by the definition of the map θ_i,

$$\theta_i(b_i) = \min_{j\in[1,k]} \theta_i(u_j),$$

it follows by our choice of the point set $\{\tau_j\}_{j\in\mathbb{N}}$ that

$$\|\tau_{\theta_i(b_i)-1} - \tau_\infty\| = \max_{j\in[1,k]} \|\tau_{\theta_i(u_j)-1} - \tau_\infty\|. \qquad (A)$$

Moreover, recall that

$$\mathrm{rep}_{U_i'}(b_i) = rep_{U_i'}(u_1) \star \Big(rep_{U_i'}(u_2) \star \big(rep_{U_i'}(u_3) \star (\dots)\big)\Big),$$

and that, by Proposition 4.12,

$$\mathrm{rep}_{U_i'}(u_j)(t) \in B\big(\tau_\infty, \|\tau_{\theta_i(u_j)-1} - \tau_\infty\|\big) \quad \forall t \in I,\ \forall\ j \in [1,k].$$

Thus, by (A),

$$\mathrm{rep}_{U_i'}(b_i)(t) \in B\big(\tau_\infty, \|\tau_{\theta_i(b_i)-1} - \tau_\infty\|\big) \quad \forall t \in I,\ \forall\ i \in \mathbb{N}.$$

In particular, this means that

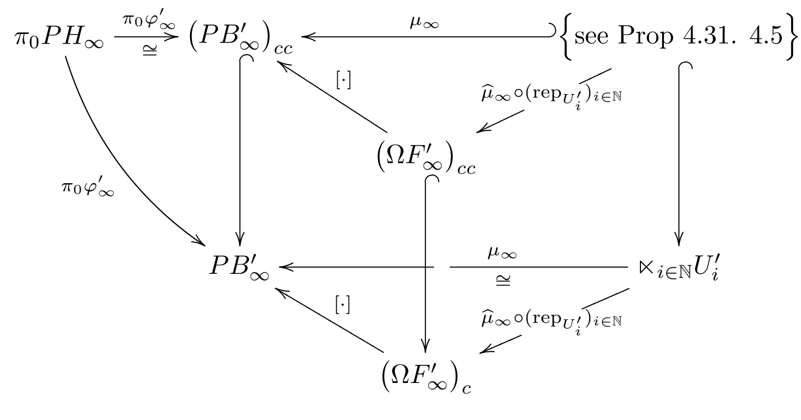
$$\lim_{i\to\infty} \mathrm{rep}_{U_i'}(b_i)(t) = \tau_\infty \quad \forall t \in I,$$

because $\lim_{i\to\infty}\theta_i(b_i) = \infty$, such that, according to Proposition 4.28,

$$\mu_\infty\big((b_i)_{i\in\mathbb{N}}\big) = \big[\widehat{\mu}_\infty\big((\mathrm{rep}_{U_i'}(b_i)\big)_{i\in\mathbb{N}}\big] \in \big(PB'_\infty\big)_{cc}.$$

□

We now can consider the following commutative diagram, which summarizes Propositions 4.21 and 4.22 and Theorem 4.31. Some maps are tacitly (co-) restricted, without changing the notation.



As we pointed out above, the image of $\pi_0\varphi_\infty$ seems to be difficult to describe within the semidirect product decomposition $\ltimes_{i\geq 2} U_i$. We now underline this by an example.

Example. Pick an element $(u_i)_{i\in\mathbb{N}} \in \ltimes_{i\in\mathbb{N}} U'_i$ given by

$$u_i = \begin{cases} \delta_n^{(n)}, & i = n, \\ 1, & i \neq n \end{cases}$$

for some $n \in \mathbb{N}$. Clearly, $(u_i)_{i\in\mathbb{N}} \in \mu_\infty^{-1}\big(PB'_\infty\big)_{cc}$, such that, by Theorem 4.5, there is a homeomorphism $h \in PH_\infty$ such that

$$\pi_0\varphi'_\infty[h] = \mu_\infty\big((u_i)_{i\in\mathbb{N}}\big).$$

In particular,

$$(\beta_i)_{i\in\mathbb{N}} := \mathrm{rep}_{PB'_\infty}\big((u_i)_{i\in\mathbb{N}}\big)$$

is a combed, convergent representative of $\pi_0\varphi'_\infty[h]$, given by

$$\beta_i = \begin{cases} \widehat{\delta}_n, & i = n \\ p_{\tau_i}, & i \neq n \end{cases},$$

i.e., all strands are straight, except the n-th strand that winds once around all points τ_i with $i > n$ (see p. 69). Writing,

$$\begin{aligned} (v_i)_{i\geq 2} &:= {\mu_\infty^s}^{-1} \circ \pi_1\imath \circ \mu_\infty\big((u_i)_{i\in\mathbb{N}}\big) \\ &= {\mu_\infty^s}^{-1} \circ \pi_1\imath\big[(\beta_i)_{i\in\mathbb{N}}\big] \\ &= {\mu_\infty^s}^{-1}\big[(\beta_i)_{i\in\mathbb{N}}\big] \end{aligned}$$

in $\ltimes_{i\geq 2} U_i$, where $\iota : F'_\infty \hookrightarrow F_\infty$ is the inclusion map, one can verify that $(v_i)_{i\geq 2}$ is given by

$$v_i := \begin{cases} 1, & i \leq n \\ A_{n,i}, & i > n. \end{cases}$$

It might be difficult to find criteria to find out reversely that the given sequence $(v_i)_{i\geq 2} \in \ltimes_{i\geq 2} U_i$ is in the image of $\pi_0\varphi_\infty$, i.e., in the image of $\pi_1\iota\big(PB'_\infty\big)_{cc}$, whence the advantage of working with the semidirect decomposition

$$PB_\infty \cong \ltimes_{i\in\mathbb{N}} U'_i$$

rather than with the (more natural) semidirect decomposition

$$PB_\infty \cong \ltimes_{i\geq 2} U_i.$$

4.4 Generalization of the choice of $\mathcal{T}_\infty$

Recall our choice of a particular basepoint $\mathcal{T}_\infty$ of the spaces F_∞ and F'_∞, given in Definition 2.1. To conclude the section, we return to an arbitrary choice of $\mathcal{T}_\infty$.
Let $\mathcal{T}^*_\infty \equiv (\tau^*_i)_{i\in\mathbb{N}} \in F_\infty$ be any infinite configuration with a single accumulation point $\tau_\infty \in \overset{\circ}{D}{}^2$, i.e., $\lim_{i\to\infty} \tau_i = \tau_\infty$, and such that

$$\tau_i \neq \tau_\infty \quad \forall i \in \mathbb{N},$$

and write

$$PH^*_\infty := \Big\{ h \in H_0 \mid h(\tau^*_i) = \tau^*_i \quad \forall i \in \mathbb{N} \Big\}.$$

Also, write

$$\begin{array}{rcl} \varphi^*_\infty : PH^*_\infty & \to & \Omega(F'_\infty, \mathcal{T}^*_\infty) \\ h & \mapsto & \big(K(h,\cdot)(\tau^*_i)\big)_{i\in\mathbb{N}}, \end{array}$$

similarly to the definition of the map φ_∞. Furthermore, according to Proposition A.7, there is a homeomorphism $h \in H_0$ such that

$$h(\tau_i) = \tau^*_i \quad \forall i \in \mathbb{N},$$

which allows us to define pointed maps

$$\begin{array}{rccc} \Psi_1 : & \big(PH_\infty, \mathrm{Id}_{D^2}\big) & \xrightarrow{\cong} & \big(PH^*_\infty, \mathrm{Id}_{D^2}\big) \\ & f & \mapsto & h \circ f \circ h^{-1} \end{array} \qquad \begin{array}{rccc} \Psi_2 : & \big(F_\infty, \mathcal{T}_\infty\big) & \xrightarrow{\cong} & \big(F_\infty, \mathcal{T}^*_\infty\big) \\ & (x_i)_{i\in\mathbb{N}} & \mapsto & \big(h(x_i)\big)_{i\in\mathbb{N}}. \end{array}$$

Furthermore, write

$$\begin{array}{rccc} \overline{\Psi}_1 : & \big(H_\infty, \mathrm{Id}_{D^2}\big) & \xrightarrow{\cong} & \big(H^*_\infty, \mathrm{Id}_{D^2}\big) \\ & f & \mapsto & h \circ f \circ h^{-1} \end{array} \qquad \begin{array}{rccc} \overline{\Psi}_2 : & \big(C_\infty, \mathcal{T}_\infty\big) & \xrightarrow{\cong} & \big(C_\infty, \mathcal{T}^*_\infty\big) \\ & \big[(x_i)_{i\in\mathbb{N}}\big] & \mapsto & \big[\big(h(x_i)\big)_{i\in\mathbb{N}}\big], \end{array}$$

where, in the lower diagrams, the maps Ψ_2 and $\overline{\Psi}_2$ are suitably (co-) restricted.

Proposition 4.32. *For any choice of $\mathcal{T}^*_\infty$ such that, in D^2, the set $\{\tau^*_i\}_{i\in\mathbb{N}}$ accumulates at a single point $\tau^*_\infty \in \overset{\circ}{D}{}^2$, and such that*

$$\tau^*_i \neq \tau^*_\infty \quad \forall i \in \mathbb{N},$$

the following diagrams commute

$$\begin{array}{ccc} \pi_0 PH_\infty & \xrightarrow{\pi_0\varphi_\infty} & \pi_1 F_\infty \\ {\scriptstyle \pi_0\Psi_1}\downarrow\cong & & \cong\downarrow{\scriptstyle \pi_1\Psi_2} \\ \pi_0 PH^*_\infty & \xrightarrow{\pi_0\varphi^*_\infty} & \pi_1(F_\infty, \mathcal{T}^*_\infty) \end{array} \qquad \begin{array}{ccc} \pi_0 H_\infty & \xrightarrow{\pi_0\overline{\varphi}_\infty} & \pi_0 \mathcal{O}C_\infty \\ {\scriptstyle \pi_0\overline{\Psi}_1}\downarrow\cong & & \cong\downarrow{\scriptstyle \pi_1\overline{\Psi}_2} \\ \pi_0 H^*_\infty & \xrightarrow{\pi_0\overline{\varphi}^*_\infty} & \pi_0(\mathcal{O}C_\infty, \mathcal{T}^*_\infty) \end{array}$$

$$\begin{array}{ccc} \pi_0 PH_\infty & \xrightarrow{\pi_0\varphi'_\infty} & \pi_1 F'_\infty \\ {\scriptstyle \pi_0\Psi_1}\downarrow\cong & & \cong\downarrow{\scriptstyle \pi_1\Psi_2} \\ \pi_0 PH^*_\infty & \xrightarrow{\pi_0{\varphi'}^*_\infty} & \pi_1(F'_\infty, \mathcal{T}^*_\infty) \end{array} \qquad \begin{array}{ccc} \pi_0 H_\infty & \xrightarrow{\pi_0\overline{\varphi}'_\infty} & \pi_0 \mathcal{O}C'_\infty \\ {\scriptstyle \pi_0\overline{\Psi}_1}\downarrow\cong & & \cong\downarrow{\scriptstyle \pi_1\overline{\Psi}_2} \\ \pi_0 H^*_\infty & \xrightarrow{\pi_0\overline{\varphi}'^*_\infty} & \pi_0(\mathcal{O}C'_\infty, \mathcal{T}^*_\infty), \end{array}$$

where the maps $\Psi_1, \Psi_2, \overline{\Psi}_1, \overline{\Psi}_2$ *are defined as above.*

Proof. To prove that the first diagram commutes, we show that the following diagram commutes up to homotopy.

$$\begin{array}{ccc} PH_\infty & \xrightarrow{\varphi_\infty} & \Omega F_\infty \\ {\scriptstyle \Psi_1}\downarrow\cong & & \cong\downarrow{\scriptstyle \Omega\Psi_2} \\ PH^*_\infty & \xrightarrow{\varphi^*_\infty} & \Omega\big(F_\infty, \mathcal{T}^*_\infty\big) \end{array}$$

Pick some $f \in PH_\infty$, recall the contracting homotopy $K : H_0 \times I \to H_0$, write $\overline{K}(\cdot, t) := K(\cdot, 1-t)$ for all $t \in I$, and verify that

$$\begin{aligned} \Omega\Psi_2 \circ \varphi_\infty(f) &= \Big(\big(h \circ K(f, \cdot)\big)(\tau_i)\Big)_{i\in\mathbb{N}} \\ &\overset{*}{\simeq} \Big(\big(K(h \circ f, \cdot) \circ \overline{K}(h, \cdot)\big)(\tau_i)\Big)_{i\in\mathbb{N}} \\ &\overset{**}{=} \Big(\big(K(h \circ f, \cdot) \circ \overline{K}(h, \cdot) \circ h^{-1}\big)(\tau^*_i)\Big)_{i\in\mathbb{N}} \\ &\overset{*}{\simeq} \Big(\big(K(h \circ f \circ h^{-1}, \cdot)\big)(\tau^*_i)\Big)_{i\in\mathbb{N}} \\ &= \varphi^*_\infty \circ \Phi_1(h), \end{aligned}$$

where $(*)$ is given by Lemma A.3, and $(**)$ holds because $h(\tau_i) = \tau^*_i$ for all $i \in \mathbb{N}$. Similarly, one can prove that the remaining diagrams commute. □

This result generalizes the main results of this section to an arbitrary choice for $\mathcal{T}^*_\infty$, as we show next. Before, we note that the definition of the spaces $\big(\Omega(F_\infty, \mathcal{T}^*_\infty)\big)_c$ and $\big(\Omega(F_\infty, \mathcal{T}^*_\infty)\big)_{cc}$ makes sense for any basepoint $\mathcal{T}^*_\infty = (\tau^*_i)_{i\in\mathbb{N}}$, as long as the sequence $(\tau^*_i)_{i\in\mathbb{N}}$ converges in $\overset{\circ}{D}{}^2$.

Theorem 4.33. *The diagram on page 82 generalizes to any choice of* $\mathcal{T}^* = (\tau^*_i)_{i\in\mathbb{N}}$ *such that*

$$\lim_{i\to\infty} \tau^*_i = \tau^*_\infty$$

for some $\tau^*_\infty \in \overset{\circ}{D}{}^2$.

Proof. The result follows directly by suitably attaching the commutative diagrams given in Proposition 4.32 to the diagram on page 82. □

Chapter 5

An application to homoclinic tangles

In this chapter, we apply the injectivity of the map

$$\pi_0\overline{\varphi}_\infty : \pi_0 H_\infty \to \pi_0 \mathcal{O}C_\infty$$

to prove a result that can be used for the study of homeomorphisms with a homoclinic fixed point (see Theorem 5.13).
Moreover, we allow the basepoint $\mathcal{T}_\infty \in F_\infty$ be any configuration $\mathcal{T} = (\tau_i)_{i\in\mathbb{N}}$ satisfying

$$\lim_{i\to\infty} \tau_i = \tau_\infty$$

for some $\tau_\infty \in \overset{\circ}{D}{}^2$. Note that, under this condition, the map

$$\pi_0\overline{\varphi}_\infty : \pi_0 H_\infty \to \pi_0 \mathcal{O}C_\infty$$

is injective, according to Theorem 4.33. The proof of the main theorem requires some preliminary results involving the winding number, which we introduce next. For a detailed introduction to this subject, see [11].

Definition 5.1. *Given any loop $\alpha \in \mathcal{C}(S^1, \mathbb{R}^2 \setminus 0)$, define a loop*

$$\begin{array}{rcl} \overline{\alpha} : S^1 & \to & S^1 \\ s & \mapsto & \dfrac{\alpha(s)}{\|\alpha(s)\|} \end{array}$$

Let $\widetilde{\alpha} : I \to \mathbb{R}$ be a lifting of $\overline{\alpha}$ with respect to the standard covering map $q : \mathbb{R} \to S^1$, and define the ***winding number*** *of α by*

$$w(\alpha) := \widetilde{\alpha}(1) - \widetilde{\alpha}(0).$$

The following two propositions give alternative ways to define the winding number.

Proposition 5.2. *[11, Lemma 66.3] For any loop* $\alpha \in \mathcal{C}(S^1, \mathbb{R}^2 \setminus 0)$,

$$w(\alpha) = \frac{1}{2\pi i} \oint_\alpha \frac{dz}{z}.$$

In other words, given a lifting $\widetilde{\alpha} : I \to \mathbb{R}$ *of the map* $\frac{\alpha}{\|\alpha\|} : S^1 \to S^1$ *with respect to the standard covering map* $q : \mathbb{R} \to S^1$*, then,*

$$w(\alpha) = \frac{1}{2\pi i} \int_0^1 \frac{d\widetilde{\alpha}/dt}{\alpha(t)} dt.$$

Proposition 5.3. *Given any* $\alpha \in \mathcal{C}(S^1, \mathbb{R}^2 \setminus 0)$*, let* $p : \mathbb{R}^2 \setminus 0 \to S^1$ *be the canonical retraction, write* A *for the generator of* $\pi_1(S^1, *)$*, where we choose* $* := p\big(\alpha(1)\big)$ *for the basepoint of* S^1*. Then,*

$$[p \circ \alpha]_* = A^{\pm w(\alpha)} \quad in \quad \pi_1(S^1, *),$$

where the sign depends on the choice of the representative of A*.*

Proof. This follows easily from the definition of the winding number. □

Three elementary properties of the winding number are given in the following proposition.

Proposition 5.4. *For all* $\beta \in \mathcal{C}(S^1, \mathbb{R}^2 \setminus 0)$,

$$w(\overline{\beta}) = -w(\beta),$$

where $\overline{\beta}$ *is the inverse path of* β*.*
If two loops $\alpha, \beta \in \mathcal{C}(S^1, \mathbb{R}^2 \setminus 0)$ *are homotopic, then,*

$$w(\alpha) = w(\beta).$$

For all $\beta, \gamma \in \mathcal{C}(S^1, \mathbb{R}^2 \setminus 0)$ *that satisfy* $\beta(1) = \gamma(1)$,

$$w(\beta \star \gamma) = w(\beta) + w(\gamma).$$

Proof. The proof of the first two facts are given in [11, Lemma 66.1]. The third fact follows easily from the definition of the winding number. □

Lemma 5.5. *Let*

$$\Gamma : I \to \mathcal{C}(S^1, \mathbb{R}^2)$$

be a path that satisfies

$$\Gamma(0) = \Gamma(1),$$

and such that $\Gamma(t) : S^1 \to R^2$ *is injective for all* $t \in I$ *(i.e., such that, for all* s, s' *in* S^1 *with* $s \neq s'$, $\Gamma(\cdot)(s) - \Gamma(\cdot)(s')$ *is a well defined element of* $\mathcal{C}(S^1, \mathbb{R}^2 \setminus 0)$*). Then, there is an* $n \in \mathbb{Z}$ *such that*

$$w\big(\Gamma(\cdot)(s) - \Gamma(\cdot)(s')\big) = n$$

for all $s, s' \in S^1$ *with* $s \neq s'$*.*

Proof. Let $\Gamma : I \to \mathcal{C}(S^1, \mathbb{R}^2)$ be a path with the required properties, and pick some $s, s' \in S^1$ with $s \neq s'$. We show that the path

$$\big(\Gamma(\cdot)(s) - \Gamma(\cdot)(s')\big) : I \to \mathbb{R}^2$$

is homotopic to the path

$$\big(\Gamma(\cdot)(1) - \Gamma(\cdot)(1/2)\big) : I \to \mathbb{R}^2,$$

where S^1 is identified with $I/\dot{I}$.
We assume that

$$s > s',$$

where the converse case is proved similarly. Observing that

$$s + t(1-s) \neq s' + t(1/2 - s') \quad \forall t \ni I,$$

there is a well defined homotopy

$$\begin{array}{rcl} G : S^1 \times I & \to & \mathbb{R}^2 \setminus 0 \\ (z,t) & \mapsto & \Gamma(z)\big(s + t(1-s)\big) - \Gamma(z)\big(s' + t(1/2 - s')\big) \end{array}$$

that satisfies

$$G(\cdot, 0) = \big(\Gamma(\cdot)(s) - \Gamma(\cdot)(s')\big), \quad G(\cdot, 1) = \big(\Gamma(\cdot)(1) - \Gamma(\cdot)(1/2)\big),$$

as required. □

In the sequel, we consider loops $(\beta_i - \beta_j) : S^1 \to D^2 \setminus 0$ for some integers $i \neq j$, where $(\beta_i)_{i\in\mathbb{N}}$ is in ΩF_∞. As, for all $i \neq j$, $\beta_i(t) \neq \beta_j(t)$ for all $t \in I$, $(\beta_i - \beta_j)$ is indeed an element of $\mathcal{C}(S^1, \mathbb{R}^2 \setminus 0)$.

Lemma 5.6. *For every* $(\beta_i)_{i\in\mathbb{N}} \in \Omega F_\infty$,

$$w(\beta_i - \beta_j) = 0 \quad \forall i, j \in \mathbb{N}, \ \ i \neq j$$

if and only if

$$\big[(\beta_i)_{i\in\mathbb{N}}\big] = \big[(p_{\tau_i})_{i\in\mathbb{N}}\big] \quad \textit{in } PB_\infty.$$

Proof. The "if"-part follows directly from Proposition 5.4 To prove the "only if"-part, pick any $(\beta_i)_{i\in\mathbb{N}} \in \Omega F_\infty$ with

$$w(\beta_i - \beta_j) = 0 \quad \forall i, j \in \mathbb{N}, \ \ i \neq j,$$

and, by contradiction, assume that

$$\big[\big(\beta_i\big)_{i\in\mathbb{N}}\big] \neq \big[\big(p_{\tau_i}\big)_{i\in\mathbb{N}}\big] \quad \text{in } PB_\infty.$$

Recalling that $PB_\infty = \lim_n PB_n$, it follows from the basic properties of inverse limits that, in PB_2,

$$\big[\big(\beta_i, \beta_j\big)\big] \neq \big[\big(p_{\tau_i}, p_{\tau_j}\big)\big]$$

for some $i, j \in \mathbb{N}$. Recalling that PB_2 has one single generator B (corresponding to σ_1^2 in Artin's presentation of B_2), there is thus an $n \in \mathbb{Z} \setminus 0$ such that, in PB_2,

$$\big[\big(\beta_i, \beta_j\big)\big] = B^n$$

From this, it is easy to see that,

$$[p \circ (\beta_i - \beta_j)]_* = A^{\pm n} \quad \text{in } \pi_1(S^1, *),$$

where A is the generator of $\pi_1(S^1, *)$, and the basepoint of S^1 is $* := p(\beta_i(1) - \beta_j(1))$. Thus, it follows by Prop 5.3 that

$$w\big(\beta_i - \beta_j\big) = \pm n,$$

which contradicts our assumption, because $n \neq 0$. □

In the sequel, we identify $\mathbb{R}^2$ canonically with the complex plane $\mathbb{C}$.

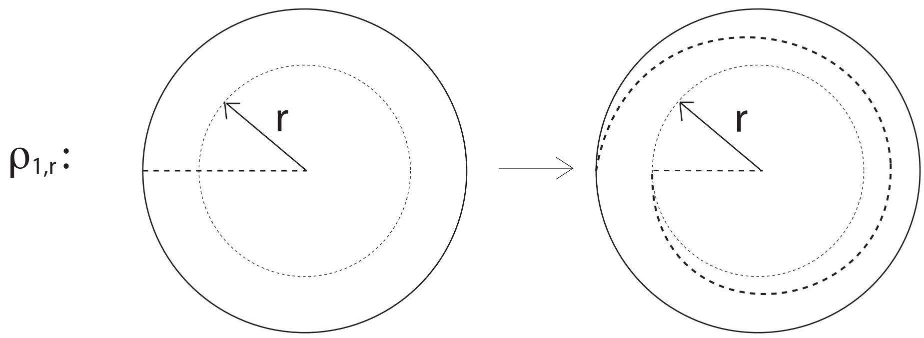
Definition 5.7. *For each $n \in \mathbb{Z}$, and for each $r \in \big]\sup_{i\in\mathbb{N}} \|\tau_i\|, 1\big[$, define an element $\rho_{n,r}$ in PH_∞ by*

$$\rho_{n,r}(x) := \begin{cases} x, & \|x\| \leq r \\ x\,exp\left(-2\pi i n \frac{\|x\|-r}{1-r}\right), & \|x\| \geq r. \end{cases}$$

Observe that, for all possible choices of n and r, $\rho_{n,r}|_{\partial D^2} = \mathrm{Id}$, and that, as $\|\tau_i\| < r$ for all $i \in \mathbb{N}$,

$$\rho_{n,r}(\tau_i) = \tau_i \quad \forall i \in \mathbb{N}.$$

Thus, indeed, $\rho_{n,r} \in PH_\infty$. The following drawing illustrates how $\rho_1(r)$ maps the given dotted line.



Writing

$$\overline{H}_\infty := \big\{h \in \mathcal{H}_\infty \mid \big\{h(\tau_i)\}_{i\in\mathbb{N}} = \{\tau_i\}_{i\in\mathbb{N}}\big\}$$

for the space of homeomorphisms that fix the set $\{\tau_i\}_{i\in\mathbb{N}}$, but that don't necessarily fix the boundary ∂D^2, we make the following observation.

Proposition 5.8. *For all $n \in \mathbb{Z}$ and $r \in \left]\sup_{i\in\mathbb{N}} \|\tau_i\|, 1\right[$,*

$$[\rho_{n,r}] = [Id_{D^2}] \quad \text{in } \pi_0\overline{H}_\infty.$$

Proof. For all $n \in \mathbb{Z}$ and $r \in \left]\sup_{i\in\mathbb{N}} \|\tau_i\|, 1\right[$, the right adjoint of the homotopy

$$\begin{aligned} R_{n,r} : D^2 \times I &\rightarrow D^2 \\ (x,t) &\mapsto \begin{cases} x, & \|x\| \leq r \\ x\exp\left(-2\pi i n t \frac{1-\|x\|}{1-r}\right), & \|x\| \geq r. \end{cases} \end{aligned}$$

is a path in $\overline{H}_\infty$ from $\rho_{n,r}$ to Id_{D^2}. □

Proposition 5.9. *For all $n \in \mathbb{Z}$ and $r \in \left]\sup_{i\in\mathbb{N}} \|\tau_i\|, 1\right[$,*

$$w\big(K(\rho_{n,r},\cdot)(\tau_i) - K(\rho_{n,r},\cdot)(\tau_j)\big) = -n \qquad \forall i,j \in \mathbb{N},\ i \neq j.$$

Proof. Fix some $n \in \mathbb{Z}$ and $r \in \left]\sup_{i\in\mathbb{N}} \|\tau_i\|, 1\right[$, and define a path $\Lambda : I \rightarrow H_0$ by

$$\Lambda(t)(x) := \begin{cases} x\ \exp\big(-2\pi i n(1-t)\big), & \|x\| \leq r \\ x\ \exp\left(-2\pi i n(1-t)\frac{\|x\|-r}{1-r}\right), & \|x\| \geq r \end{cases}$$

for all $t \in I,\ x \in D^2$. Note that, for all $n \in \mathbb{Z}$,

$$\Lambda(0) = \rho_{n,r}, \qquad \Lambda(1) = \mathrm{Id}_{D^2}.$$

Observing that $K\big(\rho_{n,r},\cdot\big) : I \rightarrow H_0$ is a path with the same start- and endpoint as Λ, it follows from Lemma A.3, that there is a homotopy

$$\Gamma : S^1 \times I \rightarrow F_\infty$$

from $\mathrm{ev}_\infty \circ K(\rho_{n,r},\cdot)$ to $\mathrm{ev}_\infty \circ \Lambda$. Its adjoint is a path

$$\gamma := (\gamma_i)_{i\in\mathbb{N}} : I \rightarrow \Omega F_\infty$$

with

$$\gamma(0) = \mathrm{ev}_\infty \circ K(\rho_{n,r},\cdot), \quad \gamma(1) = \mathrm{ev}_\infty \circ \Lambda.$$

It follows that, for any $i, j \in \mathbb{N}$ with $i \neq j$, and $r \in \left]\sup_{i\in\mathbb{N}} \|\tau_i\|, 1\right[$,

$$\begin{aligned}
w\big(K(\rho_{n,r},\cdot)(\tau_i) - K(\rho_{n,r},\cdot)(\tau_j)\big) &= w\big(\gamma_i(0) - \gamma_j(0)\big) \\
&\overset{*}{=} w\big(\gamma_i(1) - \gamma_j(1)\big) \\
&= w\big(\Lambda(\cdot)(\tau_i) - \Lambda(\cdot)(\tau_j)\big) \\
&= \frac{1}{2\pi i}\int_0^1 \frac{d\big(\Lambda(\cdot)(\tau_i) - \Lambda(\cdot)(\tau_j)\big)/dt}{\Lambda(\cdot)(\tau_i) - \Lambda(\cdot)(\tau_j)} dt \\
&= \frac{1}{2\pi i}\int_0^1 \frac{d\big((\tau_i - \tau_j)\exp(-2\pi int)\big)/dt}{(\tau_i - \tau_j)\exp(-2\pi int)} dt \\
&= \frac{1}{2\pi i}\int_0^1 \frac{(-2\pi in)(\tau_i - \tau_j)\exp(-2\pi int)}{(\tau_i - \tau_j)\exp(-2\pi int)} dt \\
&= \frac{1}{2\pi i}\int_0^1 -2\pi in dt \\
&= -n,
\end{aligned}$$

where $(*)$ is given by the fact that

$$w\big(\gamma_i(\cdot)\big) : I \to \mathbb{Z}$$

is a continuous map when I has the metric- and $\mathbb{Z}$ has the discrete topology, and thus is constant. □

Definition 5.10. *Let $(s_i)_{i\in\mathbb{N}}$ be any list of points of S^1 such that*

$$\lim_{i\to\infty} s_i = 1.$$

Definition 5.11. *Define a set*

$$\begin{aligned}
\mathcal{A} \;:=\; &\Big\{\alpha : S^1 \to \overset{\circ}{D}{}^2 \mid \alpha \textit{ is a homeomorphism onto its image, and} \\
&\{\alpha(s_i)\}_{i\in\mathbb{N}} = \{\tau_i\}_{i\in\mathbb{N}}\Big\},
\end{aligned}$$

and endow it with the subspace topology $\mathcal{A} \subset \mathcal{C}(S^1, \overset{\circ}{D}{}^2)$.

The main theorem of this chapter, Theorem 5.13, can be seen as a first application of our preceeding results on infinite mapping class groups and infinite braids to the study of a particular subspace of $\overline{H}_\infty$ that is of interest in fields other than low-dimensional topology. To give a typical example of a case where the theorem can be applied, let $f \in \overline{H}_\infty$ be a *diffeomorphism*, of which τ_∞ is a hyperbolic fixpoint, and let $\widetilde{W}^s_f$ and $\widetilde{W}^u_f$ be the corresponding stable- and unstable manifolds, which are defined by

$$\widetilde{W}^s_f := \big\{x \in \overset{\circ}{D}{}^2 \mid \lim_{i\to\infty} \|f^i(x) - \tau_\infty\| = 0\big\}, \quad \widetilde{W}^u_f := \big\{x \in \overset{\circ}{D}{}^2 \mid \lim_{i\to\infty} \|f^{-i}(x) - \tau_\infty\| = 0\big\},$$

respectively. According to [9, Thm. 10.1.6], there are C^1-embeddings $W^s_f : I \to \overset{\circ}{D}{}^2$ and $W^u_f : I \to \overset{\circ}{D}{}^2$, such that

$$\widetilde{W}^s_f = \bigcup_{i\in\mathbb{N}} f^{-i} \circ W^s_f, \quad \widetilde{W}^u_f = \bigcup_{i\in\mathbb{N}} f^{i} \circ W^u_f.$$

Write $\widetilde{W}^s_f[x,y]$ for the section on $\widetilde{W}^s_f$ between any points x and y on $\widetilde{W}^s_f$ (as $\widetilde{W}^s_f$ does not intersect itself, this notion makes sense), and similarly for $\widetilde{W}^u_f$. As f is differentiable, the following is easy to prove.

Proposition 5.12.

For any $x, y \in \widetilde{W}^s_f$, $\widetilde{W}^s_f[x,y]$ is a C^1 embedding of I in $\overset{\circ}{D}{}^2$.

For any $x, y \in \widetilde{W}^u_f$, $\widetilde{W}^u_f[x,y]$ is a C^1 embedding of I in $\overset{\circ}{D}{}^2$.

Moreover, for any points x, y in $\widetilde{W}^s_f \cap \widetilde{W}^u_f$, such that

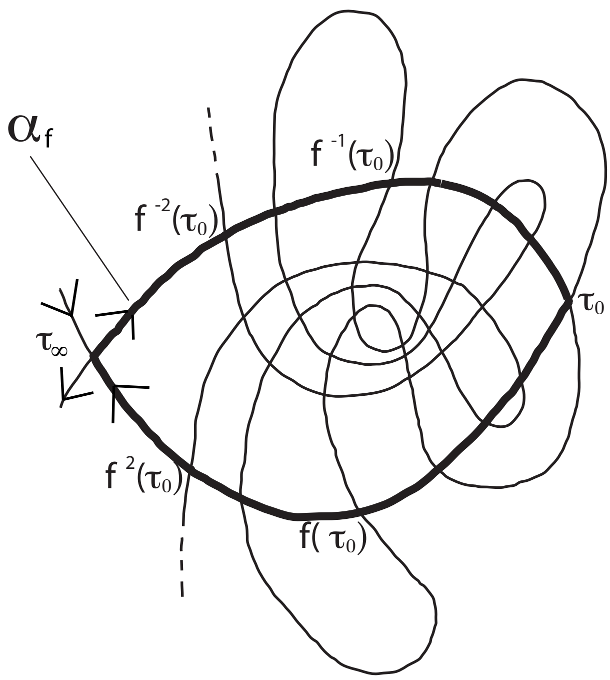
$$\widetilde{W}^s_f[x,y] \cap \widetilde{W}^u_f[x,y] = \{x, y\},$$

the union $\widetilde{W}^s_f[x,y] \cup \widetilde{W}^u_f[x,y]$ is an embedding (not differentiable in general) of S^1 in $\overset{\circ}{D}{}^2$.
In particular, assuming that τ^0 is a **primary intersection point** of $\widetilde{W}^s_f$ and $\widetilde{W}^u_f$, i.e.,

$$\widetilde{W}^s_f[\tau_\infty, \tau^0] \cap \widetilde{W}^u_f[\tau_\infty, \tau^0] = \{\tau_\infty, \tau^0\}$$

(see [15]), it follows that there is an embedding $\alpha : S^1 \to \overset{\circ}{D}{}^2$, such that

$$\text{Im } \alpha = \widetilde{W}^s_f[\tau_\infty, \tau^0] \cup \widetilde{W}^u_f[\tau_\infty, \tau^0].$$



Now, let $g \in \overline{H}_\infty$ be another *diffeomorphism* with homoclinic fixpoint τ_∞ and primary intersection point τ^0, and let $\beta : S^1 \to \overset{\circ}{D}{}^2$ be its associated embedding, defined similarly to α.
Now, assume that the orbit of τ^0 with respect to f and g coincide with the set $\{\tau_i\}_{i\in\mathbb{N}}$, i.e.,

$$\left\{f^i(\tau^0)\right\}_{i\in\mathbb{Z}} = \left\{g^i(\tau^0)\right\}_{i\in\mathbb{Z}} = \{\tau_i\}_{i\in\mathbb{N}},$$

and, furthermore, that

$$\left\{\alpha(s_i)\right\}_{i\in\mathbb{N}} = \left\{\beta(s_i)\right\}_{i\in\mathbb{N}}$$

for some sequence $(s_i)_{i\in\mathbb{N}}$ in S^1, such that, moreover,

$$\lim_{i\to\infty} s_i = 1,$$

i.e., $(s_i)_{i\in\mathbb{N}}$ satisfies the condition of Definition 5.10. Then, in particular,

$$\alpha, \beta \in \mathcal{A}.$$

Under these assumptions, Theorem 5.13 can be applied.

Theorem 5.13. *Let $f, g \in \overline{H}_\infty$. If there are elements $\alpha, \beta \in \mathcal{A}$ such that*

$$[\alpha] = [\beta], \quad [f \circ \alpha] = [g \circ \beta] \qquad \textit{in } \pi_0\mathcal{A}.$$

Then,

$$[f] = [g] \quad \in \pi_0\overline{H}_\infty.$$

Proof. Pick any $f, g \in \overline{H}_\infty$, and let $\alpha, \beta : S^1 \to \overset{\circ}{D}{}^2$ be as required. Moreover, by Proposition A.2, we can assume that

$$f, g \in H_\infty.$$

According to Proposition 5.8, it suffices to show that there is an $n \in \mathbb{Z}$ and an $r \in I$, such that

$$[g] = [f \circ \rho_{n,r}] \qquad \text{in } \pi_0 \overline{H}_\infty.$$

Claim: There is a homeomorphism $\kappa_1 \in PH_\infty$ that satisfies

$$\kappa_1 \circ \beta = \alpha, \quad [\kappa_1] = [\mathrm{Id}] \quad \text{in } \pi_0 PH_\infty. \qquad (A)$$

Proof of the claim: Observe that, as α and β are homeomorphisms onto their image, there is a homeomorphism

$$\begin{array}{ccc} \mathrm{Im}\alpha & \overset{\cong}{\to} & \mathrm{Im}\beta \\ \alpha(s) & \mapsto & \beta(s) \qquad \forall s \in S^1. \end{array}$$

Consequently, by the Schoenflies Theorem (e.g. [7, Cor. 9.25]), there is a homeomorphism $\widehat{\widehat{k}} \in \mathcal{H}(D^2)$ that satisfies

$$\widehat{\widehat{k}} \circ \alpha = \beta.$$

As we now show, we can choose this homeomorphism to be in H_0, i.e., such that it fixes ∂D^2 pointwise. Choose any $r_0 \in]0, 1[$ such that

$$\mathrm{Im}\alpha \cup \mathrm{Im}\beta \subset B(0, r_0).$$

As the Schoenflies Theorem also holds by replacing D^2 with $B(0, r_0)$, there is a homeomorphism $\widehat{k} \in \mathcal{H}\big(B(0, r_0)\big)$ such that

$$\widehat{k} \circ \alpha = \beta.$$

Now, define a homeomorphism as follows, where we use polar coordinates.

$$\begin{array}{rcl} k : D^2 & \overset{\cong}{\longrightarrow} & D^2 \\ x & \mapsto & \begin{cases} \Big(\|x\|, \arg\, x + \big(\arg \widehat{k}(r_0, \arg x) - \arg x\big)\frac{1-\|x\|}{1-r_0}\Big), & \|x\| \geq r_0, \\ \widehat{k}(x), & \|x\| \leq r_0. \end{cases} \end{array}$$

This map is well defined, because, for all $x \in D^2$ with $\|x\| = r_0$,

$$\begin{array}{rcl} k(x) & = & \big(\|x\|, \arg\, \widehat{k}(r_0, \arg x)\big) \\ & = & \big(\|x\|, \arg\, \widehat{k}(x)\big) \\ & \overset{*}{=} & \big(\|\widehat{k}(x)\|, \arg\, \widehat{k}(x)\big) \\ & = & \widehat{k}(x), \end{array}$$

where $(*)$ holds, because $\widehat{k}$ maps $\partial B(0, r_0)$ onto itself. Moreover, observe that, as $\operatorname{Im} \alpha, \operatorname{Im} \beta \subset B(0, r_0)$,

$$k \circ \alpha = \widehat{k} \circ \alpha = \beta. \qquad (B)$$

Also, note that $k|_{\partial D^2} = \mathrm{Id}$, i.e., $k \in H_0$, and that

$$k(\tau_i) = k\big(\alpha(s_i)\big) = \beta(s_i) = \tau_i \quad \forall i \in \mathbb{N},$$

which means that

$$k \in PH_\infty.$$

Now, recall that

$$[\alpha] = [\beta] \quad \text{in } \pi_0 \mathcal{A},$$

i.e., there is a path $\Lambda_1 : I \to \mathcal{A}$ such that

$$\Lambda_1(0) = \alpha, \quad \Lambda_1(1) = \beta. \qquad (C)$$

Thus, it follows by the definition of $\mathcal{A}$, that

$$\big(\Lambda_1(\cdot)(s_i)\big)_{i \in \mathbb{N}} = \big(p_{\tau_i}\big)_{i \in \mathbb{N}} \in \Omega F_\infty. \qquad (D)$$

Furthermore, define a path

$$\begin{array}{rcl} \Lambda_2 : I & \to & \mathcal{C}(S^1, \overset{\circ}{D}{}^2) \\ t & \mapsto & K(k, t) \circ \alpha(\cdot), \end{array}$$

and observe that it satisfies

$$\Lambda_2(0) = K(k, 0) \circ \alpha = k \circ \alpha = \beta,$$

and

$$\Lambda_2(1) = K(k, 1) \circ \alpha = \mathrm{Id} \circ \alpha = \alpha.$$

Recalling (C), this allows us to define a path $\Lambda : I \to \mathcal{C}(S^1, \overset{\circ}{D}{}^2)$ by

$$\Lambda = \Lambda_1 \star \Lambda_2,$$

which, in particular, satisfies

$$\Lambda(0) = \Lambda(1) = \alpha.$$

As $\Lambda(t) : S^1 \to \overset{\circ}{D}{}^2$ is a homeomorphism onto its image for all $t \in I$, Lemma 5.5 applies, which means that there is an $m \in \mathbb{Z}$ such that

$$w\big(\Lambda(\cdot)(s_i) - \Lambda(\cdot)(s_j)\big) = m \quad \forall i, j \in \mathbb{N}, \ \ i \neq j. \qquad (E)$$

Now, observe that, in PB_∞,

$$\begin{aligned}
[\varphi_\infty(\rho_{m,r_0}\circ k)] &\overset{*}{=} [\varphi_\infty(\rho_{m,r_0})]\,[\varphi_\infty(k)]\\
&= \Big[\big(K(\rho_{m,r_0},\cdot)(\tau_i)\big)_{i\in\mathbb{N}}\Big]\Big[\big(K(k,\cdot)(\tau_i)\big)_{i\in\mathbb{N}}\Big]\\
&= \Big[\big(K(\rho_{m,r_0},\cdot)(\tau_i)\big)_{i\in\mathbb{N}}\Big]\Big[\big(\Lambda_2(\cdot)(s_i)\big)_{i\in\mathbb{N}}\Big]\\
&\overset{D}{=} \Big[\big(K(\rho_{m,r_0},\cdot)(\tau_i)\big)_{i\in\mathbb{N}}\Big]\Big[\big(\Lambda_1(\cdot)(s_i)\big)_{i\in\mathbb{N}}\Big]\Big[\big(\Lambda_2(\cdot)(s_i)\big)_{i\in\mathbb{N}}\Big]\\
&= \Big[\big(K(\rho_{m,r_0},\cdot)(\tau_i)\big)_{i\in\mathbb{N}}\Big]\Big[\big(\Lambda(\cdot)(s_i)\big)_{i\in\mathbb{N}}\Big], \qquad (F)
\end{aligned}$$

where $(*)$ is given by Proposition 1.15. Also, for any integers $i\neq j$,

$$\begin{aligned}
&w\Big(K(\rho_{m,r_0},\cdot)(\tau_i)\star\Lambda(\cdot)(s_i) - K(\rho_{m,r_0},\cdot)(\tau_j)\star\Lambda(\cdot)(s_j)\Big)\\
&\qquad = w\Big(\big(K(\rho_{m,r_0},\cdot)(\tau_i) - K(\rho_{m,r_0},\cdot)(\tau_i)\big)\star\big(\Lambda(\cdot)(s_i)-\Lambda(\cdot)(s_j)\big)\Big)\\
&\qquad \overset{*}{=} w\big(K(\rho_{m,r_0},\cdot)(\tau_i) - K(\rho_{m,r_0},\cdot)(\tau_i)\big) + w\big(\Lambda(\cdot)(s_i)-\Lambda(\cdot)(s_i)\big)\\
&\qquad \overset{**}{=} m - m = 0,
\end{aligned}$$

where $(*)$ is given by Proposition 5.4, and $(**)$ follows from Proposition 5.9 and (E). Using Lemma 5.6, this allows us to conclude that

$$\begin{aligned}
\big[\varphi_\infty(\rho_{m,r_0}\circ k)\big] &\overset{F}{=} \Big[\big(K(\rho_{m,r_0},\cdot)(\tau_i)\big)_{i\in\mathbb{N}}\Big]\Big[\big(\Lambda(\cdot)(s_i)\big)_{i\in\mathbb{N}}\Big]\\
&= \Big[\Big(K(\rho_{m,r_0},\cdot)\star\Lambda(\cdot)(s_i)(\tau_i)\Big)_{i\in\mathbb{N}}\Big]\\
&= \big[(p_{\tau_i})_{i\in\mathbb{N}}\big] \quad \text{in } PB_\infty.
\end{aligned}$$

Thus, recalling that $\pi_0\varphi_\infty$ is injective, it follows that

$$[\rho_{m,r_0}\circ k] = [\mathrm{Id}] \quad \text{in } \pi_0 PH_\infty.$$

Writing

$$\kappa_1 := \rho_{m,r_0}\circ k,$$

it thus follows that

$$\kappa_1\circ\beta = \alpha, \quad [\kappa_1] = [\mathrm{Id}] \quad \text{in } \pi_0 PH_\infty,$$

which proves the claim.

Furthermore, by the fact that

$$[f\circ\alpha] = [g\circ\beta] \qquad \text{in } \pi_0\mathcal{A},$$

we can show in exactly the same way as above, that there is a homeomorphism $\kappa_2\in PH_\infty$ that satisfies

$$\kappa_2\circ\big(f\circ\alpha\big) = g\circ\beta, \quad [\kappa_2] = [\mathrm{Id}] \quad \text{in } \pi_0 PH_\infty. \qquad (G)$$

Now, write $h := g^{-1} \circ \kappa_2 \circ f \circ \kappa_1$, observe that $h \in PH_\infty$, because $\kappa_1, \kappa_2 \in PH_\infty$, and $g^{-1} \circ f \in PH_\infty$, because, as $[\alpha] = [\beta]$ in $\pi_0 \mathcal{A}$, there is, for each $i \in \mathbb{N}$, a $j \in \mathbb{N}$ with

$$\tau_i = \alpha(s_j) = \beta(s_j),$$

such that, moreover,

$$f(\tau_i) = f \circ \alpha(s_j) \stackrel{*}{=} g \circ \beta(s_j) = g(\tau_i),$$

where $(*)$ is given by the fact that $[f \circ \alpha] = [g \circ \beta]$ in $\pi_0 \mathcal{A}$. Furthermore, notice that

$$\begin{aligned} h \circ \beta &= g^{-1} \circ \kappa_2 \circ f \circ \kappa_1 \circ \beta \\ &\stackrel{A}{=} g^{-1} \circ \kappa_2 \circ f \circ \alpha \\ &\stackrel{G}{=} g^{-1} \circ g \circ \beta \\ &= \beta. \end{aligned}$$

Also, the right adjoint $\widehat{K} : I \to \mathcal{C}(S^1, D^2)$ of the homotopy

$$\begin{aligned} K(h, \cdot) \circ \beta : S^1 \times I &\to \overset{\circ}{D}{}^2 \\ (s,t) &\mapsto K(h,t)\big(\beta(s)\big) \end{aligned}$$

satisfies the condition of Lemma 5.5, which means that there is an integer n such that, for all $i, j \in \mathbb{N}$ with $i \neq j$,

$$w\big(K(h,\cdot) \circ \beta(s_i) - K(h,\cdot) \circ \beta(s_j)\big) \equiv w\big(\widehat{K}(h)(\cdot) \circ \beta(s_i) - \widehat{K}(h)(\cdot) \circ \beta(s_j)\big) = n.$$

Thus,

$$\begin{aligned} & w\Big(K(h,\cdot)(\tau_i) \star K(\rho_{n,r_0},\cdot)(\tau_i) - K(h,\cdot)(\tau_j) \star K(\rho_{n,r_0},\cdot)(\tau_j)\Big) \\ &\quad = w\Big(\big(K(h,\cdot)(\tau_i) - K(h,\cdot)(\tau_j)\big) \star \big(K(\rho_{n,r_0},\cdot)(\tau_i) - K(\rho_{n,r_0},\cdot)(\tau_j)\big)\Big) \\ &\quad \stackrel{*}{=} w\big(K(h,\cdot)(\tau_i) - K(h,\cdot)(\tau_j)\big) + w\big(K(\rho_{n,r_0},\cdot)(\tau_i) - K(\rho_{n,r_0},\cdot)(\tau_j)\big) \\ &\quad = n - n = 0, \end{aligned}$$

where $(*)$ is given by Proposition 5.4. Thus, according to Lemma 5.6,

$$\begin{aligned} \pi_0 \varphi_\infty [h \circ \rho_{n,r_0}] &\stackrel{*}{=} [\varphi_\infty(h)] \big[\varphi_\infty\big(\rho_{n,r_0}\big)\big] \\ &= \big[\varphi_\infty(h) \star \varphi_\infty\big(\rho_{n,r_0}\big)\big] \\ &= \Big[\Big(K(h,\cdot)(\tau_i) \star K(\rho_{n,r_0},\cdot)(\tau_i)\Big)_{i \in \mathbb{N}}\Big] \\ &= \Big[\big(p_{\tau_i}\big)_{i \in \mathbb{N}}\Big], \end{aligned}$$

where $(*)$ is given by Proposition 1.15. Moreover, as $\pi_0 \varphi_\infty$ is injective by Theorem 3.7, it follows that

$$\big[h \circ \rho_{n,r_0}\big] = \big[\mathrm{Id}\big] \qquad \text{in } \pi_0 PH_\infty.$$

This finishes the proof, because then,

$$\big[\mathrm{Id}\big] = \big[h \circ \rho_{n,r_0}\big] = \big[g^{-1} \circ \kappa_2 \circ f \circ \kappa_1 \circ \rho_{n,r_0}\big] \overset{A,G}{=} \big[g^{-1} \circ f \circ \rho_{n,r_0}\big] \quad \text{in } \pi_0 PH_\infty.$$

□

Appendix A

Various technical results

Proposition A.1. *For all $n \in \mathbb{N} \cup \infty$, the spaces F_n and C_n are pathwise connected.*

Proof. We prove that F_n is pathwise connected for all $n \in \mathbb{N} \cup \infty$. If, for some $n \in \mathbb{N} \cup \infty$, $\overline{x}$ and $\overline{y}$ are points in C_n, then any path in F_n between representatives x, y of $\overline{x}$ and $\overline{y}$, respectively, projects to a path in C_n between $\overline{x}$ and $\overline{y}$.
Pick two points $x := (x_i)_{i\in\mathbb{N}}, y := (y_i)_{i\in\mathbb{N}} \in F_\infty$. We prove by induction that there is a path in F_∞ from x to y. Clearly, there is a path $\gamma_1 : I \to F_1 = \overset{\circ}{D}{}^2$ from x_1 to y_1. Assume that, for some $n > 1$, there is a well defined path

$$\Gamma_n := (\gamma_i)_{i\in[1,n]} : I \to F_n$$

from $(x_i)_{i\in[1,n]}$ to $(y_i)_{i\in[1,n]}$. As both $\{x_i\}_{i\in[1,n+1]}$ and $(y_i)_{i\in[1,n+1]}$ are sets of pairwise distinct points, there is, by the separability of $\overset{\circ}{D}{}^2$, a real number $\varepsilon > 0$ such that

$$x_{n+1} \in \overset{\circ}{D}{}^2 \setminus \bigcup_{i\in[1,n]} B(x_i, \varepsilon), \quad y_{n+1} \in \overset{\circ}{D}{}^2 \setminus \bigcup_{i\in[1,n]} B(y_i, \varepsilon), \qquad (A)$$

and

$$\bigcap_{i\in[1,n]} \overline{B(x_i, \varepsilon)} = \varnothing, \quad \bigcap_{i\in[1,n]} \overline{B(y_i, \varepsilon)} = \varnothing. \qquad (B)$$

Moreover, by the continuity of the paths $\{\gamma_i\}_{i\in[1,n]}$, there is a $0 < \widehat{t} < 1/2$ such that, for all $i \in [1, n]$,

$$\gamma_i(t) \in B(x_i, \varepsilon) \quad \forall t \in [0, \widehat{t}], \qquad \gamma_i(t) \in B(y_i, \varepsilon) \quad \forall t \in [1 - \widehat{t}, 1]. \qquad (C)$$

As, by (B), $\overset{\circ}{D}{}^2 \setminus \bigcup_{i\in[1,n]} \overline{B(x_i, \varepsilon)}$ is homeomorphic to $\overset{\circ}{D}{}^2 \setminus \bigcup_{i\in[1,n]} x_i$, which is a pathwise connected space, it follows that $\overset{\circ}{D}{}^2 \setminus \bigcup_{i\in[1,n]} \overline{B(x_i, \varepsilon)}$ too is pathwise connected, which, similarly, also holds for $\overset{\circ}{D}{}^2 \setminus \bigcup_{i\in[1,n]} \overline{B(y_i,\varepsilon)}$. Consequently, by (A), there are paths

$$\gamma^0_{n+1} : [0,\widehat{t}] \to \overset{\circ}{D}{}^2, \quad \gamma^1_{n+1} : [1-\widehat{t},1] \to \overset{\circ}{D}{}^2$$

that satisfy

$$\gamma^0_{n+1}(0) = x_{n+1}, \quad \gamma^1_{n+1}(1) = y_{n+1}$$

and

$$\gamma^0_{n+1}(\widehat{t}) = \gamma^1_{n+1}(1-\widehat{t}) =: \widehat{x},$$

where $\widehat{x}$ is some point in $\overset{\circ}{D}{}^2 \setminus \bigcup_{i\in[1,n]} \overline{B(x_i,\varepsilon)} \cup \overline{B(y_i,\varepsilon)}$ that, moreover, satisfies

$$\widehat{x} \notin \bigcup_{i\in[1,n]} \bigcup_{t\in I} \gamma_i(t). \qquad (D)$$

Observe that there is a well defined path

$$\begin{aligned} \gamma_{n+1} : I \quad &\to \quad \overset{\circ}{D}{}^2 \\ t \quad &\mapsto \quad \begin{cases} \gamma^0_{n+1}(t), & t \in [0,\widehat{t}] \\ \widehat{x}, & t \in [\widehat{t}, 1-\widehat{t}] \\ \gamma^1_{n+1}(t), & t \in [1-\widehat{t},1], \end{cases} \end{aligned}$$

from x_{n+1} to y_{n+1}, such that, by (C) and (D),

$$\gamma_{n+1}(t) \neq \gamma_i(t) \quad \forall i \in [1,n], \forall t \in I,$$

i.e., there is a well defined path

$$\Gamma_{n+1} := (\gamma_i)_{i\in[1,n+1]} : I \to F_{n+1}$$

from $(x_i)_{i\in[1,n+1]}$ to $(y_i)_{i\in[1,n+1]}$.

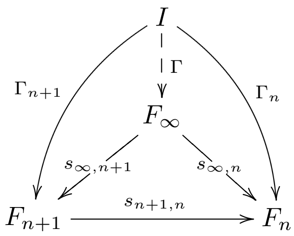
By induction, we thus can conclude that there is a sequence of paths

$$\Big(\Gamma_n : (I,0,1) \to \big(F_n, (x_i)_{i\in[1,n]}, (y_i)_{i\in[1,n]}\big)\Big)_{n\in\mathbb{N}}$$

that constitutes a map from I to the inverse system $\{F_n, s_{n+1,n}\}_{n\in\mathbb{N}}$, i.e., for each $n \in \mathbb{N}$, there is a commutative diagram

$$\begin{array}{ccccc} & & I & & \\ & \swarrow^{\Gamma_{n+1}} & & \searrow^{\Gamma_n} & \\ F_{n+1} & & \xrightarrow{s_{n+1,n}} & & F_n \end{array} .$$

Consequently, by the universal property of the inverse limit F_∞, there is a map $\Gamma : I \to F_\infty$ that makes the diagram



commute for all $n \in \mathbb{N}$. Moreover, $\Gamma(0) = x$, and $\Gamma(1) = y$, because $\Gamma_n(0) = (x_i)_{i\in[1,n]}$ and $\Gamma_n(1) = (y_i)_{i\in[1,n]}$ for all $n \in \mathbb{N}$. $\square$

Let the basepoint $\mathcal{T}_\infty \in F_\infty$ be as chosen in Definition 2.1, and recall that $\overline{H}_\infty$ is the subspace of $\mathcal{H}(D^2, D^2)$ of homeomorphisms that fix the point set $\{\tau_i\}_{i\in\mathbb{N}}$ as a set. Also, recall that

$$\varrho_i := \|\tau_i - \tau_\infty\| \quad \forall i \in \mathbb{N}.$$

Proposition A.2. *For each $f \in \overline{H}_\infty$, there is an element $\widehat{f} \in H_\infty$, such that*

$$[f] = [\widehat{f}] \quad \textit{in} \quad \pi_0\overline{H}_\infty.$$

Proof. Pick some $f \in \overline{H}_\infty$, and, in polar coordinates, extend as follows to a map

$$\begin{aligned} f_{\text{ext}} : \overline{B(\tau_\infty, 2)} &\to \overline{B(\tau_\infty, 2)} \\ (r, \varphi) &\mapsto \begin{cases} f(r, \varphi), & 0 \le r \le 1 \\ \Big(r, \varphi + \Big(\arg\ \big(f(1, \varphi)\big) - \varphi\Big)(2 - r)\Big), & 1 \le r \le 2, \end{cases} \end{aligned}$$

where $\arg\ (\widehat{r}, \widehat{\varphi}) := \widehat{\varphi}$ for all $(\widehat{r}, \widehat{\varphi}) \in \overline{B(\tau_\infty, 2)}$. Recalling that f induces a homeomorphism

$$f|_{\partial D^2} : \partial D^2 \xrightarrow{\cong} \partial D^2,$$

it can be easily verified that f_{ext} is a well defined element of $\mathcal{H}\Big(\overline{B(\tau_\infty, 2)}, \overline{B(\tau_\infty, 2)}\Big)$ that satisfies

$$f_{\text{ext}}|_{\partial\overline{B(\tau_\infty, 2)}} = \text{Id}. \qquad (A)$$

Furthermore, define a continuous map $\kappa : I \to \mathcal{C}\big(D^2, \overline{B(\tau_\infty, 2)}\big)$ by

$$\begin{aligned} \kappa(t) : D^2 &\to \overline{B(\tau_\infty, 2)} \\ (r, \varphi) &\mapsto \begin{cases} (r, \varphi), & 0 \le r \le \varrho_1 \\ \Big(r + t\frac{r - \varrho_1}{1 - \varrho_1}, \varphi\Big), & \varrho_1 \le r \le 1 \end{cases} \end{aligned}$$

for all $t \in I$. Observe that $\kappa(t)$ is an embedding for all $t \in I$, and that, moreover,

$$\kappa(0) = \mathrm{Id}_{D^2}, \quad \text{and} \quad \kappa(1) \in \mathcal{H}\big(D^2, \overline{B(\tau_\infty, 2)}\big). \qquad (B)$$

As, moreover, $\kappa(t)|_{\overline{B(\tau_\infty,\varrho_1)}} = \mathrm{Id}$, it follows that

$$\kappa^{-1}(t) \circ f_{\text{ext}} \circ \kappa(t)(\tau_i) = f(\tau_i) \quad \forall t \in I, \forall i \in \mathbb{N},$$

Thus, there is a well defined map

$$\begin{array}{rcl} \Gamma : I & \rightarrow & \overline{H}_\infty \\ t & \mapsto & \kappa^{-1}(t) \circ f_{\text{ext}} \circ \kappa(t) \end{array}$$

that satisfies $\Gamma(0) = f$, and $\Gamma(1)|_{\partial D^2} = \mathrm{Id}$, i.e., $\Gamma(1) \in H_\infty$. Writing

$$\widehat{f} := \Gamma(1)$$

finishes the proof. □

Lemma A.3. *For any $n \in \mathbb{N} \cup \infty$, let $\Gamma, \Gamma' : I \rightarrow H_0$ be paths such that*

$$\Gamma(0) = \Gamma'(0) \in PH_n, \qquad \Gamma(1) = \Gamma'(1) \in PH_n.$$

Then,

$$\big[ev_n \circ \Gamma\big] = \big[ev_n \circ \Gamma'\big] \quad \textit{in } \pi_1 F_n.$$

If, moreover,

$$\Gamma(t)(\tau_\infty) = \Gamma'(t)(\tau_\infty) = \tau_\infty \quad \forall t \in I,$$

then,

$$\big[ev_n \circ \Gamma\big] = \big[ev_n \circ \Gamma'\big] \quad \textit{in } \pi_1 F'_n.$$

Finally, for any $n \in \mathbb{N} \cup \infty$, let $\Gamma, \Gamma' : I \rightarrow H_0$ be paths that satisfy

$$\Gamma(0) = \Gamma'(0) \in H_n, \qquad \Gamma(1) = \Gamma'(1) \in H_n.$$

Then,

$$\overline{ev}_n \circ \Gamma \simeq_* \overline{ev}_n \circ \Gamma' \quad \textit{in } \pi_1 C_n \quad (\textit{or } \pi_0 \mathcal{O} C_\infty, \ \textit{if } n = \infty).$$

Proof. In this proof, we use a contracting homotopy $K : H_0 \times I \rightarrow H_0$ with the properties given in Theorem 3.3. To prove the first statement for any $n \in \mathbb{N} \cup \infty$, pick any Γ, Γ' with the required properties. Define a map

$$\begin{array}{rcl} \widehat{H} : I \times I & \rightarrow & F_n \\ (s,t) & \mapsto & \mathrm{ev}_n \circ \Gamma(s) \circ K\big((\Gamma(s)^{-1} \circ \Gamma'(s), t\big), \end{array}$$

and observe that, for all $t \in I$,

$$\begin{array}{rcl} \widehat{H}(0,t) & = & \mathrm{ev}_n \circ \Gamma(0) \circ K\big((\Gamma(0)^{-1} \circ \Gamma'(0), t\big) \\ & = & \mathrm{ev}_n \circ \Gamma(0) \circ K\big(\mathrm{Id}, t\big) \\ & \overset{*}{=} & \mathrm{ev}_n \circ \Gamma(0) \\ & = & \mathcal{T}_n, \end{array}$$

and

$$\begin{aligned}
\widehat{H}(1,t) &= \mathrm{ev}_n \circ \Gamma(1) \circ K\big((\Gamma(1)^{-1} \circ \Gamma'(1), t\big) \\
&= \mathrm{ev}_n \circ \Gamma(1) \circ K\big(\mathrm{Id}, t\big) \\
&\overset{*}{=} \mathrm{ev}_n \circ \Gamma(1) \\
&= \mathcal{T}_n,
\end{aligned}$$

where $(*)$ is given by the property (f) of Theorem 3.3. Thus, identifying $(I, \dot{I})$ with $(S^1, 0)$ leads to a well defined homotopy

$$\begin{aligned}
H : S^1 \times I &\to F_n \\
(s,t) &\mapsto \widehat{H}(s,t),
\end{aligned}$$

which has the required properties, because

$$H(\cdot, 0) = \mathrm{ev}_n \circ \Gamma'(\cdot), \qquad H(\cdot, 1) = \mathrm{ev}_n \circ \Gamma(\cdot).$$

Moreover, if both paths $\Gamma, \Gamma' : I \to H_0$ satisfy

$$\Gamma(s)(\tau_\infty) = \tau_\infty \quad \forall t \in I,$$

then, it follows from Theorem 3.3 (c) that

$$\Gamma(s) \circ K\big((\Gamma(s)^{-1} \circ \Gamma'(s), t\big)(\tau_\infty) = \tau_\infty \quad \forall s, t \in I,$$

i.e., $\widehat{H}$ is a well defined path in F_n', because

$$\Gamma(s) \circ K\big((\Gamma(s)^{-1} \circ \Gamma'(s), t\big)(x) \neq \tau_\infty \quad \forall x \in D^2, \forall s, t \in I.$$

The remaining statement is proved similarly. □

Lemma A.4. *The topology of the group of infinite permutations Σ_∞ is metric.*

Proof. As Σ_∞ is topologized as a subspace of the mapping space $\mathbb{N}^\mathbb{N}$, where $\mathbb{N}^\mathbb{N}$ has the topology of pointwise convergence, it suffices to show that $\mathbb{N}^\mathbb{N}$ is metric. Endow $\mathbb{N} \subset \mathbb{R}$ with the subspace topology (i.e., $\mathbb{N}$ has the discrete topology), and endow $\prod_{i\in\mathbb{N}} \mathbb{N}$ and $\prod_{i\in\mathbb{N}} \mathbb{R}$ with the product topology. Then, $\prod_{i\in\mathbb{N}} \mathbb{N}$ is a subspace of $\prod_{i\in\mathbb{N}} \mathbb{R}$, and, as $\prod_{i\in\mathbb{N}} \mathbb{R}$ is metric by [11, Thm 20.5], $\prod_{i\in\mathbb{N}} \mathbb{N}$ is metric too. With our choice of the topologies, $\mathbb{N}^\mathbb{N}$ is homeomorphic to $\prod_{i\in\mathbb{N}} \mathbb{N}$ by [11, p. 282], which means that $\mathbb{N}^\mathbb{N}$ is metric. □

Remark A.5. *Choose any $n \in \mathbb{N}$, or $n = \infty$, and let $\gamma = (\gamma_i)_{i\in\underline{n}} \in \mathcal{C}(I, F_n)$ be a path such that $p_n\gamma(0) = p_n\gamma(1) = \overline{\mathcal{T}}_n$, i.e., $p_n\gamma \in \Omega C_n$. As usual, we write*

$$p_n\gamma = [\gamma] = [(\gamma_i)_{i\in\underline{n}}] \qquad \in \Omega C_n,$$

where $[\gamma]$ denotes the orbit $\Sigma_n(\gamma_i)_{i\in\underline{n}}$. When we consider the class of γ in $\pi_1 C_n$, we write again

$$[\gamma] = [(\gamma_i)_{i\in\underline{n}}] \qquad \in \pi_1 C_n,$$

where $[\gamma]$ denotes the orbit in $\pi_1 C_n$ of the orbit in ΩC_n of γ.

Lemma A.6. *Pick any $n \in \mathbb{N} \cup \infty$. If a pair of paths $\gamma_1, \gamma_2 \in \mathcal{C}\big(\mathrm{I}, H_0\big)$ satisfies*

$$\gamma_1(0) = \gamma_2(0) \in PH_n \quad \text{and} \quad \gamma_1(1) = \gamma_2(1) = id_{D^2},$$

then,

$$[ev_n(\gamma_1)] = [ev_n(\gamma_2)] \quad \in \ \pi_1 F_n.$$

Moreover, any pair of paths $\gamma_1, \gamma_2 \in \mathcal{C}\big((\mathrm{I}, 0, 1), (H_0, H_n, \mathrm{Id}_{D^2})\big)$, such that

$$\gamma_1(0) = \gamma_2(0),$$

satisfies

$$[\overline{ev}_n(\gamma_1)] = [\overline{ev}_n(\gamma_2)] \quad \in \ \pi_1 C_n.$$

Proof. Recalling the contracting homotopy $K : H_0 \times I \to H_0$, there is an ambient isotopy given by

$$\begin{array}{rcl} L : \mathcal{C}(I, H_0) \times I & \to & \mathcal{C}(I, H_0) \\ (\gamma, s) & \mapsto & \Big(t \mapsto K\Big(\gamma_2(t) \circ \big(\gamma_1(t)\big)^{-1}, 1 - s\Big) \circ \gamma(t)\Big). \end{array}$$

Clearly, $L(\gamma_1, 1) = \gamma_2$, and $L(\gamma, 0) = \gamma$ for all $\gamma \in \mathcal{C}(I, H_0)$. Also, $L(\gamma_i, s)(0) = \gamma_i(0)$ and $L(\gamma_i, s)(1) = \gamma_i(1)$ for all $s \in I$, where $i = 1, 2$. It follows, that

$$\mathrm{ev}_n L(\gamma_1, -) : I \to \Omega F_n$$

is a path in ΩF_n from $\mathrm{ev}_n(\gamma_1(-))$ to $\mathrm{ev}_n(\gamma_2(-))$. This proves the first assertion. The second assertion is proved similarly. □

Proposition A.7. *For every configuration $(x_i)_{i \in \mathbb{N}} \in F_\infty$ that converges to a point $x_\infty \in \overset{\circ}{D}{}^2$, i.e.,*

$$\lim_{i \to \infty} x_i = x_\infty,$$

there is an element $h \in H_0$ such that

$$h(\tau_i) = x_i \quad \forall i \in \mathbb{N}.$$

Proof. Pick some $(x_i)_{i \in \mathbb{N}} \in F_\infty$ such that

$$\lim_{i \to \infty} x_i = x_\infty,$$

for some $x_\infty \in \overset{\circ}{D}{}^2$. Observing that there is an element $h_1 \in H_0$ such that

$$h_1(\tau_\infty) = x_\infty,$$

it remains to prove the existence of an $h_2 \in H_0$ that satisfies

$$h_2\big(h_1(\tau_i)\big) = x_i \quad \forall i \in \mathbb{N}.$$

The existence of h_1 allows us, without restricting the generality, to assume that

$$x_\infty = \tau_\infty.$$

For each $i \in \mathbb{N}$, write

$$r_i := 4 \sup_{j \geq i} \max \Big(\|\tau_\infty - x_i\|, \|\tau_\infty - \tau_i\| \Big),$$

and let $\widehat{\alpha}_i : I \to \overset{\circ}{D}{}^2$ be a continuous path satisfying the following conditions.

$$\begin{array}{ll} (i) & \widehat{\alpha}_i(0) = \tau_i \\ (ii) & \widehat{\alpha}_i(1) = x_i \\ (iii) & \widehat{\alpha}_i \cap \Big(\{x_j\}_{j \in [1, i-1]} \cup \{\tau_j\}_{j \geq i+1} \Big) = \varnothing \\ (iv) & \widehat{\alpha}_i(t) \subset B \left(\tau_\infty, \dfrac{1}{2} r_i \right) \quad \forall t \in I. \end{array}$$

Now, recall the definition

$$t_1 := 0, \quad t_i := \sum_{k=1}^{i-1} \frac{1}{2^k} \forall i \geq 2,$$

and define a path set $(\alpha_i)_{i \in \mathbb{N}}$ by

$$\alpha_i(t) = \begin{cases} \tau_i & , \forall t \in \left[0, t_i \frac{1}{2^k}\right] \\ \widehat{\alpha}_i \left(2^i \left(t - t_i \frac{1}{2^k} \right) \right) & , \forall t \in \left[t_i \frac{1}{2^k}, t_i \frac{1}{2^k} \right] \\ x_i & , \forall t \in \left[t_i \frac{1}{2^k}, 1 \right]. \end{cases}$$

By the properties $(i) - (iii)$ of the path set $\{\widehat{\alpha}_i\}_{i \in \mathbb{N}}$, it follows that these paths define a well defined path $(\alpha_i)_{i \in \mathbb{N}} : I \to F_\infty$ from $(\tau_i)_{i \in \mathbb{N}}$ to $(x_i)_{i \in \mathbb{N}}$.
Note that, by the property (iv) of the paths $\widehat{\alpha}_i$,

$$\lim_{i \to \infty} \alpha_i(t) = \tau_\infty \quad \forall t \in I,$$

because $\lim_{i \to \infty} r_i = 0$. As in the proof of Theorem 4.5, this allows us to show that, for each $i \in \mathbb{N}$, there is a path $g_i \in \mathcal{C}\big((I, 0, 1), (H_0, Id, PH_\infty)\big)$ that satisfies

$$\begin{array}{ll} (i) & g_i(t)(\tau_j) = \begin{cases} \alpha_i(t) \quad \forall t \in I & \text{if } j = i \\ \tau_j \quad \forall t \in I & j \neq i, \end{cases} \quad \text{and} \\ (ii) & g_i(t)|_{D^2 \setminus B(\tau_\infty, r_i)} = Id \quad \forall t \in I \\ (iii) & g_i(t) = \mathrm{Id} \quad \forall t \leq t_i \dfrac{1}{2^k} \\ (iv) & g_i(t) = g_i \left(t_i \dfrac{1}{2^k} \right) \quad \forall t \geq t_i \dfrac{1}{2^k}. \end{array}$$

Now, write

$$\mathcal{G}_n(-) = g_n(-) \circ \cdots \circ g_1(-)$$

for all $n \in \mathbb{N}$, and observe that, for every $n \in \mathbb{N}$,

$$\begin{aligned} \mathcal{G}_n(0) &= \mathrm{Id}_{D^2} \\ \mathcal{G}_n(t)(\tau_i) &= \alpha_i(t) \quad \forall\ i \leq n, \forall\ t \in I \end{aligned}$$

by the properties of the maps g_i. Moreover, one can show in analogy to the proof of Theorem 4.5 that the sequence $\mathcal{G}_n$ converges uniformly, which means that there is a path $\mathcal{G} \in \mathcal{C}\big((I,0,1),(H_0, Id, PH_\infty)\big)$ such that

$$\lim_{n\to\infty} \mathcal{G}_n = \mathcal{G}.$$

By the properties of the paths $\mathcal{G}_n$, it follows that $\mathcal{G}$ satisfies

$$\mathcal{G}(1)(\tau_i) = x_i \quad \forall i \in \mathbb{N}.$$

Thus, writing $h_2 := \mathcal{G}(1)$ finishes the proof. □

Next, we present two lemmas that are used in the text. The proof of the first lemma requires long calculations, whereas both lemmas can be understood quite easily by geometric interpretation.
For every $n > 1$, let $\{\sigma_i\}_{i\in[1,n-1]}$ be the set of generators of the group B_n with respect to Artin's presentation, and, for every pair of integers $i, j \in [1, n]$ with $i < j$, write, as usual,

$$A_{i,j} := \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}.$$

Recall Artin's presentation of B_n

$$\begin{aligned} \sigma_i\sigma_j &\sim \sigma_j\sigma_i \qquad \text{if } |i-j| \geq 2,\ 1 \leq i,j \leq n-1 \qquad (A1) \\ \sigma_i\sigma_{i+1}\sigma_i &\sim \sigma_{i+1}\sigma_i\sigma_{i+1}, \qquad 1 \leq i \leq n-2 \qquad (A2). \end{aligned}$$

Lemma A.8. *For every $n \in \mathbb{N}$ and $i \in [1, n-1]$, the following two word classes, with respect to Artin's presentation of B_n, are equal.*

$$A_{i,n} \sim \sigma_i^{-1}\ldots\sigma_{n-2}^{-1}\sigma_{n-1}^2\sigma_{n-2}\ldots\sigma_i.$$

Proof. Fix some $n \in \mathbb{N}$. Observe that the case $i = n-1$ is trivial, and fix some $i \in [1, n-2]$. To prove the required result, we need the following equivalences in B_n, which follow immediately from Artin's relations, valid for all $k, l \in [1, n-2]$ with $|k-l| \geq 2$.

$$\begin{aligned} \sigma_k^{-1}\sigma_l^{-1} &\sim \sigma_l^{-1}\sigma_k^{-1} \quad \forall |k-l| \geq 2 \ (A) \\ \sigma_k^{-1}\sigma_{k+1}^{-1}\sigma_k^{-1} &\sim \sigma_{k+1}^{-1}\sigma_k^{-1}\sigma_{k+1}^{-1} \qquad (B) \\ \sigma_{k+1}\sigma_k &\sim \sigma_k^{-1}\sigma_{k+1}\sigma_k\sigma_{k+1} \qquad (C) \\ \sigma_{k+1}^{-1}\sigma_k^{-1} &\sim \sigma_k^{-1}\sigma_{k+1}^{-1}\sigma_k^{-1}\sigma_{k+1}. \qquad (D) \end{aligned}$$

We need to prove that

$$\sigma_{n-1}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{n-1}^{-1}\sigma_i^{-1}\ldots\sigma_{n-2}^{-1}\sigma_{n-1}^{-2}\sigma_{n-2}\ldots\sigma_i \sim 1.$$

For every $j \in [i, n-2]$, define

$$\begin{aligned} M_j \quad := \quad & \sigma_{n-1}\ldots\sigma_{j+1}\sigma_j^2\sigma_{j+1}^{-1}\sigma_j^{-1}\sigma_{j+2}^{-1}\sigma_{j+1}^{-1}\cdots \\ & \sigma_{k+1}^{-1}\sigma_k^{-1}\sigma_{k+2}^{-1}\sigma_{k+1}^{-1}\ldots\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\sigma_{n-1}^{-2}\sigma_{n-2}\sigma_{n-3}. \end{aligned}$$

We claim that, for every $j \in [i, n-3]$,

$$M_j \sim \sigma_j^{-1} M_{j+1}. \qquad (E)$$

The claim is proved as follows, where at each stage, the term in brackets is replaced by an equivalent one.

$$\begin{aligned} M_j \quad = \quad & \sigma_{n-1}\ldots\sigma_{j+1}\sigma_j^2\left[\sigma_{j+1}^{-1}\sigma_j^{-1}\right]\sigma_{j+2}^{-1}\sigma_{j+1}^{-1}\cdots \\ & \sigma_{k+1}^{-1}\sigma_k^{-1}\sigma_{k+2}^{-1}\sigma_{k+1}^{-1}\ldots\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\sigma_{n-1}^{-2}\sigma_{n-2}\sigma_{n-3} \\ \overset{D}{\sim} \quad & \sigma_{n-1}\ldots\left[\sigma_{j+1}\sigma_j\right]\sigma_{j+1}^{-1}\sigma_j^{-1}\sigma_{j+1}\sigma_{j+2}^{-1}\sigma_{j+1}^{-1}\cdots \\ & \sigma_{k+1}^{-1}\sigma_k^{-1}\sigma_{k+2}^{-1}\sigma_{k+1}^{-1}\ldots\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\sigma_{n-1}^{-2}\sigma_{n-2}\sigma_{n-3} \\ \overset{C}{\sim} \quad & \sigma_{n-1}\ldots\sigma_{j+2}\sigma_j^{-1}\sigma_{j+1}\sigma_j\sigma_{j+1}\sigma_{j+1}^{-1}\sigma_j^{-1}\sigma_{j+1}\sigma_{j+2}^{-1}\sigma_{j+1}^{-1}\cdots \\ & \sigma_{k+1}^{-1}\sigma_k^{-1}\sigma_{k+2}^{-1}\sigma_{k+1}^{-1}\ldots\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\sigma_{n-1}^{-2}\sigma_{n-2}\sigma_{n-3} \\ \sim \quad & \sigma_{n-1}\ldots\sigma_{j+2}\sigma_j^{-1}\sigma_{j+1}^2\sigma_{j+2}^{-1}\sigma_{j+1}^{-1}\cdots \\ & \sigma_{k+1}^{-1}\sigma_k^{-1}\sigma_{k+2}^{-1}\sigma_{k+1}^{-1}\ldots\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\sigma_{n-1}^{-2}\sigma_{n-2}\sigma_{n-3} \\ \overset{A}{\sim} \quad & \ldots \\ \overset{A}{\sim} \quad & \sigma_j^{-1}\sigma_{n-1}\ldots\sigma_{j+2}\sigma_{j+1}^2\sigma_{j+2}^{-1}\sigma_{j+1}^{-1}\cdots \\ & \sigma_{k+1}^{-1}\sigma_k^{-1}\sigma_{k+2}^{-1}\sigma_{k+1}^{-1}\ldots\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\sigma_{n-1}^{-2}\sigma_{n-2}\sigma_{n-3} \\ = \quad & \sigma_j^{-1}M_{j+1} \end{aligned}$$

Also, observe that

$$\begin{aligned}
M_{n-3} &= \sigma_{n-1}\sigma_{n-2}\sigma_{n-3}^2\left[\sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\right]\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\sigma_{n-1}^{-2}\sigma_{n-2}\sigma_{n-3} \\
&\overset{D}{\sim} \sigma_{n-1}\sigma_{n-2}\sigma_{n-3}\sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\sigma_{n-2}\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\sigma_{n-1}^{-2}\sigma_{n-2}\sigma_{n-3} \\
&= \sigma_{n-1}\sigma_{n-2}\sigma_{n-3}\sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\sigma_{n-2}\left[\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\sigma_{n-1}^{-1}\right]\sigma_{n-1}^{-1}\sigma_{n-2}\sigma_{n-3} \\
&\overset{B}{\sim} \sigma_{n-1}\sigma_{n-2}\sigma_{n-3}\sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\sigma_{n-2}\sigma_{n-2}^{-1}\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\sigma_{n-1}^{-1}\sigma_{n-2}\sigma_{n-3} \\
&\sim \sigma_{n-1}\sigma_{n-2}\sigma_{n-3}\sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\left[\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\sigma_{n-1}^{-1}\right]\sigma_{n-2}\sigma_{n-3} \\
&\overset{B}{\sim} \sigma_{n-1}\sigma_{n-2}\sigma_{n-3}\sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\sigma_{n-2}^{-1}\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\sigma_{n-2}\sigma_{n-3} \\
&\sim \sigma_{n-1}\sigma_{n-2}\sigma_{n-3}\left[\sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\sigma_{n-2}^{-1}\right]\sigma_{n-1}^{-1}\sigma_{n-3} \\
&\overset{B}{\sim} \sigma_{n-1}\sigma_{n-2}\sigma_{n-3}\sigma_{n-3}^{-1}\sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\sigma_{n-1}^{-1}\sigma_{n-3} \\
&\sim \sigma_{n-1}\sigma_{n-2}\sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\sigma_{n-1}^{-1}\sigma_{n-3} \\
&\sim \sigma_{n-1}\sigma_{n-3}^{-1}\sigma_{n-1}^{-1}\sigma_{n-3} \\
&\overset{A1}{\sim} \sigma_{n-3}^{-1}\sigma_{n-1}\sigma_{n-1}^{-1}\sigma_{n-3} \\
&\sim 1. \qquad (F)
\end{aligned}$$

Now, we can prove the required result as follows.

$$\begin{aligned}
&\sigma_{n-1}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{n-1}^{-1}\sigma_i^{-1}\ldots\sigma_{n-2}^{-1}\sigma_{n-1}^{-2}\sigma_{n-2}\ldots\sigma_i \\
\overset{A}{\sim}\ &\ldots \\
\overset{A}{\sim}\ &\sigma_{n-1}\ldots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+2}^{-1}\sigma_{i+1}^{-1}\sigma_{i+3}^{-1}\sigma_{i+2}^{-1}\ldots \\
&\sigma_{k+1}^{-1}\sigma_k^{-1}\sigma_{k+2}^{-1}\sigma_{k+1}^{-1}\ldots\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\sigma_{n-1}^{-2}\sigma_{n-2}\ldots\sigma_i \\
=\ &M_i\sigma_{n-4}\ldots\sigma_i \\
\overset{E}{\sim}\ &\sigma_i^{-1}M_{i+1}\sigma_{n-4}\ldots\sigma_i \\
\overset{E}{\sim}\ &\qquad\ldots \\
\overset{E}{\sim}\ &\sigma_i^{-1}\ldots\sigma_{n-4}^{-1}M_{n-3}\sigma_{n-4}\ldots\sigma_i \\
\overset{F}{\sim}\ &\sigma_i^{-1}\ldots\sigma_{n-4}^{-1}\sigma_{n-4}\ldots\sigma_i \\
\sim\ &1
\end{aligned}$$

□

Lemma A.9. *For all integers i, n with $i \in [1, n]$, the following two terms are equivalent with respect to Artin's presentation.*

$$A_{i,i+1}\cdots A_{i,n+1} \sim \sigma_i\cdots\sigma_{n-1}\sigma_n^2\sigma_{n-1}\cdots\sigma_i.$$

Proof. Recall the definition

$$A_{i,j} := \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}$$

for all $1 \leq i < j$. Pick some $i, n \in \mathbb{N}$ with $i \in [1, n]$. If $i = n$, the required equivalence is trivial.

$$A_{n,n+1} = \sigma_n^2$$

Assume that $i < n$, and write

$$A_{ii+1} \cdots A_{i,n} \sim \sigma_i \cdots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_i$$

for the induction hypothesis. Then,

$$\begin{aligned} A_{ii+1} \cdots A_{i,n} A_{i,n+1} &\sim \sigma_i \cdots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_i A_{i,n+1} \\ &\overset{*}{\sim} \sigma_i \cdots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_i \sigma_i^{-1} \ldots \sigma_{n-1}^{-1} \sigma_n^2 \sigma_{n-1} \ldots \sigma_i \\ &\sim \sigma_i \cdots \sigma_{n-2} \sigma_{n-1} \sigma_n^2 \sigma_{n-1} \ldots \sigma_i, \end{aligned}$$

where $(*)$ is given by Lemma A.8. □

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