Split-Cuts and the Stable Set Polytope of Quasi-Line Graphs

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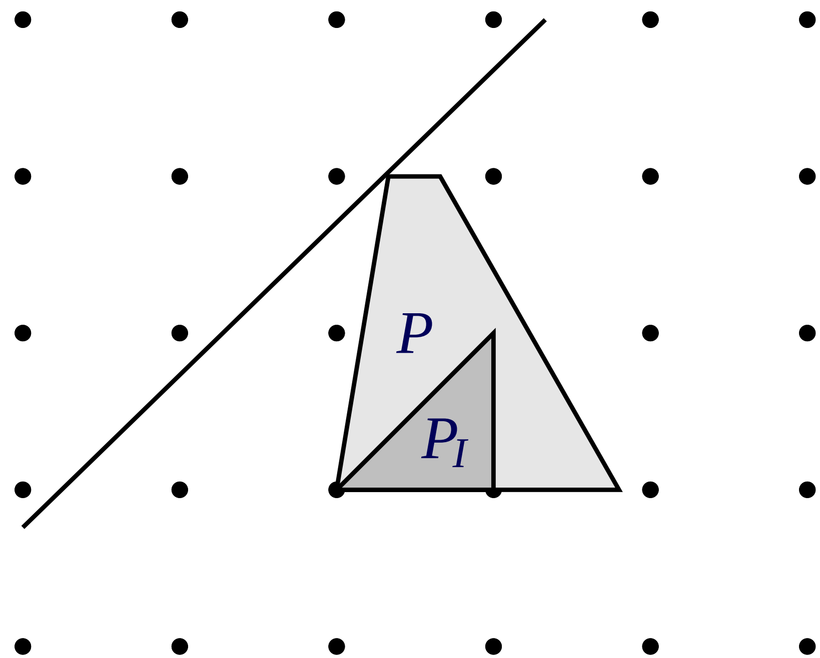
Friedrich Eisenbrand

Joint work with G. Oriolo, P. Ventura and G. Stauffer

Gomory Cutting Planes

Gomory cutting planes

$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$ polyhedron, $c^T x \leqslant \delta$, $c \in \mathbb{Z}^n$ valid for P.
Then: $c^T x \leqslant \lfloor \delta \rfloor$ valid for integer hull P_I of P. (Gomory 1958, Chvátal 1973)



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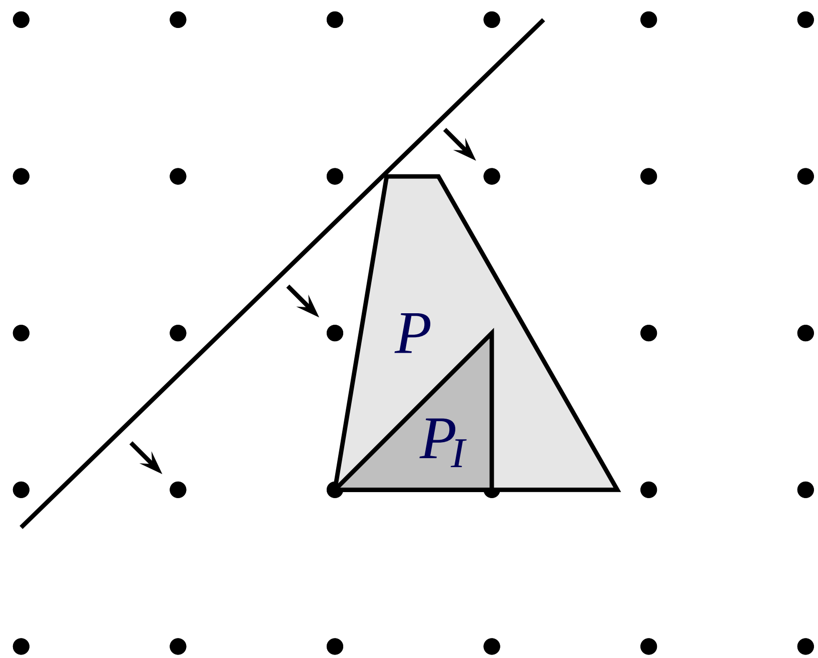

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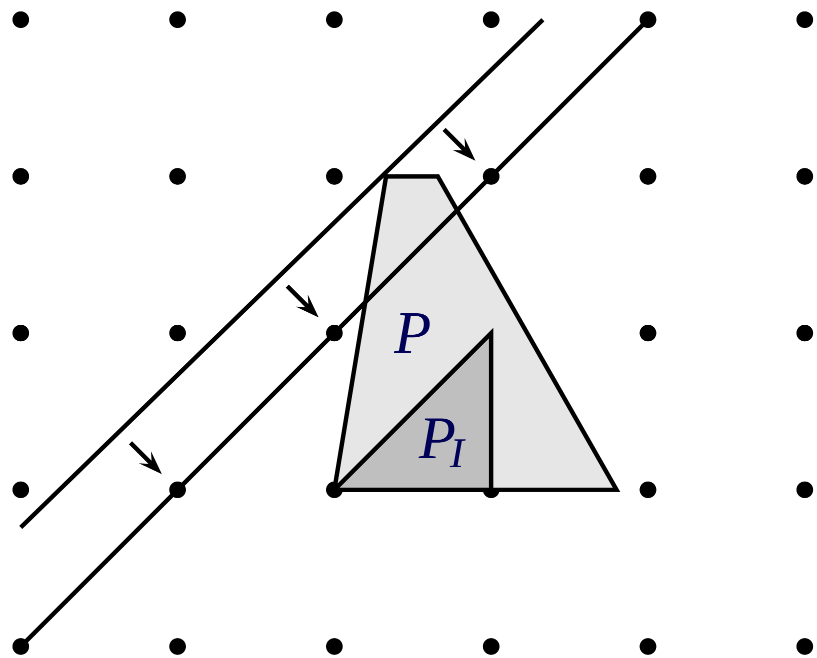


Chvátal closure

Inequality $c^T x \leqslant \delta$ valid for $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$ if and only if there exists $\lambda \in \mathbb{R}^n_{\geqslant 0}$ with $\lambda^T A = c$ and $\lambda^T b \leqslant \delta$

Chvátal closure:

$$P' = P \bigcap_{\substack{\lambda \geqslant 0 \\ \lambda^T A \in \mathbb{Z}^n}} \lambda^T Ax \leqslant \lfloor \lambda^T b \rfloor.$$

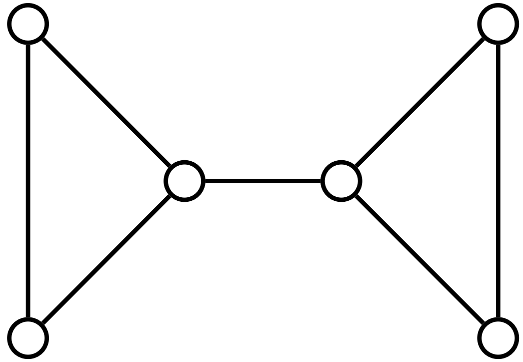
(Chvátal 1973)

P' is polyhedron, if P is rational (Schrijver 1980), (Chvátal 1973)

Optimizing over P' is NP-hard (E. 1999)

Matching: A case where $P_I = P'$

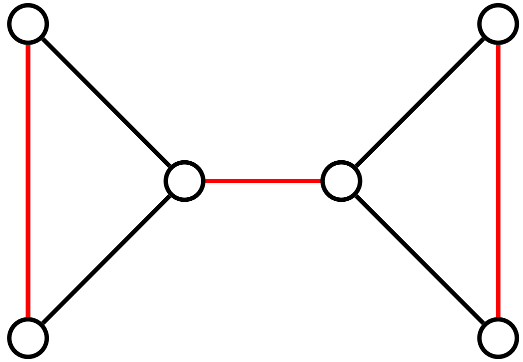
Matching



Matching polytope $P(G)$: Convex hull of incidence vectors of matchings of G.

Which kind of inequalities describe $P(G)$?

Matching



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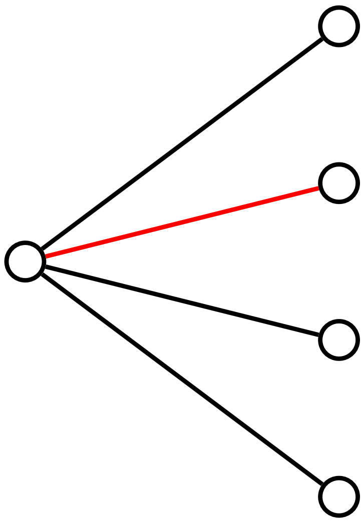
Which kind of inequalities describe $P(G)$?

The fractional matching polytope



- $\forall e \in E : x(e) \geqslant 0$
- $\forall v \in V : \sum_{e \in \delta(v)} x(e) \leqslant 1$

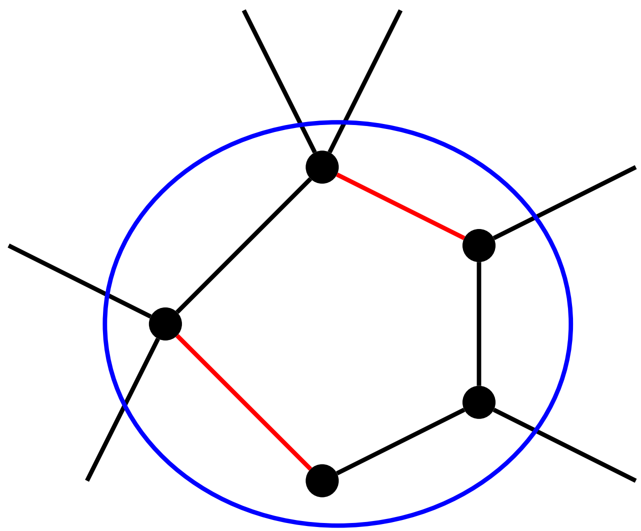
The matching polytope

Theorem (Edmonds 65). *The matching polytope is described by the following inequalities:*

i) $x(e) \geqslant 0$ *for each* $e \in E$,

ii) $\sum_{e \in \delta(v)} x(e) \leqslant 1$ *for each* $v \in V$,

iii) $\sum_{e \in E(U)} x(e) \leqslant \lfloor |U|/2 \rfloor$ *for each* $U \subseteq V$



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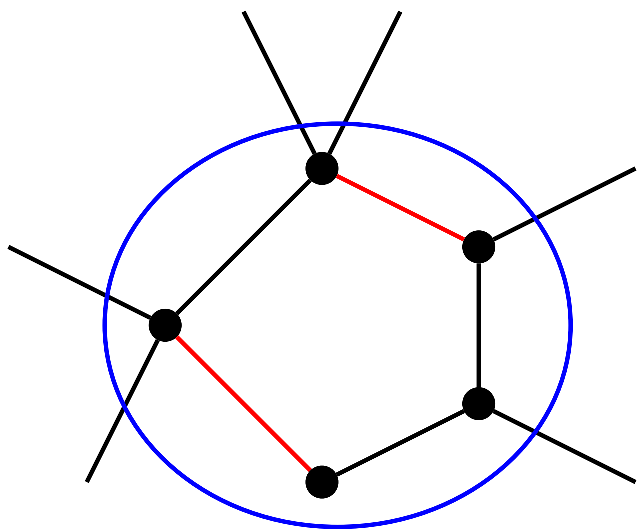
iii) $\sum_{e \in E(U)} x(e) \leqslant \lfloor |U|/2 \rfloor$ *for each* $U \subseteq V$ *Gomory Cut!*

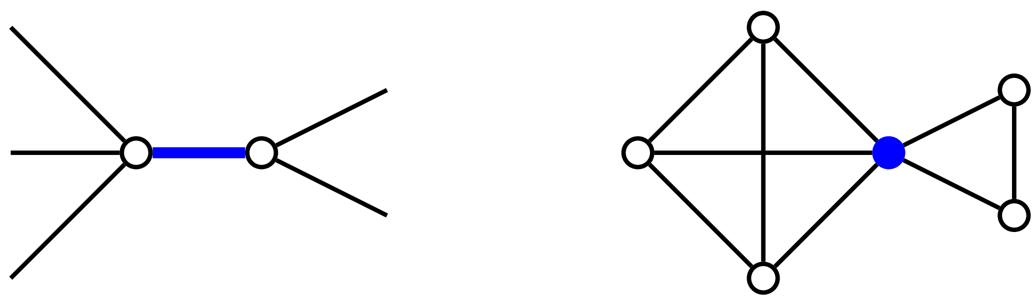


Stable Sets: A generalization of Matching

Stable Sets

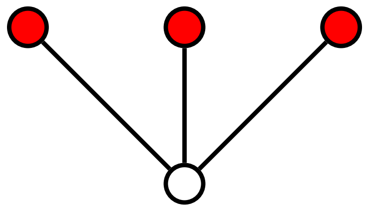
Stable set: Subset of pairwise non-adjacent nodes

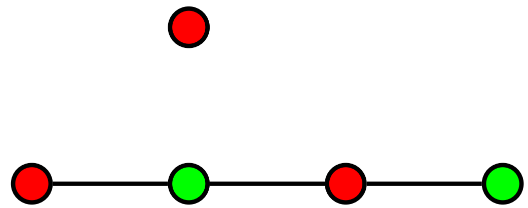
A stable set of a line graph is a matching of the original graph.



Claw-Free Graphs

Theorem (Minty 1980, Nakamura & Tamura 2001). *The maximum weight stable set problem of a claw-free graph can be solved in polynomial time.*





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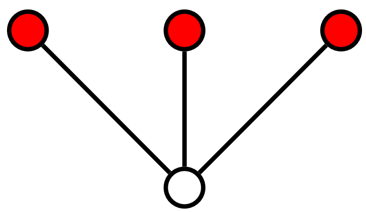

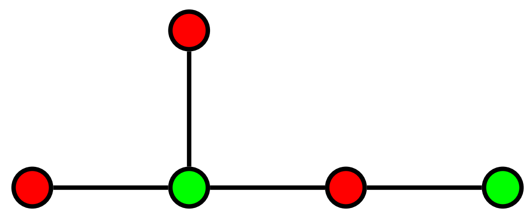

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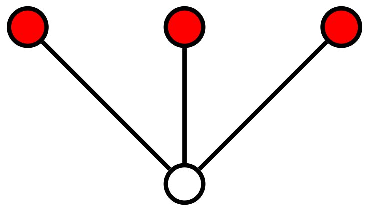

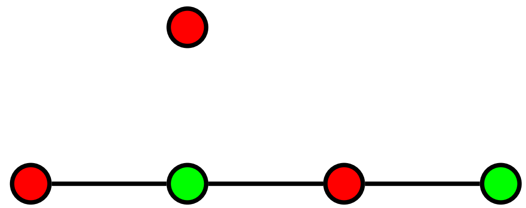

Which kind of inequalities describe $STAB(G)$ if G is claw-free ? (Gröschtel, Lovász & Schrijver 1987)

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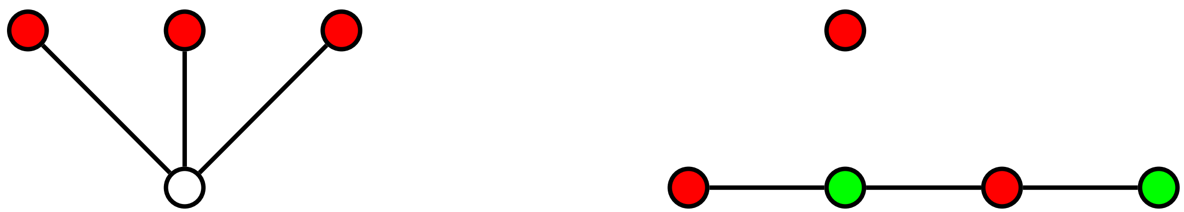
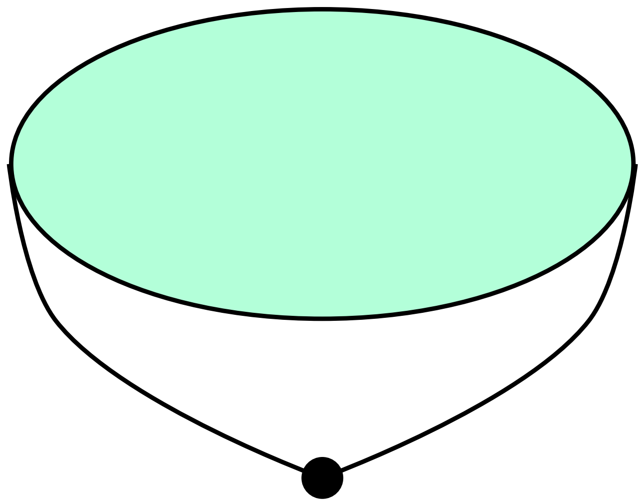


Which kind of inequalities describe $STAB(G)$ if G is claw-free ? (Gröschtel, Lovász & Schrijver 1987)

As of today: There is even no conjecture !

Quasi-Line Graphs

G is quasi-line if neighborhoods of nodes decompose into two cliques



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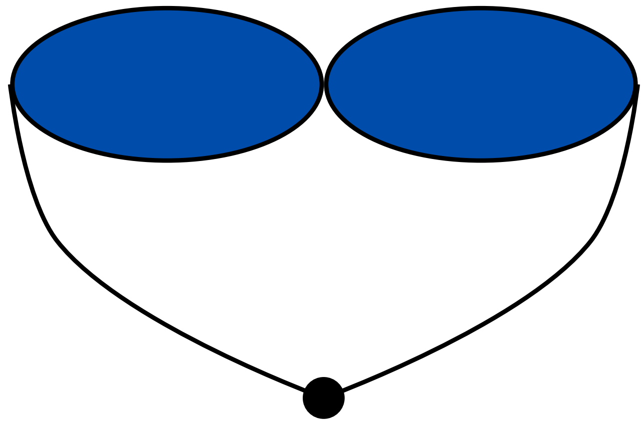

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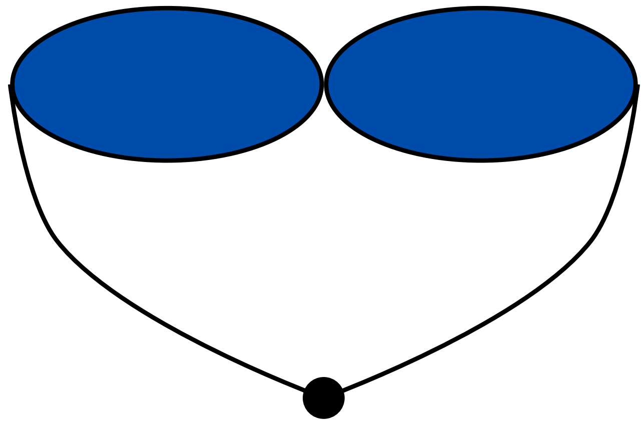

Line graphs $\subset$ quasi-line graphs $\subset$ claw-free graphs

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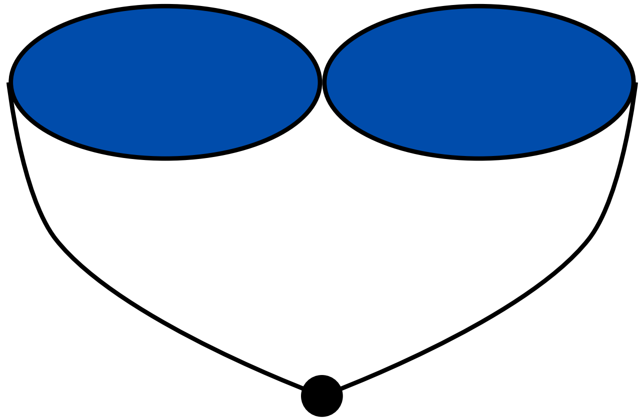


Line graphs $\subset$ quasi-line graphs $\subset$ claw-free graphs

Which kind of inequalities describe $STAB(G)$ if G is quasi-line ?

Gomory cuts are not enough!

Bad news first: Inequalities are not 0/1

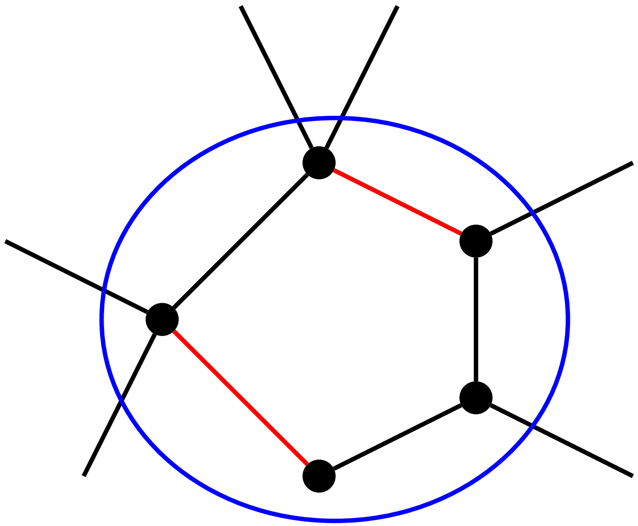
- For each natural number a, there exists a quasi-line graph, whose stable set polytope has a normal vector with coefficients $a/a+1$ (Giles & Trotter 1981)
- Facets cannot be interpreted as subsets of nodes

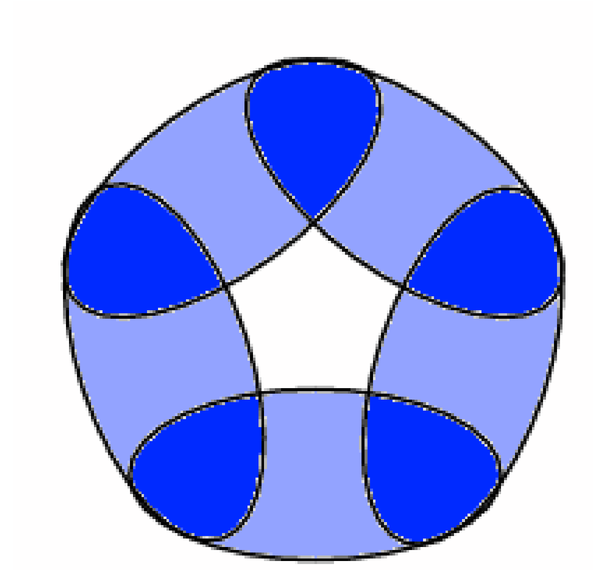
Clique family inequalities

- F is family of cliques
- $p \geqslant 1$ a natural number and r remainder of $|F|/p$
- V_{p-1}: Vertices contained in $p-1$ cliques of F
- $V_{\geqslant p}$: Vertices contained in at least p cliques of F

$$(p-r-1)\cdot \sum_{v\in V_{p-1}} x(v) + (p-r)\cdot \sum_{v\in V_{\geqslant p}} x(v) \leqslant (p-r)\cdot \lfloor \frac{|F|}{p} \rfloor$$

A generalization of Edmond's inequalities





$$(2-1-1)\cdot\sum_{v\in{\color{cyan}\bullet}} x(v) + (2-1)\cdot\sum_{v\in\bullet} x(v) \leqslant (2-1)\cdot\lfloor\frac{|F|}{2}\rfloor$$

Ben Rebea's conjecture

Conjecture. *If G is quasi-line, then $STAB(G)$ can be characterized by positiveness inequalities, clique inequalitites and clique family inequalities.*

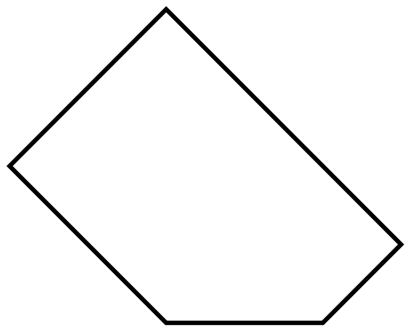
Clique family inequalities are split cuts!

Split Cuts

Splits

Split: Tuple (π, π_0), $\pi \in \mathbb{Z}^n$, $\pi_0 \in \mathbb{Z}$.

$P \subseteq \mathbb{R}^n$ polyhedron: $P^{(\pi,\pi_0)} = \mathrm{conv}\left(P \cap (\pi x \leqslant \pi_0), P \cap (\pi x \geqslant \pi_0 + 1)\right)$.



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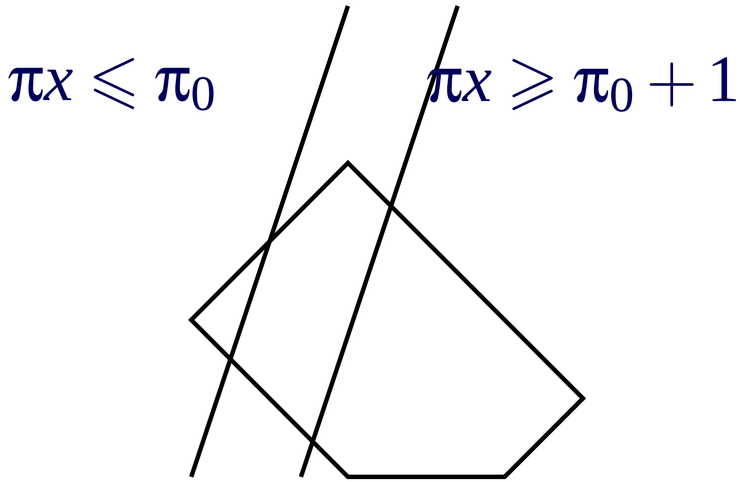

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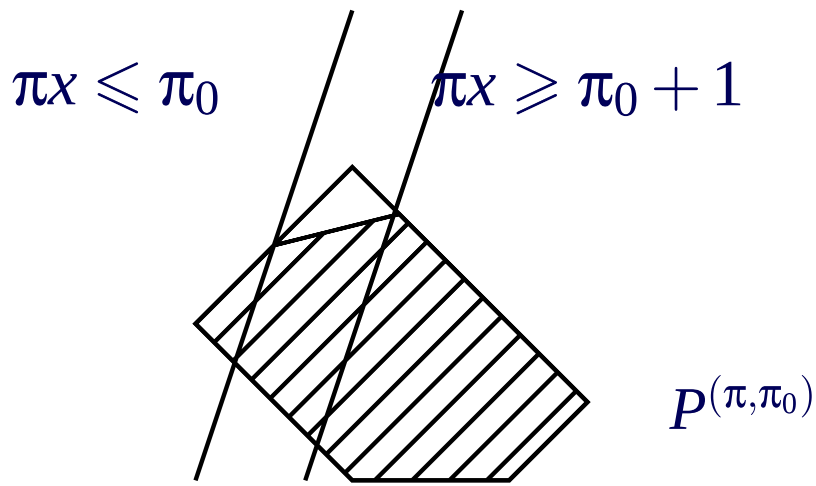

Split closure

Split closure:

$$P^s = \bigcap_{(\pi,\pi_0) \text{ split}} P^{(\pi,\pi_0)}.$$

P^s is polyhedron if P is rational
Split cut: Inequality valid for P^s (Cook, Kannan & Schrijver (1990)

Special case of disjunctive cut (Balas 1979)

AKA: Gomory mixed integer cut, Mixed integer rounding cut.
(Nehmhauser & Wolsey 1988, 1990), (Cornuéjols & Li 2001), (Andersen, Cornuéjols & Li 2002)

Separating split-cuts is NP-hard (Caprara & Letchford 2001)

Proving Ben Rebea’s conjecture

Outline

- Decomposition theorem for quasi-line graphs
  (Chudnovsky, Seymour 04)
  - Either a composition of fuzzy linear interval graphs (1)
  - or a fuzzy circular interval graph (2)
- Edmonds' inequalities are enough for class (1)
  (Chudnovsky, Seymour 04)
- Reduction from class (2) to the class of circular interval graphs
  A facet of a fuzzy circular interval graph $G = (V, E)$ is also a facet of $G' = (V, E' \subset E)$ with G' a circular interval graph
- Establish the conjecture for circular interval graphs

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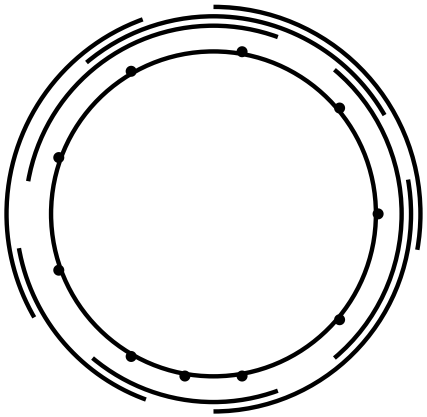
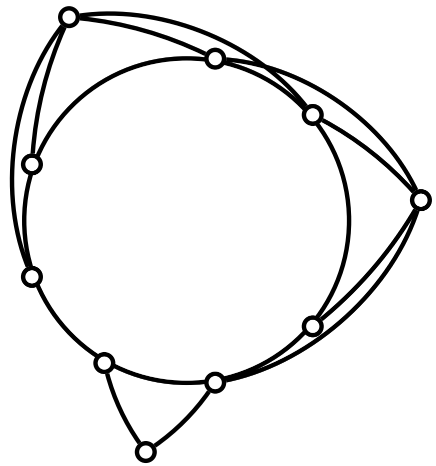
Stable sets of circular interval graphs

- Circular interval graphs





- Packing problem

$$\begin{array}{llll} \max & \sum_{v \in V} c(v) x(v) & & \\ s.t. & Ax & \leqslant & 1 \\ & x(v) & \in & \{0,1\} \quad \forall v \in V \end{array}$$

e.g. $A = \begin{pmatrix} 1 & 1 & 1 & & & & & & & \\ & 1 & 1 & 1 & & & & & & \\ & & 1 & 1 & 1 & & & & & \\ & & & 1 & 1 & 1 & & & & \\ & & & & & 1 & 1 & 1 & & \\ & & & & & & & 1 & 1 & \\ 1 & & & & & & & & 1 & 1 \\ 1 & 1 & & & & & & & & 1 \end{pmatrix}$

where the matrix A is a circular ones matrix

Totally unimodular transformation

- Totally unimodular transformation (Bartholdi, Orlin, Ratliff 80)

$$x = Ty, \text{ where } T = \begin{pmatrix} 1 & & & & & \\ -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & -1 & 1 \end{pmatrix}$$

Let $P = \{x \in \mathbb{R}^n \mid \binom{A}{-I} x \leqslant \binom{1}{0}\}$, $Q = \{y \in \mathbb{R}^n \mid \binom{A}{-I} T y \leqslant \binom{1}{0}\}$

- Effect of T on a circular ones matrix

$$\underbrace{\begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \dots & 0 & 0 & 1 & 1 & 1 & 0 & \dots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}}_{A} \longrightarrow \underbrace{\begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \dots & 0 & -1 & 0 & 0 & 1 & 0 & \dots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}}_{A \cdot T}$$

Slicing the polytope

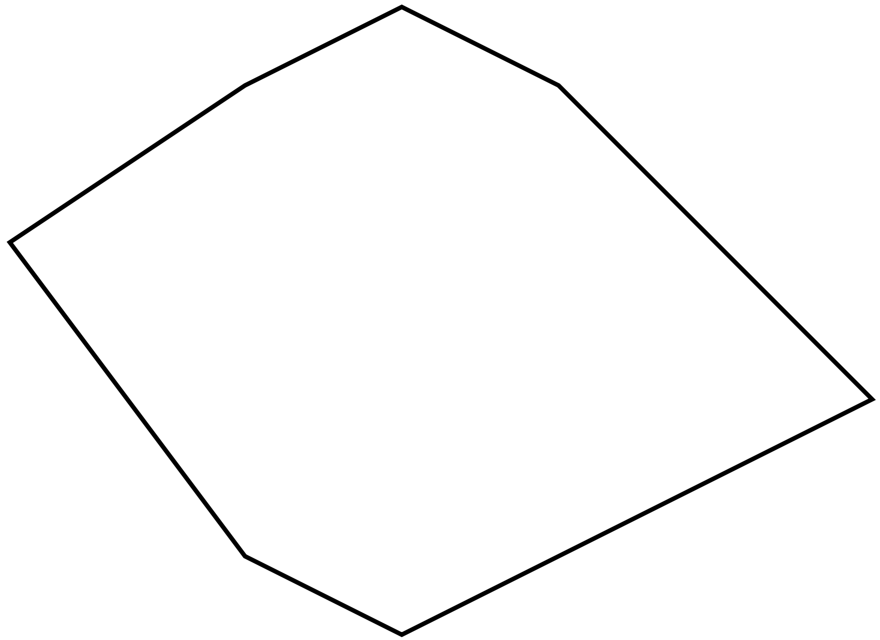
- The structure of Q

  $Q = \{y \in \mathbb{R}^n \mid \binom{A}{-I} T y \leqslant \binom{1}{0}\} = \{y \in \mathbb{R}^n \mid (N|v) y \leqslant \binom{1}{0}\}$
  where N is "almost" a arc-node incidence matrix

- Slicing Q and P

  - N totally unimodular $\Longrightarrow$
    $\forall \beta \in \mathbb{N}, Q_\beta = \{y \in \mathbb{R}^n \mid (N|v) y \leqslant \binom{1}{0}, y(n) = \beta\}$ is integral
  - T totally unimodular $\Longrightarrow$
    $\forall \beta \in \mathbb{N}, P \cap \{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = \beta\}$ is integral

Slicing the polytope



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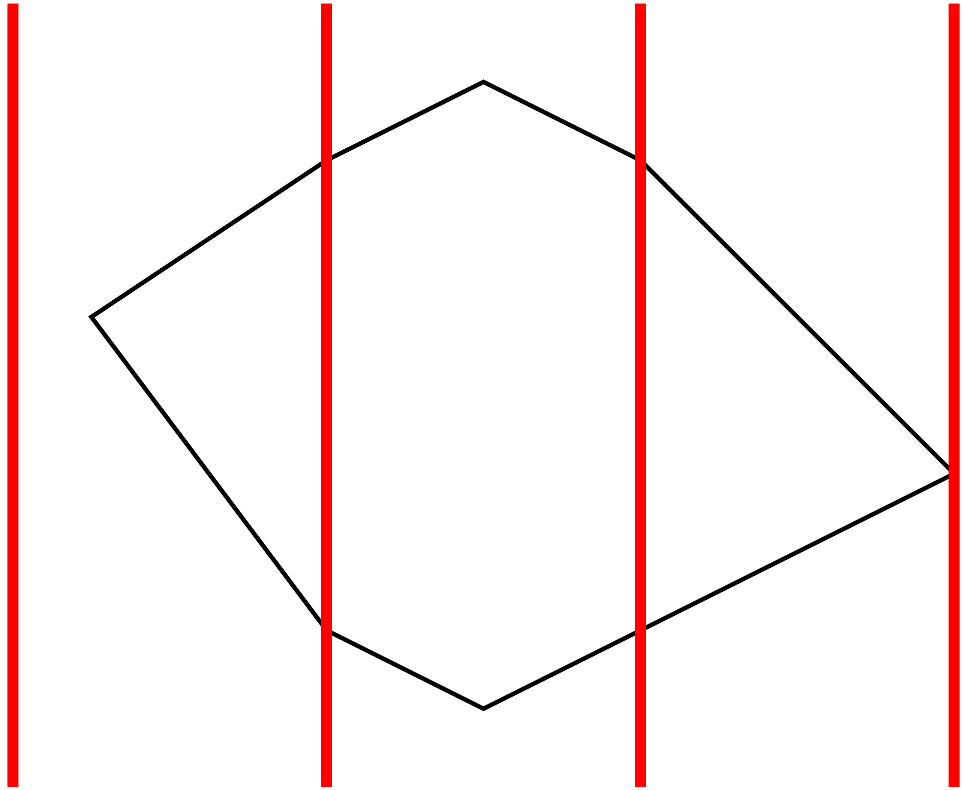
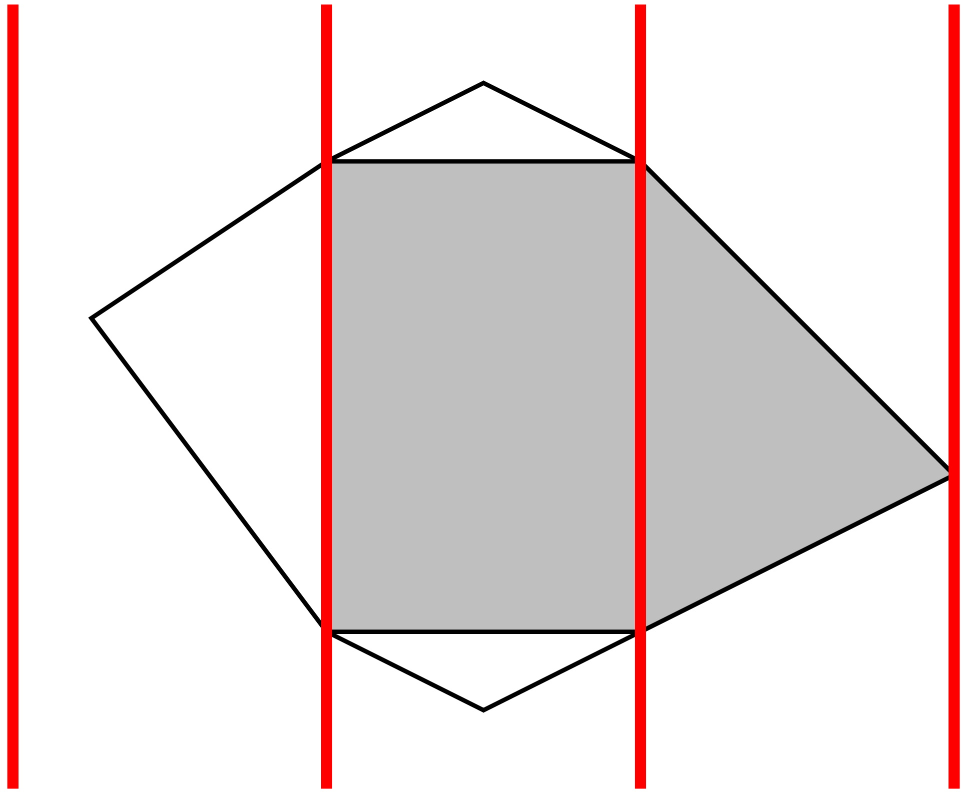


$$Q_\beta = Q \cap y(n) = \beta$$

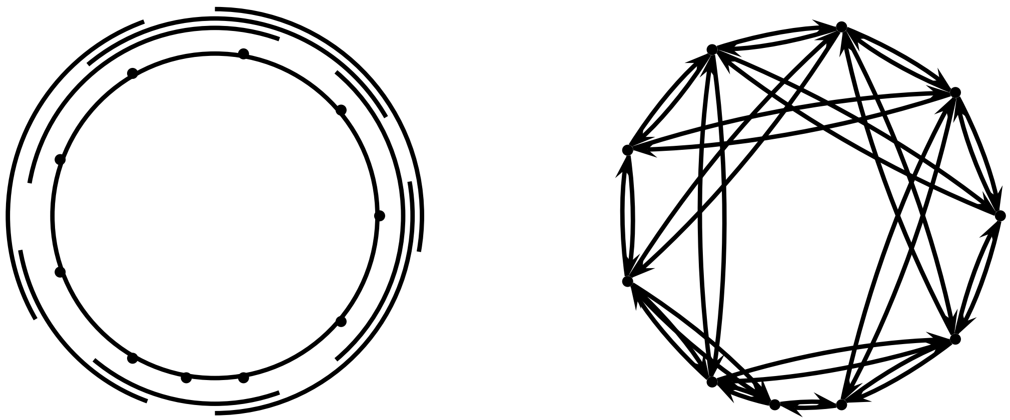
Slicing the polytope



$$Q_\beta = Q \cap y(n) = \beta$$

Separation problem

- Separating a point y lying between two slices Q_β and $Q_{\beta+1}$.
- Reduce to a minimum circulation problem in the auxiliary graph defined by $(N, -N)$



- Detection of negative cost simple cycles

The separation problem

- Given y^* with $y^*(n) = \beta + (1-\mu)$, $0 < \mu < 1$
- $y^* \in Q_I$ if and only if there exist $y_L \in Q_\beta$ and $y_R \in Q_{\beta+1}$ such that

$$y^* = \mu y_L + (1-\mu) y_R$$

- If and only if following system is feasible:

$$\begin{array}{rcl} \overline{y^*} & = & \overline{y_L} + \overline{y_R} \\ N\overline{y_L} & \leqslant & \mu d_L \\ N\overline{y_R} & \leqslant & (1-\mu) d_R \end{array} ,$$

where $d_L = \binom{1}{0} - \beta v$ and $d_R = \binom{1}{0} - (\beta+1)v$.

Using duality

- If and only if the following flow problem has no negative solution

$$\min -f_L N \overline{y^*} + \mu f_L d_L + (1-\mu) f_R d_R$$

$$\begin{aligned} f_L N &= f_R N \\ f_L, f_R &\geqslant 0. \end{aligned}$$

The structure of the facets

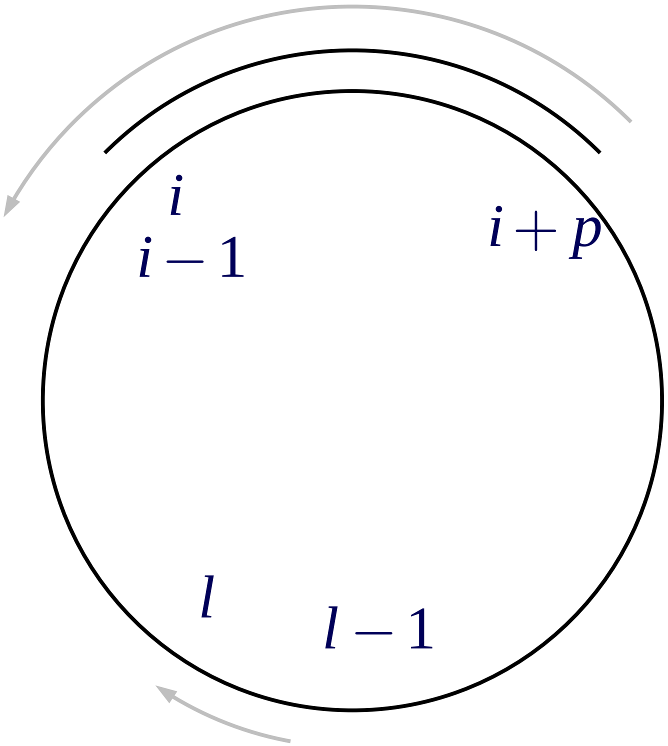
- Let x^* in the relative interior of facet with $\sum x^*(i) = \beta + 1/2$
- Facet is uniquely determined by cycle of zero weight with edge weights given by

$$\left(s^* + \tfrac{1}{2}v\right) f_L + \left(s^* - \tfrac{1}{2}v\right) f_R$$

where s^* is the slack vector

$$s^* = \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix} - \begin{pmatrix} A \\ -I \end{pmatrix} x^* \geqslant \mathbf{0}.$$

The network

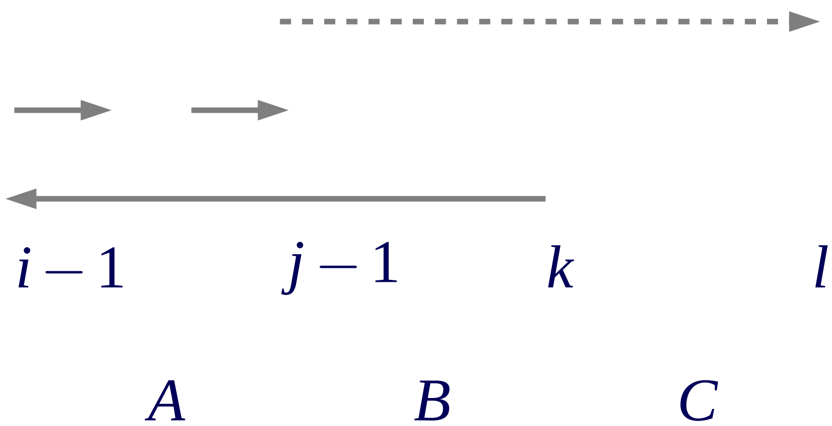


- Row:
  $A = \{i, i+1, \ldots, i+p\}$
  Arc in $\mathcal{S}_L$: $(i+p, i-1)$
  Weight:
  $1 - \sum_{j \in A} x^*(j)$ if $n \notin A$
  $1 - \sum_{j \in A} x^*(j) + 1/2$ if $n \in A$
- Lower bound: $x(l) \geqslant 0$
  Arc in $\mathcal{T}_L$: $(l-1, l)$
  Weight:
  $x^*(l)$ if $l \neq n$
  $x^*(l) - 1/2$ if $l \neq n$

The main lemma

Lemma. *If there exists a simple cycle C of cost 0, then there exists a simple cycle C' of cost 0 such that it does not contain any arc stemming from a clique in the left derivation f_L.*

Sketch of proof



$$1-A-B+A+1-B-C<C$$

if and only if

$$1-B-C<0$$

x^* does not satisfy clique inequalities.

SSP of quasi-line graphs

Theorem (E, Oriolo, Ventura, Stauffer 2005). *The stable set polytope of quasi-line graphs can be characterized by:*

(i) positiveness inequalities

(ii) clique inequalitites

(iii) clique family inequalities

SSP of quasi-line graphs is split closure of fractional SSP.

Split-Cut
Mysteries

Number of facets

- Chvátal closure has polynomial number of facets in fixed dimension (Bockmayr & Eisenbrand 2001)
- Separation of Gomory-Chvátal cutting planes is a MIP (Fischetti & Lodi 2005)

Is P^s polynomial in fixed dimension ?
Can we efficiently separate split cuts in fixed dimension ?

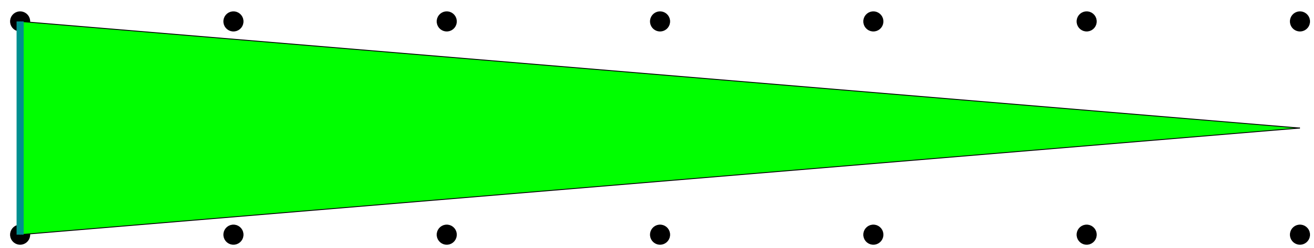
Claw-Free graphs

Still there is no conjecture about the SSP of claw-free graphs.

What is the split-rank of claw-free graphs ?
Is it one ?

More rank questions

Chvátal rank is finite (Chvátal 1973, Schrijver 1980) but can be arbitrarily large; already in dimension 2



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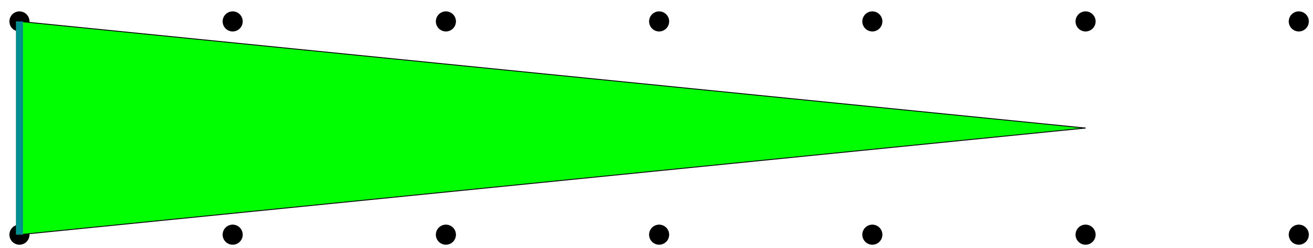

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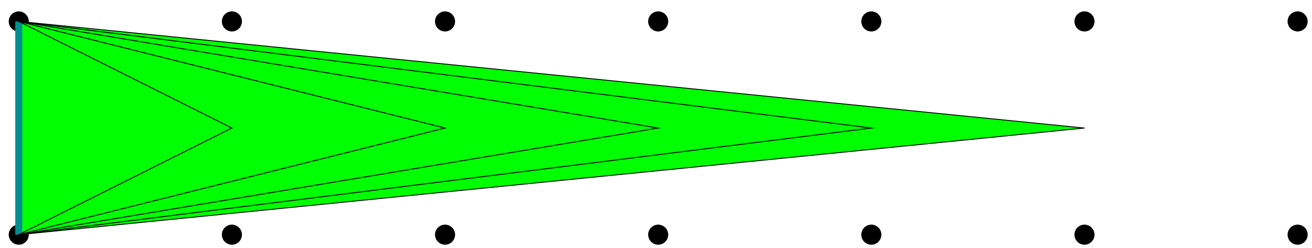

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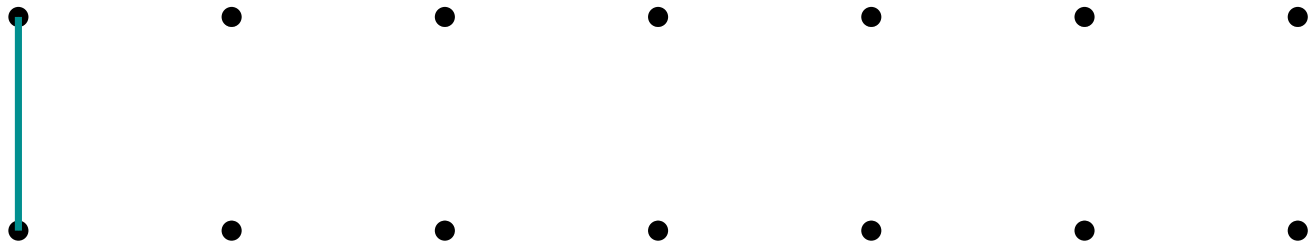

Are there such bad examples in dimension 2 for the split closure ?