Carathéodory Bounds for Integer Cones

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Abstract

Let $b \in \mathbb{Z}^d$ be an integer conic combination of a finite set of integer vectors $X \subset \mathbb{Z}^d$. In this note we provide upper bounds on the size of a smallest subset $\widetilde{X} \subseteq X$ such that b is an integer conic combination of elements of $\widetilde{X}$. We apply our bounds to general integer programming and to the cutting stock problem and provide an NP certificate for the latter, whose existence has not been known so far.

Keywords: Carathéodory's Theorem; Integer Cone; Integer Programming; Cutting Stock; Bin Packing

1 Introduction

The *conic hull* of a finite set $X \subset \mathbb{R}^d$ is the set

$$\mathrm{cone}(X) = \{\lambda_1 x_1 + \cdots + \lambda_t x_t \mid t \geq 0; x_1, \ldots, x_t \in X; \lambda_1, \ldots, \lambda_t \geq 0\}.$$

Carathéodory's theorem (see, e.g. [7]) states that if $b \in \mathrm{cone}(X)$, then $b \in \mathrm{cone}(\widetilde{X})$, where $\widetilde{X} \subseteq X$ is a subset of X whose cardinality is bounded by the maximum number of linearly independent points of X and thus is bounded by d.

The *integer conic hull* of a finite set $X \subset \mathbb{R}^d$ is the set

$$\mathrm{int\_cone}(X) = \{\lambda_1 x_1 + \cdots + \lambda_t x_t \mid t \geq 0; x_1, \ldots, x_t \in X; \lambda_1, \ldots, \lambda_t \in \mathbb{Z}_{\geq 0}\}.$$

The analogous question here is the following: Given $b \in \mathrm{int\_cone}(X)$, how large is the smallest subset $\widetilde{X}$ of X such that $b \in \mathrm{int\_cone}(\widetilde{X})$?

It is easy to see that one cannot give an upper bound for the cardinality of a smallest subset $\widetilde{X}$ in terms of the dimension d. For example, if $X \subset \mathbb{Z}^d$ consists of the vectors $x_{ij} = 2^i e_j + e_d$ for $i = 0, \ldots, n-1$ and $j = 1, \ldots, d-1$, where e_j is the j-th unit vector, then all these vectors are needed to represent the vector $b \in \mathbb{Z}^d$ with $b_j = (2^n - 1)$, $j = 1, \ldots, d-1$, $b_d = n(d-1)$. Indeed, $b = \sum_{i=0}^{n-1} \sum_{j=1}^{d-1} x_{ij}$ is the unique representation.

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The *size* of an integer $a \in \mathbb{Z}$ is the number of bits which are needed to represent a in binary encoding. It is convenient to define it as $\text{size}(a) = \log(|a|+1)$. The *size of a vector* $v \in \mathbb{Z}^n$, denoted $\text{size}(v)$, is the sum of the sizes of its components. In the example above, we have $\text{size}(b) = n(d-1) + \log(n(d-1))$, and hence the number $n(d-1)$ of nonzero terms that are necessary in order to provide b as a result of an integer conic combination can be arbitrary large comparing to the dimension; this number has the same order of magnitude as $\text{size}(b)$.

In this note, we show the following integer analogues of Carathéodory's theorem. The first theorem provides polynomial bounds in the dimension and the largest size of a component of a vector in X.

Theorem 1. *Let $X \subset \mathbb{Z}^d$ be a finite set of integer vectors and let $b \in \text{int\_cone}(X)$. Then there exists a subset $\widetilde{X} \subseteq X$ such that $b \in \text{int\_cone}(\widetilde{X})$ and the following holds for the cardinality of $\widetilde{X}$.*

(i) If all vectors in X are nonnegative, then $|\widetilde{X}| \leq \text{size}(b)$.

(ii) If $M = \max_{x \in X} \|x\|_\infty$, then $|\widetilde{X}| \leq 2d\log(4dM)$.

In the second theorem, we suppose that X is *closed under convex combinations*. The latter means that every integer point in the convex hull of X also belongs to X. The theorem provides then an exponential bound in the dimension.

Theorem 2. *Let $X \subset \mathbb{Z}^d$ be a finite set of integer vectors which is closed under convex combinations and let $b \in \text{int\_cone}(X)$. Then there exists a subset $\widetilde{X} \subseteq X$ of cardinality $|\widetilde{X}| \leq 2^d$ such that $b \in \text{int\_cone}(\widetilde{X})$. Furthermore if $b = \sum_{x \in X} \lambda_x x$, $\lambda_x \in \mathbb{Z}_{\geq 0}$ for all $x \in X$, then there exist $\mu_x \in \mathbb{Z}_{\geq 0}$, $x \in \widetilde{X}$, such that $b = \sum_{x \in \widetilde{X}} \mu_x x$ and $\sum_{x \in X} \lambda_x = \sum_{x \in \widetilde{X}} \mu_x$.*

As an application of these results we prove that the *cutting stock problem* has an optimal solution whose binary encoding length is polynomial in the input length and that the cutting stock problem with a fixed number of item sizes has an optimal solution with a fixed number of *patterns*.

Related work

A finite set of vectors X is called a *Hilbert basis*, if each integer vector $b \in \text{cone}(X)$ is a *nonnegative integer combination* of elements in X. Cook, Fonlupt and Schrijver [2] provided the following integer analogue of Carathéodory's theorem: If $\text{cone}(X)$ is pointed and if $X \subseteq \mathbb{Z}^d$ is an *integral* Hilbert basis, then for every $b \in \text{cone}(X)$ there exists a subset $\widetilde{X} \subseteq X$ with $|\widetilde{X}| \leq 2d-1$ such that b is an integer conic combination of vectors in $\widetilde{X}$. Sebő [8] improved this bound to $2d-2$. Bruns et al. [1] have shown that the bound d is not valid. If X is an integral Hilbert basis but $\text{cone}(X)$ is not pointed, then an upper bound in terms of the dimension d cannot be given [2]. Cook et al. [2] also considered integer programs $\min\{\mathbf{1}^T y \mid Ay = b, y \in \mathbb{Z}^m_{\geq 0}\}$, where $A \in \{0,1\}^{d \times m}$ and $A^T x \leq \mathbf{1}$ has the integer rounding property. The authors showed that, if there exists an optimal solution to such an integer program, then there exists an optimal solution with at most $2d-2$ nonzero coefficients.

2 Proofs of the theorems

To prove the above stated assertions, we exploit the fact that, if the sum of some vectors in X can be expressed as the sum of other vectors in X, then we can eliminate one (or more) vector used in the integer conic combination. A similar idea has also been applied by Sebő [8] to prove his conjectures for Hilbert bases in particular cases.

Lemma 3. *Let $X \subset \mathbb{Z}^d_{\geq 0}$ be a finite set of nonnegative integer vectors and let $b \in \text{int\_cone}(X)$. If $|X| > \sum_{i=1}^d \log(b_i+1)$, then there exists a proper subset $\widetilde{X}$ of X such that $b \in \text{int\_cone}(\widetilde{X})$.*

Proof. Let $b = \sum_{x\in X} \lambda_x x$ with $\lambda_x > 0$ integer for all $x \in X$. Clearly, $\sum_{x\in\widetilde{X}} x \le b$ for any subset $\widetilde{X} \subseteq X$. This implies that the number of different vectors which are representable as the sum of vectors of a subset $\widetilde{X}$ of X is bounded by $\prod_{i=1}^{d}(b_i+1)$. If $2^{|X|} > \prod_{i=1}^{d}(b_i+1)$, there exist two subsets $A,B \subseteq X$, $A \neq B$, with $\sum_{x\in A} x = \sum_{x\in B} x$. Consequently, there exist two disjoint subsets of X, $A' = A\setminus B$ and $B' = B\setminus A$, with $\sum_{x\in A'} x = \sum_{x\in B'} x$. Suppose that $A' \neq \emptyset$ and set $\lambda = \min\{\lambda_x : x \in A'\}$. Then we can rewrite

$$\begin{aligned}
\sum_{x\in X}\lambda_x x &= \sum_{x\in X\setminus A'}\lambda_x x + \sum_{x\in A'}\lambda_x x \\
&= \sum_{x\in X\setminus A'}\lambda_x x + \sum_{x\in A'}(\lambda_x-\lambda)x + \lambda\sum_{x\in A'} x \\
&= \sum_{x\in X\setminus A'}\lambda_x x + \sum_{x\in A'}(\lambda_x-\lambda)x + \lambda\sum_{x\in B'} x \\
&= \sum_{x\in X}\mu_x x,
\end{aligned}$$

where $\mu_x = \lambda_x$ if $x \in X\setminus(A'\cup B')$, $\mu_x = \lambda_x+\lambda$ if $x \in B'$ and $\mu_x = \lambda_x-\lambda$ if $x\in A'$. Thus, $\mu_x \ge 0$ for all $x \in X$ and at least one of the μ_x, $x\in A'$, is zero. Thus if $2^{|X|} > \prod_{i=1}^{d}(b_i+1)$, one can find a proper subset $\widetilde{X}$ of X such that $b \in \text{int\_cone}(\widetilde{X})$. □

Lemma 4. *Let $X \subseteq \mathbb{Z}^d$ be a finite set of integer vectors and let $b \in \text{int\_cone}(X)$. If*

$$|X| > d\,\log(2\,|X|\,\max_{x\in X}\|x\|_\infty + 1), \tag{1}$$

then there exists a proper subset $\widetilde{X}$ of X such that $b \in \text{int\_cone}(\widetilde{X})$.

Proof. Suppose that $b = \sum_{x\in X}\lambda_x x$ with $\lambda_x > 0$ integer for all $x\in X$. Let n denote the cardinality of X, $n = |X|$. Suppose that $n > d\,\log(2n\max_{x\in X}\|x\|_\infty+1)$. For every subset $\widetilde{X}\subseteq X$, $\|\sum_{x\in\widetilde{X}} x\|_\infty$ is bounded by $n\max_{x\in X}\|x\|_\infty$. This implies that the number of different vectors which are representable as the sum of vectors of a subset $\widetilde{X}$ of X is bounded by $(2n\max_{x\in X}\|x\|_\infty+1)^d$. By our assumption we have $2^n > (2n\max_{x\in X}\|x\|_\infty+1)^d$. Therefore there exist two subsets $A,B\subseteq X$, $A\neq B$, with $\sum_{x\in A} x = \sum_{x\in B} x$ and we can proceed as in the proof of Lemma 3. □

Proof of Theorem 1. Part (i) follows immediately from Lemma 3. To prove part (ii), it suffices to show that $|X| > 2\,d\,\log(4\,d\,M)$ implies (1). Suppose that $|X| > 2\,d\,\log(4\,d\,M)$, that is, $M < 2^{|X|/(2d)}/(4d)$. Then

$$\begin{aligned}
d\,\log(2\,|X|\,M+1) &< d\,\log\left(\tfrac{|X|}{2d}\,2^{|X|/(2d)}+1\right) \\
&\le d\,\log\left(2^{|X|/(2d)}\left(\tfrac{|X|}{2d}+1\right)\right) \\
&= \tfrac{|X|}{2} + d\,\log\left(\tfrac{|X|}{2d}+1\right) \\
&\le \tfrac{|X|}{2} + d\,\tfrac{|X|}{2d} \\
&= |X|.
\end{aligned}$$

□

To prove Theorem 2 we apply the rewriting technique used in the above proof of Theorem 1 together with a technique which was used by Hayes and Larman [4] to prove that the integer hull of knapsack polytopes has a polynomial number of vertices in fixed dimension. Recall that our set $X \subseteq \mathbb{Z}^d$ is closed under convexity. This implies that if two points $x_1, x_2 \in X$ are congruent to each other modulo 2, then $1/2(x_1+x_2)$ is integer and hence also contained in X. We say that $b = \sum_{x\in X}\lambda_x x$, $\lambda_x \in \mathbb{Z}_{\ge 0}$, is a *representation* of b of *value* $\sum_{x\in X}\lambda_x$ with *potential* $\sum_{x\in X}\lambda_x\|\binom{1}{x}\|$, where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^{d+1}$.

Proof of Theorem 2. We show that, if there exists a representation of b of value γ, then there exists a representation with the same value and at most 2^d nonzero coefficients. Let $b = \sum_{x\in X} \lambda_x x$, be the representation of b of value γ with the smallest potential. If the number of nonzero coefficients is greater than 2^d, then there exists $x_1 \neq x_2 \in X$ such that $\lambda_{x_1} > 0$, $\lambda_{x_2} > 0$ and $x_1 \equiv x_2 \pmod 2$. Since X is closed under convex combinations, $1/2(x_1+x_2)$ belongs to X. Suppose without loss of generality that $\lambda_{x_1} \geq \lambda_{x_2}$. Then $\lambda_{x_1}x_1 + \lambda_{x_2}x_2 = (\lambda_{x_1} - \lambda_{x_2})x_1 + 2\lambda_{x_2}(1/2(x_1+x_2))$. Since $\binom{1}{x_1}$ and $\binom{1}{x_2}$ are not co-linear, we have

$$\begin{aligned}(\lambda_{x_1} - \lambda_{x_2})\|\tbinom{1}{x_1}\| + 2\lambda_{x_2}\|\tbinom{1}{1/2(x_1+x_2)}\| &= (\lambda_{x_1} - \lambda_{x_2})\|\tbinom{1}{x_1}\| + \lambda_{x_2}\|\tbinom{1}{x_1} + \tbinom{1}{x_2}\| \\ &< (\lambda_{x_1} - \lambda_{x_2})\|\tbinom{1}{x_1}\| + \lambda_{x_2}\left(\|\tbinom{1}{x_1}\| + \|\tbinom{1}{x_2}\|\right) \\ &= \lambda_{x_1}\|\tbinom{1}{x_1}\| + \lambda_{x_2}\|\tbinom{1}{x_2}\|.\end{aligned}$$

Thus replacing $\lambda_{1/2(x_1+x_2)}$ by $\lambda_{1/2(x_1+x_2)} + 2\lambda_{x_2}$, λ_{x_1} by $\lambda_{x_1} - \lambda_{x_2}$ and λ_{x_2} by 0 yields a representation of b with the same value and smaller potential, which is a contradiction. Therefore λ has at most 2^d nonzero components. $\square$

3 Applications

Integer programming

We can now prove a corollary concerning integer programs with equations and nonnegativity constraints on the variables. The result states that, if there exists an optimal solution to such an integer program, then there exists one which is polynomial in the number of equations and the maximum binary encoding length of an integer in the objective function vector and the constraint matrix.

Corollary 5. *Let* $\min\{c^T y \mid Ay = b, y \geq 0, y \text{ integer}\}$ *be an integer program, where* $A \in \mathbb{Z}^{d\times n}$ *and* $c \in \mathbb{Z}^n$*. If this integer program has a finite optimum with optimal value* γ*, then there exists an optimal solution* $y^* \in \mathbb{Z}^m_{\geq 0}$ *which satisfies the following.*

(i) The number of nonzero components of y^* *is at most* $\mathrm{size}(b)$ + $\mathrm{size}(\gamma)$*, if A and c are nonnegative.*

(ii) The number of nonzero components of y^* *is at most* $2(d+1)(\log(d+1)+s+2)$*, where s is the largest size of a coefficient of A and c.*

Proof. The integer vector $\binom{\gamma}{b}$ is an integer conic combination of the column vectors of the matrix $\binom{c^T}{A}$. The optimal solutions correspond to the coefficients in the integer conic combinations for $\binom{\gamma}{b}$. The assertion follows thus from Theorem 1. $\square$

Cutting stock

Let $a, b \in \mathbb{Z}^d_{>0}$ and $\beta \in \mathbb{Z}_{>0}$. The *cutting stock problem* defined by a, b and β is the integer program

$$\begin{array}{ll} \min & \mathbf{1}^T\lambda \\ \text{s.t} & M\lambda = b, \\ & \lambda \geq 0 \text{ integer}, \end{array} \tag{2}$$

where the columns of the matrix M are exactly the integer solutions to the knapsack constraint $a^T x \leq \beta, x \geq 0$, called *patterns*. The above integer program was introduced by Gilmore and Gomory [3]. The cutting stock problem is NP-hard, however, it was not known, whether the problem has an optimal solution whose encoding length is polynomial in the input, see for example [5, 6].

A polynomial algorithm for the case $d = 2$ was given by McCormick, Smallwood and Spieksma [6]. The authors prove that there exists an optimal solution in this case, which has at most 3 nonzero entries in λ. It is open, whether the cutting stock problem can be solved in polynomial time, if the number of sizes d is fixed.

It is easy to see that the cutting stock problem defined by a, b and β has a feasible solution if and only if $a_i \leq \beta$ for all $i = 1, \ldots, d$. In this case, the unit vectors e_i, $i = 1, \ldots, d$, are among the columns of matrix M in (2). Using these vectors only, one can obtain a feasible solution λ with $\mathbf{1}^T \lambda = \sum_{i=1}^d b_i$. Thus the problem is feasible and bounded and therefore has an optimal solution.

Corollary 6. *Let $a \in \mathbb{Z}^d_{>0}$, $\beta \in \mathbb{Z}_{>0}$ and $b \in \mathbb{Z}^d_{>0}$ define a cutting stock problem* (2) *and let $a_i \leq \beta$, $i = 1, \ldots, d$. Then there exists an optimal solution to* (2) *with at most* $\min\{2\,\mathrm{size}(b), 2^d\}$ *nonzero components.*

Proof. As mentioned above, $a_i \leq \beta$, $i = 1, \ldots, d$, implies that an optimal solution exists. Let γ denote the optimal value of (2). Then $\gamma \leq \sum_{i=1}^d b_i$, and therefore $\log(\gamma + 1) \leq \sum_{i=1}^d \log(b_i + 1) = \mathrm{size}(b)$. By Corollary 5 (i), there exists an optimal solution λ such that the number of nonzero components is at most $\mathrm{size}(\gamma) + \mathrm{size}(b) \leq 2\,\mathrm{size}(b)$. The second bound follows immediately from Theorem 2, since the set of patterns is closed under convex combinations. $\square$

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