BDDs in a Branch & Cut Framework*

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Abstract. Branch & Cut is today's state-of-the-art method to solve $0/1$-integer linear programs. Important for the success of this method is the generation of strong valid inequalities, which tighten the linear programming relaxation of $0/1$-IPs and thus allow for early pruning of parts of the search tree.
In this paper we present a novel approach to generate valid inequalities for $0/1$-IPs which is based on *Binary Decision Diagrams* (BDDs). BDDs are a data-structure which represents $0/1$-vectors as paths of a certain acyclic graph. They have been successfully applied in computational logic, hardware verification and synthesis.
We implemented our BDD cutting plane generator in a branch-and-cut framework and tested it on several instances of the *MAX-ONES* problem and randomly generated $0/1$-IPs. Our computational results show that we have developed competitive code for these problems, on which state-of-the-art MIP-solvers fall short.

1 Introduction

Many industrial optimization problems can be formulated as an integer program. Formally, an integer program deals with the maximization of a linear objective function $c(1)x(1) + \cdots + c(n)x(n)$, where the variables $x(1), \ldots, x(n)$ have to be integers and have to satisfy m given linear inequalities $a_{i1}x(1) + \cdots a_{in}x(n) \leq b_i$ for $1 \leq i \leq m$. A special case of integer programming is ***0/1 integer programming*** ($0/1$-IP), which arises if the variables are additionally restricted to attain values in $\{0, 1\}$. It is a particularly important special case, since most combinatorial optimization problems are modeled with decision variables and thus are $0/1$-IPs.

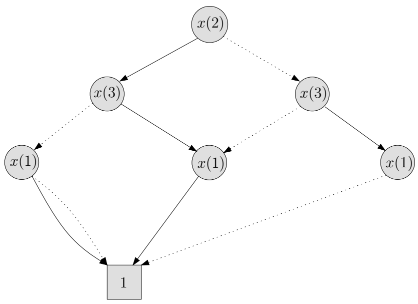
The most successful method for $0/1$-IP, which is applied by all competitive commercial codes is *branch-and-cut*. This variant of branch-and-bound relies on the fact that the *linear relaxation* of a given $0/1$-IP can be efficiently solved. The linear relaxation is the linear program which is obtained from the $0/1$-IP by *relaxing* the condition $x(i) \in \{0, 1\}$ to the condition $0 \leq x(i) \leq 1$ for each $i \in \{1, \ldots, n\}$. The value of the linear programming relaxation can then be used as an upper bound in a branch-and-bound approach to solve the $0/1$-IP. In branch-and-cut, one additionally applies *cutting planes* [8, 26] to improve the quality of the linear programming relaxation. Cutting

* This work was partly supported by the German Research Council (DFG) as part of the Transregional Collaborative Research Center "Automatic Verification and Analysis of Complex Systems" (SFB/TR 14 AVACS). See www.avacs.org for more information.

planes are inequalities which are valid for all feasible integer points, but not necessarily valid for the rational points which are feasible for the linear programming relaxation. Thus the incorporation of cutting planes improves the *tightness* of the linear relaxation and helps to prune parts of the branch-and-bound tree.

In theory a cutting plane can be easily infered from a fractional optimal solution to the linear programming relaxation. The *strength* of the cutting plane is however crucial for the performance of the branch-and-cut process. Classes of strong valid inequalities are for example *knapsack-cover inequalities* [2, 6, 11, 29], *clique inequalities* [21] the *flow-cover inequalities* [23, 24] or the *mixed integer rounding* cuts [21]. Knapsack-cover and flow-cover inequalities in particular are inequalities which are valid for the $0/1$-points which satisfy *one single* constraint of the $0/1$-IP. Up to now there is no satisfactory method available which generates valid inequalities for the $0/1$-solutions of two or more inequalities. This paper aims at a method for this algorithmic problem which is based on *Binary Decision Diagrams*, a datastructure which is widely used in computational logic, hardware verification and logic synthesis.

A Binary Decision Diagram (*BDD*) represents a set of $0/1$-vectors in a compact way, see Fig. 1. We provide a short definition of BDDs as they are used in this paper. A BDD for a set of variables $x(1), \cdots, x(n)$ is a directed acyclic graph $G = (V, A)$ with a labeling $\ell : V \longrightarrow \{x(1), \ldots, x(n)\}$ and a parity function $\mathrm{par} : A \longrightarrow \{0, 1\}$. The graph has one node with in-degree zero, called the *root* and one node with out-degree zero, called *leaf 1*. Each path from root to $\mathrm{leaf}\ 1$ contains exactly n edges and each $x(i)$, $1 \le i \le n$ is the label of a starting node of an edge on this path, thus the BDD is called *complete*. All nodes labelled with $x(i)$ lie on the same level, which means, we have an *ordered* BDD (OBDD). A path $e_1, \ldots, e_n$ from the root to the leaf represents a variable assignment, where the label of the starting node of e_i is assigned to the value $\mathrm{par}(e_i)$. In this way, the BDD *represents* a set of vectors in $\{0, 1\}^n$.



(a) BDD

$x(1)$	$x(2)$	$x(3)$
0	0	1
0	1	0
1	0	0
1	1	0
1	1	1

(b) Represented $0/1$-points

Fig. 1. A simple BDD represented as a directed graph. Edges with parity 0 are dashed.

BDDs were first proposed by Lee in 1959 [20]. Bryant [3] presented efficient algorithms for the synthesis of BDDs. After that, BDDs became very popular in the area of hardware verification, and computational logics, see e.g. [28]. Lai et. al. [19, 18] have developed a branch-and-bound algorithm for $0/1$-IP that uses an extension of BDDs called *EVBDDs*. EVBDDs represent functions $f : \{0,1\}^n \longrightarrow \mathbb{Z}$. So the EVBDDs are used not only to represent the characteristic functions of the constraints but also for the constraints themselves. In this approach however, one has to build an EVBDD for the conjunction of *all* the constraints of the $0/1$-IP. In many cases this leads to an explosion in memory requirement.

We incorporate BDDs into a cutting plane engine and apply it in an integer programming solver. We use BDDs to represent the feasible solutions of a small *subset* of the given constraints and derive valid inequalities for the polytope which is described by these solutions. Thereby we avoid the explosion of the size of the BDD which happens if the BDD represents all the constraints. The *separation problem* is solved with a sequence of shortest path problems with Lagrangean relaxation techniques. For this we use a standard BDD-package and apply our own efficient implementation of an acyclic shortest path algorithm on the BDD-datastructure. We apply our cutting plane framework to the *MAX-ONES* problem and to *randomly generated* $0/1$-IPs. Our computational results show that we could develop competitive code to solve hard $0/1$-integer programming problems, on which state-of-the-art commercial branch-and-cut codes fall short.

Currently there is active and promising research in the field of combining techniques from computational logic and constraint programming with integer programming, see e.g. [5, 13]. We contribute further to this development by using BDDs sucessfully and for the first time in a cutting plane engine.

1.1 Preliminaries from Polyhedral Theory

Before we proceed we review some terminology from polyhedral theory, see e.g. [21, 26]. A *polyhedron* P is a set of vectors of the form $P = \{x \in \mathbb{R}^n \mid Ax \le b\}$, for some matrix $A \in \mathbb{R}^{m \times n}$ and some vector $b \in \mathbb{R}^m$. The polyhedron is *rational* if both A and b can be chosen to be rational. If P is bounded, then P is called a *polytope*. An *integral* $0/1$-*polytope* is a polytope that is the convex hull of a set of $0/1$-vectors $S \subseteq \{0,1\}^n$. The *integer hull* P_{I} of a polytope P is the convex hull of the integral vectors in P. The *dimension* $\dim(P)$ of P is the dimension of its affine hull and $P \subseteq \mathbb{R}^n$ is *full-dimensional* if $\dim(P) = n$.

An inequality $c^T x \le \delta$ is *valid* for P if it is satisfied by all points in P. If $c^T x \le \delta$ is valid and $\delta = \max\{c^T x \mid x \in P\}$, it defines a *face* $F = \{x \in P \mid c^T x = \delta\}$ of P. The face F is a *facet* of P, if $\dim(F) = \dim(P) - 1$.

2 Using BDDs to Generate Cutting Planes

Suppose we have to solve a $0/1$-integer programming problem,

$$\max\{c^T x : Ax \le b,\ x \in \{0,1\}^n\} \tag{1}$$

where $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$ and $c \in \mathbb{Z}^n$. Our idea is now to choose a subset $A'x \le b'$ of the constraints in $Ax \le b$ and to build the BDD which represents all $0/1$-points which satisfy $A'x \le b'$. We next distinguish between two $0/1$-polytopes P_{I} and P_{BDD}. The polytope $P_{\mathrm{I}} = \mathrm{conv}\{x \in \{0,1\}^n \mid Ax \le b\}$ is the convex hull of the feasible $0/1$-points of the $0/1$-IP. The polytope $P_{\mathrm{BDD}} = \mathrm{conv}\{x \in \{0,1\}^n \mid A'x \le b'\}$ is the convex hull of the $0/1$-points which are feasible for $A'x \le b'$. Clearly $P_{\mathrm{BDD}} \supseteq P_{\mathrm{I}}$. We are now interested in an efficient *handling* of the constraints which define P_{BDD}. In a branch-and-cut framework, we want to decide whether our current optimal solution x^* to the linear programming relaxation lies in P_{BDD}. If not, we want to compute an inequality which is valid for P_{BDD} but not valid for x^*. This is the so-called *separation problem* for P_{BDD}.

BDD-SEP Given $x^* \in \mathbb{Q}^n$ and a BDD (G, ℓ, par), decide, whether $x^* \in P_{\mathrm{BDD}}$ and if not, compute a valid inequality for P_{BDD} which is not valid for x^*.

2.1 Polynomial Time Solvability of BDD-SEP

In the 1980's, several authors[9, 17, 22] showed that the linear optimization problem over polyhedra and the separation problem over polyhedra are polynomial time equivalent. This *equivalence of separation and optimization* is a central result in combinatorial optimization. It implies that one can solve the separation problem for P_{BDD} in polynomial time, if one can solve the *optimization problem for* P_{BDD} in polynomial time.

BDD-OPT Given $c \in \mathbb{Q}^n$ and a BDD (G, ℓ, par), compute a $0/1$-point which is represented by (G, ℓ, par) and is maximal w.r.t. the linear objective function $c^T x$.

BDD-OPT is easily shown to be the following longest path problem on G with edge weights $w : E \to \mathbb{R}$, where

$$w(e) = \begin{cases} c(i) & \text{if } \mathrm{par}(e) = 1 \text{ and } \ell(\mathrm{head}(e)) = x(i), \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

It is very easy to see that the optimal solutions to BDD-OPT are exactly the $0/1$-points which are represented by a longest path from root to $\mathrm{leaf}\ 1$. Since G is acyclic, the longest path problem can be solved in linear time. Using the equivalence of separation and optimization, we can thus conclude that BDD-SEP can, in theory, also be efficiently solved.

Theorem 1. *The problems* BDD-SEP *and* BDD-OPT *can be solved in polynomial time.*

2.2 Separation with the Subgradient Method

The point $x^* \notin P_{\mathrm{BDD}}$ if and only if there exists a $\lambda \in \mathbb{R}^n$ such that

$$\lambda^T x^* > \delta_\lambda. \tag{3}$$

The value δ_λ is the length of the longest path from root to $\mathrm{leaf}\ 1$ w.r.t. the edge weights

$$w_\lambda(e) = \begin{cases} \lambda(i) & \text{if } \mathrm{par}(e) = 1 \text{ and } \ell(\mathrm{head}(e)) = x(i), \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Since G is acyclic, δ_λ can be computed in linear time. If (3) does not hold, we update λ as in the subgradient method to solve the Lagrangean relaxation, see e.g. [26, p. 367ff.].

In words, Alg. 1 does the following. The first guess for a normalvector of a separating

Algorithm 1 Subgradient separation routine

(1) $k := 1$
(2) $\lambda^{(k)} := c$
(3) Compute a longest path $p^{(k)}$ from root to $\mathrm{leaf}\ 1$ w.r.t. the edge lengths $w_{\lambda^{(k)}}$.
(4) If $\lambda^T x^* > \delta_\lambda$ then return the separating hyperplane $\lambda^T x \leq \delta_\lambda$
(5) $t^{(k)} := \frac{1}{k}$
(6) $\lambda^{(k+1)} := \lambda^{(k)} + t^{(k)}(x^* - x_{p^{(k)}})$
(7) k := k + 1;
(8) GOTO (3)

hyperplane is the objective function vector c, which is why λ is initialized with this vector. Let $\lambda^{(k)}$ be the normalvector in the k-th iteration such that ${\lambda^{(k)}}^T x^* \leq \delta_{\lambda^{(k)}} = {\lambda^{(k)}}^T x_{p^{(k)}}$. After the update one has ${\lambda^{(k+1)}}^T (x^* - x_{p^{(k)}}) = {\lambda^{(k)}}^T (x^* - x_{p^{(k)}}) + t^{(k)} \|x^* - x_{p^{(k)}}\|^2$. If $t^{(k)} > 0$ is small enough then there exists a longest path w.r.t. $w_{\lambda^{(k)}}$, which is also a longest path w.r.t. $w_{\lambda^{(k+1)}}$. Then ${\lambda^{(k+1)}}^T x^* - \delta_{\lambda^{(k+1)}} > {\lambda^{(k)}}^T x^* - \delta_{\lambda^{(k)}}$. It is known, see [26], that for any $t^{(k)}$ with $\lim_{k\to\infty} t^{(k)} = 0$ and $\sum_{k=1}^{\infty} t^{(k)} = \infty$ the subgradient method terminates. This is the case for $t^{(k)} = 1/k$.

Geometrically, step 6 can be interpreted as a rotation of the hyperplane ${\lambda^{(k)}}^T x \leq \delta_\lambda$ in the direction of the vector $x^* - x_{p^{(k)}}$. Although Alg. 1 cannot be guaranteed to run in polynomial time, we observed that it outperforms linear programming methods for BDD-SEP by far. This is why we implemented this method in our BDD cut-separator.

3 Heuristics for Strengthening Inequalities with BDDs

The inequalities generated with the subgradient method naturally define faces of P_{BDD} with a low dimension. We want to increase their dimension in order to increase the "quality" of the hyperplanes, i.e., we are interested in facets of P_{BDD}. Using facet-defining inequalities in branch-and-cut has led to an enourmous progress in solving large-scale optimization problems, see e.g. [16]. The standard way to turn a separating hyperplane into a facet-defining inequality, see e.g. [10], turned out to be too expensive. Therefore, we developed some heuristics to strengthen inequalities which do not guarantee to produce facets, but can be efficiently implemented.

In the following let $c^T x \leq \delta$ be a valid inequality for P_{BDD}. The right hand side can be set to $\delta = \max\{c^T x \mid x \in P_{\mathrm{BDD}}\}$. We compute the maximum in linear time via optimizing over P_{BDD} with edge weights set to w_c as in (2). Note that with this method every inequality can be made tight at at least one vertex of the BDD-polytope.

3.1 Increasing the Number of Tight Vertices

By increasing the number of vertices of P_{BDD} that are tight at $c^T x \leq \delta$, chances are high to also increase the dimension of the induced face. In the following we try to strengthen an inequality along the unit vectors. Remember that every path in the BDD-graph from the root to $\mathrm{leaf}\ 1$ corresponds to a vertex of P_{BDD}. W.l.o.g. assume that for all $i \in \{1, \ldots, n\}$ the variable $x(i)$ lies in level i.

Given $i \in \{1, \ldots, n\}$ we want to find a new $c(i)$ so that the number of longest paths w.r.t. the edge weights w_c increases. For that we compute the sets of all longest paths, that use a 0-edge resp. 1-edge in level i. Be α_0 resp. α_1 their costs. If $\alpha_0 \neq \alpha_1$ setting $c(i) := c(i) + \alpha_0 - \alpha_1$ and $\delta := \alpha_0$ increases the number of shortest paths.

The strengthened c depends on the order of the indices which we took to strengthen it. Different permutations of $\{1, \ldots, n\}$ can lead to different strengthened hyperplanes. For the computation of $c(i)$ we look at each edge of G once. As we strengthen every coefficient of c the total running time is $\mathrm{O}(n|A|)$. If we do not consider permutations of the indices but take the canonical order $\{1, \ldots, n\}$ we only have to use each edge three times, so the running time can be reduced to $\mathrm{O}(|A|)$.

In a branch-and-cut framework it may occur that a hyperplane separating a given x^* does not separate x^* after strengthening for an index i. In this case we do not change $c(i)$.

3.2 Improving Coefficients

In the following we adapted a strategy known for lifting cover inequalities for the knapsack problem, see e.g. [21]. W.l.o.g. assume that $c \geq 0$ holds. If there exists a $c(i) < 0$ replace $\delta := \delta - c(i)$ and $c(i) := -c(i)$. For simplicity reasons assume we want to strengthen $c(1)$ which means, as we have $x \geq 0$, increasing its value. Rewriting the inequality to $c(1)x(1) \leq \delta - \sum_{i=2}^{n} c(i)x(i)$ shows that we can set $c(1) := \delta - \max\{\sum_{i=2}^{n} c(i)x(i) \mid x \in P_{\mathrm{BDD}}, x(1) = 1\}$. Again we use the fact that we can optimize over P_{BDD} in linear time. Different permutations of indices again lead to different strengthened inequalities.

4 Computational Results

The cuts that we developed in this paper can be used for any $0/1$-integer program, even for those, where nothing is known about their structure. We investigated the practical strength of our theory by doing computational experiments. Our results with MAX-ONES problems and randomly generated IPs show, that one can achieve a considerable speedup on hard and small $0/1$-IPs. We report on some techniques that we developed to build BDDs fast and to keep their sizes small.

4.1 Heuristics for the Variable Order of the BDD

It is well-known that the variable order used in a BDD has a great influence on its size [28]. Before we start to build BDDs for any subset $A'x \leq b'$, we choose an appropriate variable order, which considers all constraints $A\,x \leq b$.

Experiments have shown that heuristics that do not take the structure of the problem into account will produce bad variable orders. We adapted an algorithm for partitioning the outputs of circuits that was presented in [12]. It proceeds as follows: First the constraint set $A\,x \leq b$ is partitioned into subsets with a similar support. Then, for every subset a partial variable order is computed. These partial orders are merged into one total order using a technique called interleaving [7].

For the partitioning of the constraints generate a new initially empty block. Delete that constraint from the set of constraints that has the largest support and insert it into the new block. This constraint is called the leader of the block. Then all constraints satisfying a certain criterion are moved to the new block. We iterate until the set of constraints becomes empty.

The two citeria we used are:

1. Add a constraint if its support is a subset of the support of the leader (WOG).
2. Add a constraint if its support is a subset of the supports of all constraints already contained in the block (BOG).

For the partial orders we used a simple heuristics. For every variable $x(j)$ we computed $h_j = \sum_{i=1}^{n} |a_{ij}|$ and sorted the variables in every block according to decreasing h_j value. Before we apply the interleaving algorithm we sort the blocks increasingly by the number of variables contained in them.

Besides this algorithm that computes an inital variable order, we use sifting [25] to improve the order dynamically while we build the BDDs.

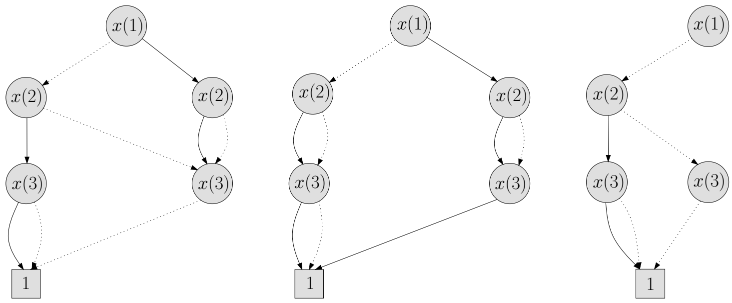
4.2 Building a BDD for a Subset of Constraints

We mainly use the following two operations for BDDs: *computing* the BDD for the *characteristic function of a single constraint* and the *conjunction* of two BDDs. Both BDD algorithms work recursively. The top-most variable is set to 0 and 1 and the algorithm is called recursively on the two branches. Figure 2 shows an example of a conjunction of two BDDs.

4.3 Decreasing the Size of the BDD

Let $a^T x \leq b$ be a constraint of the $0/1$-IP, that has not been used to build the BDD. We set the edge weights in the BDD-graph to w_a as in (2) and optimize over it. A point $x_p \in P_{\text{BDD}}$ satifies the constraint $a^T x \leq b$ if and only if the costs of its corresponding path p are less or equal b. For a given node, consider the costs of all paths that cross it. If the minimum of these costs is greater than b, we can delete that node together with its incident edges, since we are only interested in those points of the BDD-polytope, that satisfy the given constraint.

This algorithm runs in linear time in the number of nodes of the BDD and it can be applied while the BDD is built.



(a) First Operand (b) Second Operand (c) Result

Fig. 2. Conjunction of BDDs: The first operand is the BDD of the characteristic function of the constraint $2x(1) - x(2) + 3x(3) \leq 2$, the second of $-x(1) + x(3) \geq 0$.

4.4 Implementation

To evaluate the effectiveness, we implemented our methods in C++. For building BDDs we used the CUDD 2.4.0 package [27].

Before we build a BDD for the first time, we use our WOG-heuristics for finding a good variable order on the initial $0/1$-IP. In terms of finding a variable order which decreases the size of the BDDs, WOG seems to be slightly better than BOG. If the number of nodes exceeds a given limit while building the BDD, we turn on sifting occasionally. 60.000 turned out to be a good node limit for sifting. If the size of the BDD gets too large, which means, more than 1 million nodes, we stop building it.

The BDDs that we use in our implementation differ from those that we use for theory. In practice we work on reduced BDDs. There are no redundant nodes but long edges that cross levels so that not every path from the root to the $\mathrm{leaf}\ 1$ contains exactly n edges. This reduces memory and time consumption. On the BDD-datastructure used in CUDD we implemented an efficient version of an acyclic shortest path algorithm.

Our separation routine is called in a node of the branch-and-bound tree. To simplify building the BDD, we fix the variables according to the fixation in the branching that led us to this node. In addition to that we restrict the constraints that will be used for building the BDD to some of those of the $0/1$-IP that are tight at the LP solution in the current node. The BDD is a compact representation of all $0/1$-points that are feasible for the given constraints and possibly given fixations. If there are no fixations, the BDD gives an overapproximation of all $0/1$-points that are feasible for the $0/1$-IP. If some variables were fixed while building the BDD, the generated cuts are only valid for that face of P_{BDD}, which corresponds to the given fixations. To make these cuts valid for P_{BDD} we lift them by sequentially solving LPs (see e.g. [21]).

We embedded our separation routines in the cutcallback function of the CPLEX 9.0 Branch & Cut framework [15]. Algorithm 2 sketches our separation routine.

Due to numerical problems the subgradient method sometimes does not terminate. We investigated the steplength of the rotation $t^{(k)}$ and increased the denominator by 1

Algorithm 2 Separation via BDDs as cutcallback function

(1) *Restriction:* Fix some variables according to the branching decisions.
(2) *Build the BDD* for some of the constraints that are tight at the LP solution.
(3) *Solve the separation problem* with the subgradient method.
(4) *Strengthen* the cuts.
(5) *Lift* the strengthened cuts into the original full space and return them.

not in every but in every s'th iteration where $s = 5$ showed to be a good value for most of the cases. If we cannot find a separating hyperplane after 2000 iterations we stop. To make the hyperplanes integer, we multiply them with an adequate integer value and round them. After that we compute a new right-hand-side via a shortest path computation. In almost all of the cases the resulting integer hyperplanes are still separating the current LP solution from the $0/1$-IP.

4.5 Benchmarks

MAX-ONES Satisfiability problems notoriously produce hard to solve IPs [1]. Therefore we investigated SAT instances and converted them to MAX-ONES problems. A given SAT-instance over n boolean variables and a set of clauses $C_1, \ldots, C_k$ can easily be transformed into a $0/1$-IP representing a MAX-ONES problem by converting each clause to a linear constraint of the form $\sum_i x(i) + \sum_j (1 - x(j)) \geq 1$ and adding the objective function $\max \sum_{i=1}^{n} x(i)$. From a SAT competition held in 1992 [4], we took the hfo instances. The 5cnf instances are competition benchmarks of SAT-02 [14], and the remaining instances are competition benchmarks from SAT-03 [14].

Randomly Generated IPs Additionally we are interested in how our code performs on problems with less or without any structure. We randomly generated $0/1$-IPs the following way: an entry in the matrix A, the right-hand-side b and the objective function c gets a nonzero value with probability p. This value is randomly chosen from the integers with absolute value less or equal $c_{\max}$. The instances that we generated are available on request.

4.6 Results

Our experiments have been performed on a PC Intel Xeon CPU 3.06 GHz with 4 GByte RAM on GNU/Linux (kernel 2.6) operating system. Every investigated problem was solved to optimality or proved to be infeasible. On the one hand we run CPLEX 9.0 with the default values, i.e. it did presolving and used all types of built-in separation cuts. On the other hand we used the CPLEX Branch & Cut framework with our separation routines (bcBDD), but switched off presolve and all built-in cuts. We switched off presolve since we sometimes encountered problems working on the presolved model. We also tried to switch off presolve for the benchmarks made with CPLEX standalone. It showed, that presolving the randomly generated IPs does not really influence the running time but switching off presolve for the MAX-ONES instances increased CPLEX running times.

For the *MAX-ONES* instances we found out, that generating nearly all of our cuts in the root node is the most promising strategy. Using too few constraints to build the BDD resulted in weaker cutting planes. In practice it showed that 70% of the constraints, that are tight at the current LP solution, is a good threshold for generating cuts with an adequate quality while building the BDD does not consume too much time.

For *randomly generated IPs* building the BDDs is harder as the constraints have no structure. We generated our cuts deeper in the branch-and-bound tree and lifted them. Furthermore we only used 20% of the constraints, that belong to the basis of the LP solution in the current branch-and-bound node.

Table 1. Results for the SAT-02 / SAT-03 instances

Name	#Var.	#Constr.	solvable	CPLEX(s)	bcBDD(s)	Speedup (%)
5cnf_3800_50f1	50	760	yes	55.59	35.21	36.66
5cnf_3900_060	60	936	no	5000.01	3519.79	29.60
5cnf_3900_070	70	1092	yes	4523.13	3524.89	22.07
5cnf_4000_50f1	50	800	no	183.55	180.21	1.82
5cnf_4000_50f7	50	800	no	252.49	240.19	4.87
5cnf_4000_50t1	50	800	yes	24.21	12.81	47.09
5cnf_4000_50t3	50	800	yes	106.74	89.66	16.00
5cnf_4000_50t8	50	800	yes	125.63	109.32	12.98
5cnf_4000_60t5	60	960	yes	3905.54	3458.66	11.44
5cnf_4100_50f1	50	820	no	291.00	206.07	29.19
5cnf_4100_50f2	50	820	no	237.47	171.19	27.91
5cnf_4100_50f3	50	820	no	253.30	153.63	39.35
5cnf_4100_50f5	50	820	no	259.19	166.43	35.79
5cnf_4100_50f7	50	820	no	380.19	257.91	32.16
5cnf_4100_50t1	50	820	no	242.31	134.71	44.41
icosahedron	30	192	no	184.35	186.81	-1.39
marg2x5	35	120	no	22.52	23.51	-4.40
marg2x6	42	144	no	207.22	237.38	-14.55
marg2x7	49	168	no	3371.32	3330.37	1.21
marg3x3add4	37	160	no	453.39	414.66	8.54
urqh1c2x4	35	216	no	492.38	464.90	5.58
urqh2x3	31	240	no	465.25	413.98	11.02

In all tables the running times are the total user times given in seconds. We computed the speedup as 1 minus the ratio of our running time divided by the CPLEX running time. The values for the hfo instances are average values taken over 20 different instances of each type. The standard deviation is in brackets. For 109 of the 120 hfo-instances we obtain faster running times compared to CPLEX default MIP-solver. The average of the overall speedup for the hfo-instances is 18.31% with a standard deviation of 14.44%. For the randomly generated IPs we achieved an average speedup of 34.23%.

Table 2. Results for the hfo instances

Name	#Var.	#Constr.	solvable	CPLEX(s)	bcBDD(s)	Speedup(%)
hfo5	55	1163	no	3615.93 (770.64)	2510.11 (437.74)	27.32 (21.36)
hfo5	55	1163	yes	1917.11 (1108.12)	1399.99 (764.53)	22.88 (17.60)
hfo6	40	1745	no	966.84 (77.72)	771.48 (58.14)	19.97 (5.94)
hfo6	40	1745	yes	529.44 (256.00)	417.65 (222.52)	21.71 (19.69)
hfo7	32	2807	no	662.65 (60.98)	557.29 (23.68)	15.33 (7.47)
hfo7	32	2807	yes	346.32 (193.71)	302.31 (156.45)	8.93 (10.37)
hfo8	27	4831	no	690.39 (36.73)	592.29 (19.20)	14.02 (4.66)
hfo8	27	4831	yes	352.31 (211.31)	297.77 (171.71)	16.29 (11.10)

Table 3. Results for the random IP instances

Name	#Var.	#Constr.	solvable	p	$c_{\max}$	CPLEX(s)	bcBDD(s)	Speedup (%)
rand50_00	50	40	no	0.6	15	28.30	17.05	39.75
rand50_01	50	50	yes	0.6	13	49.17	17.17	65.08
rand50_02	55	50	no	0.6	15	41.53	33.43	19.50
rand55_00	55	55	no	0.7	17	137.50	108.69	20.95
rand55_01	55	55	yes	0.7	17	85.20	78.61	7.73
rand60_00	60	60	no	0.6	13	151.65	85.60	43.55
rand60_01	60	60	no	0.6	13	104.58	92.49	11.56
rand60_02	60	60	no	0.6	13	237.59	173.62	26.92
rand60_03	60	60	no	0.6	13	191.10	134.90	29.41
rand60_04	60	60	no	0.6	13	155.90	106.82	31.48
rand60_05	60	60	no	0.6	13	285.83	155.83	45.48
rand60_06	60	60	no	0.6	13	678.75	406.58	40.10
rand60_07	60	60	yes	0.6	13	84.33	56.26	33.29
rand60_08	60	60	no	0.6	13	79.10	78.04	1.34
rand70_00	70	70	no	0.6	12	511.62	280.85	45.11
rand80_00	80	80	yes	0.6	4	89.47	31.30	65.02
rand90_00	90	90	yes	0.4	4	192.36	85.45	55.58

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