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POSITIVE EIGENFUNCTIONS OF A SCHRÖDINGER OPERATOR

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Abstract

The paper considers the eigenvalue problem

$$-\Delta u - \alpha u + \lambda g(x)u = 0 \quad \text{with } u \in H^1(\mathbb{R}^N),\ u \neq 0,$$

where $\alpha, \lambda \in \mathbb{R}$ and

$$g(x) \equiv 0 \text{ on } \overline{\Omega}, \quad g(x) \in (0,1] \text{ on } \mathbb{R}^N \setminus \overline{\Omega} \quad \text{and} \quad \lim_{|x| \to +\infty} g(x) = 1$$

for some bounded open set $\Omega \in \mathbb{R}^N$.

Given $\alpha > 0$, does there exist a value of $\lambda > 0$ for which the problem has a positive solution? It is shown that this occurs if and only if α lies in a certain interval (Γ, ξ_1) and that in this case the value of λ is unique, $\lambda = \Lambda(\alpha)$. The properties of the function $\Lambda(\alpha)$ are also discussed.

1. *Introduction*

In this paper we discuss the eigenvalue problem

$$\begin{cases} -\Delta u - \alpha u + \lambda g u = 0 & \text{in } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N), & u \neq 0, \end{cases} \tag{1.1}$$

where the function g has the following properties.

$g \in L^\infty(\mathbb{R}^N, \mathbb{R})$, and there exists a non-empty bounded open set $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary such that $g(x) \equiv 0$ on $\bar{\Omega}$, $g(x) \in (0,1]$ on $\mathbb{R}^N \setminus \overline{\Omega}$ and $\lim_{|x| \to +\infty} g(x) = 1$. (G1)

Thus g represents a potential well that deepens as $\lambda > 0$ increases. In (1.1), both α and λ are real numbers and we are concerned with the following question. Given $\alpha > 0$, does there exist a value of λ for which the problem has a positive solution? More precisely, a number λ is said to be an *eigenvalue* of (1.1) whenever there exists $u \in H^1(\mathbb{R}^N) \backslash \{0\}$ such that

$$\int_{\mathbb{R}^N} [\nabla u \cdot \nabla v - \alpha u v + \lambda g u v]\, dx = 0 \quad \text{for all } v \in H^1(\mathbb{R}^N).$$

In our discussion we take advantage of the additional regularity of eigenfunctions that follows from our assumptions.

Proposition 1.1. *If g satisfies* (G1) *and $v \in H^1(\mathbb{R}^N)$ is an eigenfunction of* (1.1), *then $v \in W^{2,p}(\mathbb{R}^N)$ for all $p \in [2, \infty)$. Hence $v \in C^1(\mathbb{R}^N)$.*

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Proof. See [**9**, Corollary 2.15] for example, or [**7**] for a deeper treatment. □

There are values of α for which (1.1) has no eigenvalues and the following quantities enable us to clarify the situation. Let ξ_1 be the first eigenvalue of the Dirichlet problem

$$\begin{cases} -\Delta\varphi = \xi\varphi & \text{in } \Omega \\ \varphi \in H_0^1(\Omega), & \Omega \text{ is given by (G1).} \end{cases} \tag{1.2}$$

As is well known, $\xi_1 > 0$, and there is a unique eigenfunction satisfying the conditions

$$\int_\Omega \varphi^2\,dx = 1 \quad \text{and} \quad \varphi > 0 \text{ on } \Omega. \tag{1.3}$$

Next set

$$\Gamma = \inf\left\{\int_{\mathbb{R}^N} |\nabla u|^2\,dx : u \in H^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} (1-g)u^2\,dx = 1\right\}. \tag{1.4}$$

We begin by establishing the following result concerning the quantity Γ. □

LEMMA 1.2. *Let* (G1) *be satisfied.*

(i) $0 \leqslant \Gamma < \xi_1$.

(ii) *If* $N = 1, 2$, *then* $\Gamma = 0$.

(iii) *If* $N \geqslant 3$ *and*

$$\ell = \liminf_{|x|\to+\infty} [1-g(x)]|x|^2 > 0,$$

then $\Gamma \leqslant ((N-2)/2)^2/\ell$. *In particular,* $\Gamma = 0$ *if* $\ell = \infty$.

(iv) *If* $N \geqslant 3$ *and* $\|1-g\|_{L^{N/2}(\mathbb{R}^N)} < \infty$, *then* $\Gamma \geqslant S_N/\|1-g\|_{L^{N/2}(\mathbb{R}^N)}$, *where* $S_N := \inf\{\int_{\mathbb{R}^N} |\nabla u|^2\,dx : u \in H^1(\mathbb{R}^N)$ *and* $\int_{\mathbb{R}^N} |u|^{2^*}\,dx = 1\}$ *and* $2^* = 2N/(N-2)$.

REMARK 1.3. Observe that, if there exists $\gamma > 2$ such that

$$\lim_{|x|\to+\infty} \sup[1-g(x)]|x|^\gamma < \infty,$$

then $\|1-g\|_{L^{N/2}(\mathbb{R}^N)} < \infty$, whereas if

$$\ell = \lim_{|x|\to+\infty} \inf[1-g(x)]|x|^2 > 0,$$

then $\|1-g\|_{L^{N/2}(\mathbb{R}^N)} = \infty$.

Furthermore, the value of S_N can be found in [**6**], for example.

Problem (1.1) may have no eigenvalues λ in the interval $(-\infty, \alpha)$. In order to formulate a precise result of this kind, we introduce the following condition.

$$\begin{aligned} &\int_{-\infty}^{\infty} \{1-g(x)\}\,dx < \infty && N = 1 \\ &\lim_{|x|\to\infty} |x|\{1-g(x)\} = 0 && N \geqslant 2. \end{aligned} \tag{G2}$$

We use this condition in the next result to ensure that the Schrödinger operator $-\Delta - \lambda(1-g)$ has no L^2-eigenvalues in the interval $(0,\infty)$. It can be replaced by any other hypothesis that yields the same conclusion, such as [**8**, Theorem XIII.58].

LEMMA 1.4. *Under the hypotheses* (G1) *and* (G2), *problem* (1.1) *has no eigenvalues* λ *in the interval* $(-\infty, \alpha]$.

Proof. If u satisfies (1.1), then

$$-\Delta u - \lambda(1-g)u = (\alpha - \lambda)u,$$

and so $\alpha - \lambda$ is an L^2-eigenvalue of the Schrödinger operator $-\Delta - \lambda(1-g)$. Using (G2) and [**2**, Proposition 10.10], this implies that $\lambda > \alpha$ if $N \geqslant 2$. For $N = 1$, the same conclusion follows from the asymptotic form of all solutions of the differential equation; see the proof of [**8**, Theorem XIII.56] for example. $\square$

Henceforth, we concentrate on the existence of eigenvalues of (1.1) in the interval (α, ∞). Our main results concerning problem (1.1) can be summarized as follows.

THEOREM 1.5. *Let the condition* (G1) *be satisfied.*

(i) *If $\alpha \geqslant \xi_1$, then there is no eigenvalue of* (1.1) *in* $[\alpha, \infty)$ *with a non-negative eigenfunction.*

(ii) *If $\Gamma < \alpha < \xi_1$, then there exists a unique eigenvalue $\lambda = \Lambda(\alpha)$ of* (1.1) *having a positive eigenfunction. Furthermore, $\Lambda(\alpha) > \alpha$, and it is simple in the sense that* $\ker(-\Delta - \alpha + \Lambda(\alpha)g) = \operatorname{span}\{u_{\Lambda(\alpha)}\}$, *where $u_{\Lambda(\alpha)} > 0$ on $\mathbb{R}^N$. All other eigenvalues of* (1.1) *are less than* $\Lambda(\alpha), 1$ *and their eigenfunctions change sign.*

(iii) *The function $\Lambda \in C^\infty((\Gamma, \xi_1))$ and is strictly increasing with*

$$\lim_{\alpha \to \Gamma+} \Lambda(\alpha) = \Gamma \quad \textit{and} \quad \lim_{\alpha \to \xi_1 -} \Lambda(\alpha) = +\infty.$$

(iv) *For $\Gamma < \alpha < \xi_1$, $\Lambda(\alpha)$ is characterized as the unique value of λ for which $\Sigma^\alpha(\lambda) = 0$, where*

$$\Sigma^\alpha(\lambda) = \inf\left\{ a_\lambda(u) : u \in H^1(\mathbb{R}^N) \textit{ and} \int_{\mathbb{R}^N} u^2\, dx = 1 \right\} \tag{1.5}$$

and

$$a_\lambda(u) = \int_{\mathbb{R}^N} |\nabla u|^2 - \alpha u^2 + \lambda g u^2\, dx.$$

In other words, $\Lambda(\alpha)$ is the unique value of λ for which 0 is the infimum of the spectrum of the Schrödinger operator

$$A^\alpha_\lambda u = -\Delta u - (\alpha - \lambda g)u. \tag{1.6}$$

(v) *If $\alpha \leqslant \Gamma$, then problem* (1.1) *has no eigenvalues λ in the interval* (α, ∞).

REMARK 1.6. Of course the alternative point of view in which λ is fixed and we seek values of α for which (1.1) has a solution is the standard eigenvalue for the Schrödinger operator $-\Delta + \lambda g(x)$, and it is well understood. However, even for this problem, our work yields the following non-trivial conclusion. If $\alpha(\lambda)$ denotes the lowest eigenvalue of $-\Delta + \lambda g(x)$, then $\alpha(\lambda)$ increases from Γ to ξ_1 as λ increases from Γ to ∞. A more intuitive form of this result is obtained by shifting the top of the potential well to the level zero. In this case, (1.1) can be written as

$$-\Delta u + \lambda(g-1)u = \rho u,$$

where $\rho = \alpha - \lambda$, and we have

$$\rho(\lambda) = -\lambda + \xi_1 + \circ\left(\frac{1}{\lambda}\right) \qquad \text{as } \lambda \to \infty,$$

where $\rho(\lambda)$ is the lowest eigenvalue of this problem.

Our work involves describing the eigenvalue λ as a function of the parameter α rather than the eigenvalue α as a function of the parameter λ in the traditional treatment. We were confronted by this form of the problem in our work [**10**] on the following nonlinear eigenvalue problem, which has (1.1) as its asymptotic linearization.

$$\begin{cases} -\Delta u + u + \lambda g(x)u = f(u) & \text{in } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N) & \text{with } u \not\equiv 0,\ N \geqslant 1, \end{cases} \tag{1.7}$$

where g satisfies (G1) and f has the following properties.

(F1) $f \in C^1(\mathbb{R}, \mathbb{R})$ and $f(s)/s \to 0$ as $s \to 0$.

(F2) There exists $\alpha > 0$ such that $f(s)/s \to \alpha+1$ as $|s| \to +\infty$ and $0 \leqslant f(s)/s \leqslant \alpha + 1$ for all $s \neq 0$.

Replacing $f(u)$ by its asymptotic linearization $(\alpha+1)u$ leads to (1.1) with $\alpha > 0$.

2. *Proof of Lemma* 1.2

(i) Let $\varphi \in H_0^1(\Omega)$ be an eigenfunction of (1.2) corresponding to ξ_1 with $\int_\Omega \varphi^2\, dx = 1$. Extending φ by zero outside Ω, we construct a function $\widetilde{\varphi} \in H^1(\mathbb{R}^N)$ such that $g\widetilde{\varphi} \equiv 0$, and hence $\int_{\mathbb{R}^N} (1-g)\widetilde{\varphi}^2\, dx = 1$. Thus

$$\int_{\mathbb{R}^N} |\nabla\widetilde{\varphi}|^2\, dx = \int_\Omega |\nabla\varphi|^2\, dx = \xi_1 \int_\Omega \varphi^2\, dx = \xi_1 \int_{\mathbb{R}^N} (1-g)\widetilde{\varphi}^2\, dx,$$

showing that $\Gamma \leqslant \xi_1$. However, if $\Gamma = \xi_1$, it follows that $\widetilde{\varphi} \in H^1(\mathbb{R}^N)$ minimizes $\int_{\mathbb{R}^N} |\nabla u|^2\, dx$ under the constraint $\int_{\mathbb{R}^N} (1-g)u^2\, dx = 1$ and consequently

$$\int_{\mathbb{R}^N} \nabla\widetilde{\varphi} \cdot \nabla v\, dx = \xi_1 \int_{\mathbb{R}^N} (1-g)\widetilde{\varphi} v\, dx \qquad \text{for all } v \in H^1(\mathbb{R}^N).$$

Since $g\widetilde{\varphi} \equiv 0$, on $\mathbb{R}^N$, this implies that $\widetilde{\varphi}$ is an L^2-eigenfunction of $-\Delta$ on $\mathbb{R}^N$. However, as is well known (see [**9**, Theorem 3.8] for example), $-\Delta$ has no such eigenfunctions and hence $\Gamma < \xi_1$.

(ii) By (G1), there exists a function $\psi \in C_0^\infty(\mathbb{R}^N)$ such that $\psi \not\equiv 0$ and $g - 1 \leqslant \psi \leqslant 0$ on $\mathbb{R}^N$. Given any $\varepsilon > 0$, it follows from [**8**, Theorem XIII.11] that there exists $v_\varepsilon \in H^2(\mathbb{R}^N)\backslash\{0\}$ and $\mu_\varepsilon < 0$ such that $(-\Delta + \varepsilon\psi)v_\varepsilon = \mu_\varepsilon v_\varepsilon$. Hence

$$\int_{\mathbb{R}^N} \left[|\nabla v_\varepsilon|^2 + \varepsilon(g-1)v_\varepsilon^2\right] dx \leqslant \int_{\mathbb{R}^N} \left(|\nabla v_\varepsilon|^2 + \varepsilon\psi v_\varepsilon^2\right) dx = \mu_\varepsilon \int_{\mathbb{R}^N} v_\varepsilon^2\, dx < 0,$$

showing that $\Gamma \leqslant \varepsilon$.

(iii) Consider any $T > ((N-2)/2)^2/\ell$. We can choose $\varepsilon \in (0,1)$ and $C = C(\varepsilon) \in (0, \ell)$ such that

$$\left[\frac{N-2}{2} + \varepsilon\right]^2 < TC.$$

There exists $R = R(C) > 0$ such that

$$(1 - g(x))|x|^2 \geqslant C \qquad \text{for all } |x| \geqslant R.$$

Then we set

$$\psi(x) = \begin{cases} 1 & |x| \leqslant R \\ (|x|/R)^{-[(N-2/2)+\varepsilon]} & |x| > R. \end{cases}$$

Now $\psi \notin H^1(\mathbb{R}^N)$, but $\nabla\psi$ and $\psi/|x| \in L^2(\mathbb{R}^N)$ with

$$\int_{|x|\geqslant R} |x|^{-2}\psi(x)^2\,dx = \omega_N R^{N-2+2\varepsilon}\int_R^\infty r^{-1-2\varepsilon}\,dr$$

$$\int_{\mathbb{R}^N} |\nabla\psi(x)|^2\;dx = \omega_N R^{N-2+2\varepsilon}\left[\frac{N-2}{2}+\varepsilon\right]^2\int_R^\infty r^{-1-2\varepsilon}\,dr,$$

where ω_N denotes the $(N-1)$-dimensional measure of the unit sphere in $\mathbb{R}^N$. Hence

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla\psi(x)|^2\;dx\, &-TC\int_{|x|\geqslant R} |x|^{-2}\psi(x)^2\,dx\\ &= \omega_N R^{N-2+2\varepsilon}\left\{\left(\frac{N-2}{2}+\varepsilon\right)^2-TC\right\}\int_R^\infty r^{-1-2\varepsilon}\,dr<0.\end{aligned}$$

Let $\zeta \in C^1(\mathbb{R}^N)$ be such that

$$\zeta(x)\equiv 1 \text{ for } |x|\leqslant 1 \quad\text{ and }\quad \zeta(x)\equiv 0 \text{ for } |x|\geqslant 2,$$

and set $\psi_k(x)=\zeta(x/k)\psi(x)$. It follows that $\psi_k\in H^1(\mathbb{R}^N)$ for any fixed $k\in\mathbb{N}$ with

$$\int_{|x|\geqslant R}|x|^{-2}\psi_k(x)^2\,dx \to \int_{|x|\geqslant R}|x|^{-2}\,\psi(x)^2\,dx$$

as $k\to\infty$. Furthermore,

$$\nabla\psi_k(x)=\frac{1}{k}\psi(x)\nabla\zeta\left(\frac{x}{k}\right)+\zeta\left(\frac{x}{k}\right)\nabla\psi,$$

where

$$\int_{\mathbb{R}^N}\zeta\left(\frac{x}{k}\right)^2|\nabla\psi(x)|^2\,dx\xrightarrow{k}\int_{\mathbb{R}^N}|\nabla\psi(x)|^2\,dx$$

by dominated convergence, and

$$\int_{\mathbb{R}^N}\left[\frac{1}{k}\psi(x)\nabla\zeta\left(\frac{x}{k}\right)\right]^2 dx\xrightarrow{k}0,$$

since

$$\begin{aligned}\int_{\mathbb{R}^N}&\left[\frac{1}{k}\;\psi(x)\nabla\zeta\left(\frac{x}{k}\right)\right]^2 dx\\ &=\left(\int_{|x|\leqslant R}+\int_{|x|\geqslant R}\right)\left[\frac{1}{k}\psi(x)\nabla\zeta\left(\frac{x}{k}\right)\right]^2 dx\\ &\leqslant\frac{C^2}{k^2}\int_{|x|\leqslant R}dx+\frac{1}{k^2}k^N\int_{R/k\leqslant|y|\leqslant 2}|\nabla\zeta(y)|^2\left(\frac{k|y|}{R}\right)^{-N+2-2\varepsilon}dy\\ &\leqslant\frac{C^2}{k^2}\int_{|x|\leqslant R}dx+k^{-2\varepsilon}R^{N-2+2\varepsilon}\int_{1\leqslant|y|\leqslant 2}|\nabla\zeta(y)|^2|y|^{-N+2-2\varepsilon}\,dy\xrightarrow{k}0.\end{aligned}$$

Hence

$$\int_{\mathbb{R}^N}|\nabla\psi_k|^2\;dx\xrightarrow{k}\int_{\mathbb{R}^N}|\nabla\psi|^2\;dx.$$

Therefore there exists k_0 such that

$$\int_{\mathbb{R}^N}|\nabla\psi_k|^2\;dx-TC\int_{|x|\geqslant R}|x|^{-2}{\psi_k}^2\,dx<0\quad\text{for all } k\geqslant k_0.$$

It follows that

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla\psi_k|^2\,dx &- T\int_{\mathbb{R}^N}(1-g){\psi_k}^2\,dx\\ &\leqslant \int_{\mathbb{R}^N} |\nabla\psi_k|^2\,dx - T\int_{|x|\geqslant R}(1-g){\psi_k}^2\,dx\\ &\leqslant \int_{\mathbb{R}^N} |\nabla\psi_k|^2\,dx - TC\int_{|x|\geqslant R}|x|^{-2}{\psi_k}^2\,dx < 0\end{aligned}$$

for all $k \geqslant k_0$, showing that $\Gamma \leqslant T$. Hence $\Gamma \leqslant ((N-2)/2)^2/\ell$. Clearly $\Gamma = 0$ if $\ell = +\infty$.

(iv) For all $u \in H^1(\mathbb{R}^N)$,

$$\begin{aligned}0 \leqslant \int_{\mathbb{R}^N}(1-g)u^2\,dx &\leqslant \left(\int_{\mathbb{R}^N}|1-g|^{N/2}\,dx\right)^{2/N}\left(\int_{\mathbb{R}^N}|u|^{2^*}\,dx\right)^{(N-2)/N}\\ &\leqslant \|1-g\|_{L^{N/2}(\mathbb{R}^N)}\|u\|^2_{L^{2^*}(\mathbb{R}^N)}\\ &\leqslant \|1-g\|_{L^{N/2}(\mathbb{R}^N)}S_N^{-1}\int_{\mathbb{R}^N}|\nabla u|^2\,dx,\end{aligned}$$

and the proof of (iv) is complete. □

3. *Existence and properties of* $\Lambda(\alpha)$

It follows from Proposition 1.1 that any eigenfunction u of problem (1.1) belongs to $C(\mathbb{R}^N)\cap H^2(\mathbb{R}^N)$, and this leads us to introduce a Schrödinger operator having u as an eigenfunction. Define

$$A_\lambda : D(A_\lambda) = H^2(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \longrightarrow L^2(\mathbb{R}^N)$$

by

$$A_\lambda u = -\Delta u - \alpha u + \lambda g u = -\Delta u - (\alpha - \lambda g)u. \tag{3.1}$$

Then A_λ is a self-adjoint operator in $L^2(\mathbb{R}^N)$ with spectrum $\sigma(A_\lambda)$ and essential spectrum $\sigma_e(A_\lambda) = [\lambda-\alpha,\infty)$ (see [**9**, Section 3] for example). Furthermore, setting

$$\Sigma(\lambda) = \inf\sigma(A_\lambda),$$

we have

$$\Sigma(\lambda) = \inf\left\{a_\lambda(u) : u \in H^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} u^2\,dx = 1\right\} > -\infty, \tag{3.2}$$

where

$$a_\lambda(u) = \int_{\mathbb{R}^N}[|\nabla u|^2 - \alpha u^2 + \lambda g u^2]\,dx$$

(see [**9**, Theorem 3.10] for example). In fact, all the quantities just mentioned depend on α as well as λ. In most of the discussion, the value of α is fixed and it is the variation with respect to λ that is of interest. However, when the dependence on α is relevant, we use the more explicit notation

$$A_\lambda^\alpha, \quad a_\lambda^\alpha(u) \quad \text{and} \quad \Sigma^\alpha(\lambda).$$

If we set

$$S_\alpha := \{\lambda \geqslant \alpha : \Sigma^\alpha(\lambda) < 0\} \quad \text{and} \quad T_\alpha := \{\lambda \geqslant \alpha : \Sigma^\alpha(\lambda) > 0\},$$

it is clear from (3.2) that S_α and T_α are intervals since $\Sigma^\alpha(\lambda)$ is non-decreasing in λ.

Lemma 3.1. *If* (G1) *holds and* $\lambda > \alpha$, *we have* $\Sigma(\lambda) = 0$ *if and only if* λ *is an eigenvalue of* (1.1) *with a non-negative eigenfunction* u_λ. *In this case,* 0 *is a simple eigenvalue of* A_λ, $\ker A_\lambda = \operatorname{span}\{u_\lambda\}$ *and* $u_\lambda > 0$ *on* $\mathbb{R}^N$.

Proof. Suppose first that $\Sigma(\lambda) = 0$. Then $0 = \inf \sigma(A_\lambda)$ by (3.2) and $0 < \lambda - \alpha = \inf \sigma_e(A_\lambda)$. Hence 0 is an eigenvalue of A_λ and there exists $u_\lambda \in C(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ such that $\ker A_\lambda = \operatorname{span}\{u_\lambda\}$ and $u_\lambda > 0$ on $\mathbb{R}^N$ (see [**9**, Theorem 3.20] for example). Thus λ is an eigenvalue of (1.1) with eigenfunction u_λ.

Conversely, if λ is an eigenvalue of (1.1) with an eigenfunction $u_\lambda \geqslant 0$, then we have already observed that $u_\lambda \in C(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ and $A_\lambda u_\lambda = 0$. Thus $0 \in \sigma(A_\lambda)$, and so $\Sigma(\lambda) \leqslant 0 < \inf \sigma_e(A_\lambda)$. By [**9**, Theorem 3.20], this implies that $\Sigma(\lambda)$ is a simple eigenvalue of A_λ with a positive eigenfunction $v \in H^2(\mathbb{R}^N)$. Thus

$$\Sigma(\lambda)\langle u_\lambda, v\rangle = \langle u_\lambda, A_\lambda v\rangle = \langle A_\lambda u_\lambda, v\rangle = 0 \quad \text{and} \quad \langle u_\lambda, v\rangle > 0,$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product in $L^2(\mathbb{R}^N)$, showing that $\Sigma(\lambda) = 0$. □

Lemma 3.2. *If* (G1) *holds, then* $\alpha \in S_\alpha$ *if and only if* $\Gamma < \alpha$.

Proof. If $\Sigma^\alpha(\alpha) < 0$, then

$$\inf\left\{\int_{\mathbb{R}^N} |\nabla u|^2 - \alpha(1-g)u^2\,dx : u \in H^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} u^2\,dx = 1\right\} = \Sigma^\alpha(\alpha) < 0,$$

and so there exists $u \in H^1(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} u^2\,dx = 1 \quad \text{and} \quad \int_{\mathbb{R}^N} [|\nabla u|^2 - \alpha(1-g)u^2]\,dx < 0.$$

It follows that $\int_{\mathbb{R}^N} (1-g)u^2\,dx > 0$ and that $\Gamma < \alpha$.

On the other hand, if $\Gamma < \alpha$, then there exists $u \in H^1(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} |\nabla u|^2\,dx < \alpha \int_{\mathbb{R}^N} (1-g)u^2\,dx$, and hence $\Sigma^\alpha(\alpha) < 0$. □

Lemma 3.3. *Let* (G1) *hold.*

(i) S_α *and* T_α *are open subsets of* $[\alpha, +\infty)$.

(ii) *If* $\alpha \geqslant \xi_1$, *then* $S_\alpha = [\alpha, \infty)$.

(iii) *If* $\Gamma < \alpha < \xi_1$, *then there exists* $\Lambda(\alpha) \in (\alpha, +\infty)$ *such that* $S_\alpha = [\alpha, \Lambda(\alpha))$, *where* $\alpha < \Lambda(\alpha) < \infty$.

Proof. (i) By the definition of a_λ, we see that, for all $\lambda, \mu \in \mathbb{R}$ and $u \in H^1(\mathbb{R}^N)$,

$$a_\lambda(u) - a_\mu(u) = (\lambda - \mu)\int_{\mathbb{R}^N} g(x)u^2\,dx. \tag{3.3}$$

Suppose that $\lambda \in S_\alpha$. Then there exists $u \in H^1(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} u(x)^2\,dx = 1 \quad \text{and} \quad a_\lambda(u) < 0.$$

Since

$$a_\mu(u) \leqslant a_\lambda(u) + |\lambda - \mu| \int_{\mathbb{R}^N} gu^2\,dx \leqslant a_\lambda(u) + |\lambda - \mu|,$$

it follows that $\Sigma(\mu) < 0$ for all $\mu \geqslant \alpha$ such that $|\lambda - \mu| \leqslant \frac{1}{2}|a_\lambda(u)|$, showing that S_α is open.

Suppose now that $\lambda \in T_\alpha$. Then for all $u \in H^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} u(x)^2\,dx = 1$, we have

$$a_\mu(u) \geqslant a_\lambda(u) - |\lambda - \mu| \geqslant \Sigma(\lambda) - |\lambda - \mu| \geqslant \tfrac{1}{2}\Sigma(\lambda) > 0$$

for all μ such that $|\lambda - \mu| \leqslant \frac{1}{2}\Sigma(\lambda)$. Thus $\Sigma(\mu) \geqslant \frac{1}{2}\Sigma(\lambda) > 0$ for all μ such that $|\lambda - \mu| \leqslant \frac{1}{2}\Sigma(\lambda)$, showing that T_α is open.

(ii) Let $\varphi_1 \in H_0^1(\Omega)$ be the eigenfunction of (1.2) satisfying (1.3), and set

$$\varphi = \varphi_1 \text{ in } \Omega, \quad \varphi \equiv 0 \text{ in } \mathbb{R}^N \backslash \Omega.$$

We now have $\varphi \in H^1(\mathbb{R}^N)$ and

$$a_\lambda(\varphi) = \int_\Omega \left(|\nabla\varphi_1|^2 - \alpha\varphi_1^2\right) dx = \xi_1 - \alpha \quad \text{and} \quad \int_{\mathbb{R}^N} \varphi^2\,dx = 1,$$

showing that $\Sigma(\lambda) < 0$ if $\alpha > \xi_1$. Furthermore, if $\alpha = \xi_1$ and $\Sigma(\lambda) = 0$, then

$$0 = a_\lambda(\varphi) = \min\left\{\int_{\mathbb{R}^N} a_\lambda(u)\,dx : u \in H^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} u^2\,dx = 1\right\}.$$

Hence there is a Lagrange multiplier $\xi \in \mathbb{R}$ such that

$$\int_{\mathbb{R}^N} \{\nabla\varphi \cdot \nabla v - [\alpha - \lambda g]\varphi v\}\,dx = \xi \int_{\mathbb{R}^N} \varphi v\,dx \qquad \text{for all } v \in H^1(\mathbb{R}^N).$$

Putting $v = \varphi$, we see that $\xi = \xi_1 - \alpha = 0$, and then

$$\int_{\mathbb{R}^N} (\nabla\varphi \cdot \nabla v - \xi_1\varphi v)\,dx = 0 \qquad \text{for all } v \in H^1(\mathbb{R}^N)$$

since $g\varphi \equiv 0$ in $\mathbb{R}^N$. As in the proof of Lemma 1.2(iv), this is in contradiction to the fact that $-\Delta$ has no eigenfunctions in $L^2(\mathbb{R}^N)$. Hence $\Sigma(\lambda) < 0$ if $\alpha = \xi_1$ too.

(iii) Suppose now that $\Gamma < \alpha < \xi_1$. Then $\alpha \in S_\alpha$ by Lemma 3.2, and there exists $\Lambda(\alpha) > \alpha$ such that $S_\alpha = [\alpha, \Lambda(\alpha))$ since S_α is an open subset (interval) of $[\alpha, \infty)$. If $\Lambda(\alpha) = \infty$, then $S_\alpha = [\alpha, +\infty)$, and for any integer $n \geqslant \alpha$, there exists $u_n \in H^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} u_n^2\,dx = 1$ such that

$$a_n(u_n) = \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 - [\alpha - ng]u_n^2\right) dx < 0. \tag{3.4}$$

Since $g(x) \geqslant 0$, this implies that

$$\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx \leqslant \alpha \int_{\mathbb{R}^N} u_n^2\,dx = \alpha,$$

and so $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Passing to a subsequence, still denoted by u_n, we may assume that, for some $u \in H^1(\mathbb{R}^N)$,

$$u_n \overset{n}{\rightharpoonup} u \text{ weakly in } H^1(\mathbb{R}^N), \quad u_n \overset{n}{\to} u \text{ strongly in } L^2_{\mathrm{loc}}(\mathbb{R}^N). \tag{3.5}$$

By (3.4),

$$n\int_{\mathbb{R}^N} g u_n^2\,dx < \alpha \int_{\mathbb{R}^N} u_n^2\,dx = \alpha. \tag{3.6}$$

Since $\lim_{|x|\to+\infty} g(x) = 1$, there exists a compact set $K \subset \mathbb{R}^N$ such that $g(x) \geqslant \frac{1}{2}$ for almost all $x \notin K$. By (3.6), we have

$$\frac{n}{2}\int_{\mathbb{R}^N\setminus K} u_n^2\,dx \leqslant n\int_{\mathbb{R}^N\setminus K} gu_n^2\,dx \leqslant n\int_{\mathbb{R}^N} gu_n^2\,dx < \alpha,$$

that is,

$$\int_{\mathbb{R}^N\setminus K} u_n^2\,dx < \frac{2\alpha}{n},$$

and so

$$1 = \int_{\mathbb{R}^N} u_n^2\,dx = \int_K u_n^2\,dx + \int_{\mathbb{R}^N\setminus K} u_n^2\,dx < \int_K u_n^2\,dx + \frac{2\alpha}{n}.$$

Since K is compact, this implies that

$$1 \leqslant \lim_{n\to\infty}\int_K u_n^2\,dx = \int_K u^2\,dx \leqslant \int_{\mathbb{R}^N} u^2\,dx.$$

However,

$$\int_{\mathbb{R}^N} u^2\,dx \leqslant \liminf_{n\to\infty}\int_{\mathbb{R}^N} u_n^2\,dx = 1$$

and hence

$$\int_{\mathbb{R}^N} u^2\,dx = \int_K u^2\,dx = 1.$$

However,

$$a_n(u_n) = \int_{\mathbb{R}^N}\left(|\nabla u_n|^2 - [\alpha - ng]u_n^2\right)dx \geqslant \int_{\mathbb{R}^N}|\nabla u_n|^2\,dx - \alpha\int_{\mathbb{R}^N} u_n^2\,dx,$$

and, by (3.4),

$$0 \geqslant \liminf_{n\to+\infty} a_n(u_n) \geqslant \int_{\mathbb{R}^N}|\nabla u|^2\,dx - \alpha. \tag{3.7}$$

On the other hand, by (3.6),

$$0 \leqslant \int_{\mathbb{R}^N} gu^2\,dx \leqslant \liminf_{n\to\infty}\int_{\mathbb{R}^N} gu_n^2\,dx \leqslant \liminf_{n\to\infty}\frac{\alpha}{n} = 0.$$

However, $g(x) \equiv 0$ in $\bar{\Omega}$ and $g(x) > 0$ in $\mathbb{R}^N\setminus\overline{\Omega}$ by (G1). Hence this implies that

$$u = 0 \text{ a.e. on } \mathbb{R}^N\setminus\overline{\Omega} \quad \text{and} \quad u = 0 \text{ a.e. on } \mathbb{R}^N\setminus\Omega.$$

Since Ω has a Lipschitz boundary, we have $\widetilde{u} \in H_0^1(\Omega)$, where $\widetilde{u}$ is the restriction of u to Ω (see [**1**, Lemma A 5.11] for example). By (1.2), $\int_\Omega(|\nabla\widetilde{u}|^2 - \xi_1\widetilde{u}^2)\,dx \geqslant 0$. Thus

$$0 \leqslant \int_\Omega(|\nabla\widetilde{u}|^2 - \xi_1\widetilde{u}^2)\,dx = \int_{\mathbb{R}^N}|\nabla u|^2\,dx - \xi_1 < \int_{\mathbb{R}^N}|\nabla u|^2\,dx - \alpha,$$

since $\int_{\mathbb{R}^N} u^2\,dx = 1$ and $\alpha < \xi_1$, which contradicts (3.7). Thus $\Lambda(\alpha) = \sup S_\alpha < +\infty$. □

Lemma 3.4. *Let* (G1) *be satisfied with* $\Gamma < \alpha < \xi_1$, *and consider* $\lambda \geqslant \alpha$. *Then* $\Sigma(\lambda) = 0$ *if and only if* $\lambda = \Lambda(\alpha)$, *where* $\Lambda(\alpha)$ *is given by Lemma* 3.3(iii). *Furthermore,* $\Lambda(\alpha) < \Lambda(\beta)$ *for* $\Gamma < \alpha < \beta < \xi_1$.

Proof. By Lemma 3.2, $\alpha \in S_\alpha$. If $\lambda \geqslant \alpha$ and $\Sigma(\lambda) = 0$, then $\lambda \notin S_\alpha$ and $\lambda > \alpha$. By Lemma 3.1, there exists $u_\lambda \in C(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ with

$$u_\lambda > 0, \quad A_\lambda u_\lambda = 0 \quad \text{and} \quad \int_{\mathbb{R}^N} u_\lambda^2 \, dx = 1.$$

Since $g(x) > 0$ on $\mathbb{R}^N \backslash \overline{\Omega}$,

$$\int_{\mathbb{R}^N} g u_\lambda^2 \, dx \neq 0.$$

For any $\varepsilon > 0$, it follows from (3.3) that

$$a_{\lambda-\varepsilon}(u_\lambda) = a_\lambda(u_\lambda) - \varepsilon \int_{\mathbb{R}^N} g u_\lambda^2 \, dx = -\varepsilon \int_{\mathbb{R}^N} g u_\lambda^2 \, dx < 0,$$

and this means that $\lambda - \varepsilon \in S_\alpha$ for any $\varepsilon > 0$. Therefore $\lambda = \sup S_\alpha = \Lambda(\alpha)$.

Conversely, if $\lambda = \Lambda(\alpha)$, it follows from Lemma 3.3 that $\lambda \notin S_\alpha \cup T_\alpha$, and, since $\lambda \geqslant \alpha$, we must have $\Sigma(\lambda) = 0$.

Consider $\alpha, \beta \in (\Gamma, \xi_1)$ with $\alpha < \beta$. Since $\Sigma^\alpha(\Lambda(\alpha)) = 0$, it follows from Lemma 3.1 that there exists $z_\alpha \in H^2(\mathbb{R}^N)\backslash\{0\}$ such that $\ker A^\alpha_{\Lambda(\alpha)} = \operatorname{span}\{z_\alpha\}$ and hence $a^\alpha_{\Lambda(\alpha)}(z_\alpha) = 0$. However,

$$a^\beta_{\Lambda(\alpha)}(z_\alpha) = a^\alpha_{\Lambda(\alpha)}(z_\alpha) + (\alpha - \beta) \int_{\mathbb{R}^N} z_\alpha^2 \, dx = (\alpha - \beta) \int_{\mathbb{R}^N} z_\alpha^2 \, dx < 0,$$

showing that $\Lambda(\alpha) \in S_\beta$ and consequently $\Lambda(\beta) > \Lambda(\alpha)$. □

Lemma 3.5. *Let $L : X = W^{2,p}(\mathbb{R}^N) \longrightarrow L^p(\mathbb{R}^N)$, where $p \in [2, \infty)$ is a Fredholm operator of index zero. Let $\{v_n\} \subset X$, $v_n \xrightharpoonup{n} v$ weakly in X, and let $\{Lv_n\}$ converge strongly in $L^p(\mathbb{R}^N)$. Then $v_n \xrightarrow{n} v$ strongly in X.*

Proof. Since $L : X \longrightarrow L^p(\mathbb{R}^N)$ is a Fredholm operator of index zero, by [**3**, Chapter I, Theorem 3.15], there exists $T \in \mathcal{B}(L^p(\mathbb{R}^N), X)$ such that

$$TL = I + K,$$

where $K : X \longrightarrow X$ is a compact linear operator. Let $Lv_n \xrightarrow{n} w$ strongly in $L^p(\mathbb{R}^N)$ for some $w \in L^p(\mathbb{R}^N)$; then $(I + K)v_n = TLv_n \xrightarrow{n} Tw$ strongly in X. Since K is compact, it follows that $Kv_n \xrightarrow{n} Kv$ strongly in X. Therefore, $v_n \xrightarrow{n} Tw - Kv$ strongly in X, and hence that $v_n \xrightarrow{n} v = Tw - Kv$ strongly in X. □

4. *Proof of Theorem* 1.5

(i) If $\alpha \geqslant \xi_1$, it follows from Lemma 3.3 that $\Sigma(\lambda) < 0$ for all $\lambda \geqslant \alpha$. Thus

$$\inf \sigma(A_\lambda) = \Sigma(\lambda) < 0 \quad \text{and} \quad \inf \sigma_e(A_\lambda) = \lambda - \alpha \geqslant 0 \quad \text{for} \quad \lambda \geqslant \alpha.$$

Hence there exists $v_\lambda \in C(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ such that $A_\lambda v_\lambda = \Sigma(\lambda) v_\lambda$ and $v_\lambda > 0$ on $\mathbb{R}^N$ (see [**9**, Theorem 3.20] for example). However, if $u \geqslant 0$ satisfies (1.1), it follows from Proposition 1.1 that $u \in C(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ and $A_\lambda u = 0$ on $\mathbb{R}^N$. As in the proof of Lemma 3.1, this leads to a contradiction. Hence (1.1) has no non-negative eigenfunction with $\lambda \geqslant \alpha$.

(ii) We now have $0 \leqslant \Gamma < \alpha < \xi_1$. It follows from Lemma 3.3(iii) and 3.4 that $S_\alpha = [\alpha, \Lambda(\alpha))$, $T_\alpha = (\Lambda(\alpha), \infty)$ and $\lambda = \Lambda(\alpha) > \alpha$ is the unique point in $[\alpha, \infty)$

such that $\Sigma(\lambda) = 0$. By Lemma 3.1, $\Lambda(\alpha)$ is an eigenvalue of (1.1) and 0 is a simple eigenvalue of $A_{\Lambda(\alpha)}$ with $\ker A_{\Lambda(\alpha)} = \mathrm{span}\{z_\alpha\}$, where $z_\alpha = u_{\Lambda(\alpha)} > 0$ on $\mathbb{R}^N$. Suppose now that $\mu \neq \Lambda(\alpha)$ is also an eigenvalue of (1.1) with eigenfunction $w \in H^1(\mathbb{R}^N)$. Then, by Proposition 1.1, $w \in H^2(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ and so 0 is an eigenvalue of A_μ. Since $\Sigma(\mu) = \inf \sigma(A_\mu)$, this shows that $\Sigma(\mu) \leqslant 0$ and hence $\mu \leqslant \sup S_\alpha = \Lambda(\alpha)$. Therefore $\Lambda(\alpha)$ is the largest eigenvalue of (1.1). Furthermore,

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} \{\nabla z_\alpha \cdot \nabla w - \alpha z_\alpha w + \Lambda(\alpha) g(x) z_\alpha w\}\, dx \\ &= \int_{\mathbb{R}^N} \{\nabla w \cdot \nabla z_\alpha - \alpha w z_\alpha + \mu g(x) w z_\alpha\}\, dx \end{aligned}$$

so that

$$(\Lambda(\alpha) - \mu) \int_{\mathbb{R}^N} g(x) z_\alpha w\, dx = 0.$$

For $\mu < \Lambda(\alpha)$, this implies that

$$\int_{\mathbb{R}^N \setminus \overline{\Omega}} g(x) z_\alpha w\, dx = 0.$$

Since $z_\alpha > 0$ and $g(x) > 0$ on $\mathbb{R}^N \setminus \overline{\Omega}$, it follows that either $w \equiv 0$ on $\mathbb{R}^N \setminus \overline{\Omega}$ or w must change sign. However, if $w \equiv 0$ on $\mathbb{R}^N \setminus \overline{\Omega}$, then its restriction $\widetilde{w}$ to Ω belongs to $H^2(\Omega) \cap H_0^1(\Omega) \setminus \{0\}$, since $\partial\Omega$ is Lipschitz (see [**1**, Lemma A 5.11]) and satisfies $-\Delta\widetilde{w} - \alpha\widetilde{w} = 0$ on Ω. However, $\alpha < \xi_1$, so this is impossible, and consequently w must change sign on $\mathbb{R}^N \setminus \overline{\Omega}$.

(iii) By part (ii), we know that for any $\alpha \in (\Gamma, \xi_1)$, there exists $\Lambda(\alpha) \in (\alpha, +\infty)$ such that $\Sigma^\alpha(\Lambda(\alpha)) = 0$, and it is a strictly increasing function of α by Lemma 3.4.

Suppose that $\{\alpha_n\} \subset (\Gamma, \xi_1)$ is an increasing sequence such that $\alpha_n \xrightarrow{n} \xi_1$. Then $\Lambda(\alpha_n) \xrightarrow{n} \Lambda$, where $\Lambda \geqslant \xi_1$, since $\Lambda(\alpha_n) > \alpha_n$. If $\Lambda < \infty$, for any $u \in H^1(\mathbb{R}^N)$, $a^{\alpha_n}_{\Lambda(\alpha_n)}(u) \xrightarrow{n} a^{\xi_1}_\Lambda(u)$. However, by Lemma 3.4, for all $n \in \mathbb{N}$, $0 = \Sigma^{\alpha_n}(\Lambda(\alpha_n)) = \inf\{a^{\alpha_n}_{\Lambda(\alpha_n)}(u) : u \in H^1(\mathbb{R}^N) \text{ and } |u|_2 = 1\}$, and so $a^{\alpha_n}_{\Lambda(\alpha_n)}(u) \geqslant 0$ for all $u \in H^1(\mathbb{R}^N)$. This implies that $a^{\xi_1}_\Lambda(u) \geqslant 0$ for all $u \in H^1(\mathbb{R}^N)$ and hence that $\Sigma^{\xi_1}(\Lambda) = \inf\{a^{\xi_1}_\Lambda(u) : u \in H^1(\mathbb{R}^N) \text{ and } |u|_2 = 1\} \geqslant 0$. This means that $\Lambda \notin S_{\xi_1}$, contradicting the fact that $S_{\xi_1} = [\xi_1, \infty)$, which was established in Lemma 3.3. Thus $\lim_{\alpha \to \xi_1 -} \Lambda(\alpha) = \infty$.

Let $\tau = \lim_{\alpha \to \Gamma +} \Lambda(\alpha)$, and observe that since $\Lambda(\alpha) > \alpha$, we must have $\tau \geqslant \Gamma$. Let us suppose that $\tau > \Gamma$. Consider a decreasing sequence $\{\alpha_n\}$ such that $\alpha_n \xrightarrow{n} \Gamma$. As in part (ii), there exists $\{z_n\} \subset H^2(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ such that $|z_n|_2 = 1$ and

$$-\Delta z_n - \alpha_n z_n + \Lambda(\alpha_n) g z_n = 0 \quad \text{on } \mathbb{R}^N.$$

Hence $\{\Delta z_n\}$ is bounded in $L^2(\mathbb{R}^N)$, from which it follows that $\{z_n\}$ is bounded in $H^2(\mathbb{R}^N)$. Passing to a subsequence, we suppose henceforth that $z_n \overset{n}{\rightharpoonup} z$ weakly in $H^2(\mathbb{R}^N)$. However,

$$-\Delta z_n - \Gamma z_n + \tau g z_n = (\alpha_n - \Gamma) z_n + (\tau - \Lambda(\alpha_n)) g z_n \quad \text{on } \mathbb{R}^N,$$

where $(\alpha_n - \Gamma) z_n + (\tau - \Lambda(\alpha_n)) g z_n \xrightarrow{n} 0$ strongly in $L^2(\mathbb{R}^N)$ and $-\Delta - \Gamma + \tau g : H^2(\mathbb{R}^N) \longrightarrow L^2(\mathbb{R}^N)$ is a Fredholm operator of index zero since $\lim_{|x| \to \infty}\{-\Gamma + \tau g(x)\} = -\Gamma + \tau > 0$ [**5**, Theorem 2.3]. Then Lemma 3.5 implies that $z_n \xrightarrow{n} z$ strongly in $H^2(\mathbb{R}^N)$, and hence $-\Delta z - \Gamma z + \tau g z = 0$ with $|z|_2 = 1$. Furthermore, $\int_{\mathbb{R}^N} g z^2\, dx > 0$, since otherwise $z \equiv 0$ on $\mathbb{R}^N \setminus \Omega$, and we would then have $-\Delta u = \Gamma u$

on $\mathbb{R}^N$, contradicting the fact that $-\Delta$ has no L^2-eigenfunctions on $\mathbb{R}^N$. However, by the definition of Γ, we have

$$0 \leqslant \int_{\mathbb{R}^N} [|\nabla z|^2 - \Gamma(1-g)z^2]\,dx = \int_{\mathbb{R}^N} [\Gamma z^2 - \tau g z^2 - \Gamma(1-g)z^2]\,dx$$
$$= (\Gamma - \tau)\int_{\mathbb{R}^N} gz^2\,dx < 0.$$

This contradiction means that our assumption $\tau > \Gamma$ must be rejected, and so $\tau = \Gamma$.

The smoothness of the function $\Lambda : (\Gamma, \xi_1) \longrightarrow \mathbb{R}$ follows by a standard application of the implicit function theorem to the mapping $\Phi : H^2(\mathbb{R}^N)\times\mathbb{R}\times\mathbb{R} \longrightarrow L^2(\mathbb{R}^N)\times\mathbb{R}$ defined by

$$\Phi(u,\alpha,\lambda) = \Big(-\Delta u - \alpha u + \lambda g u, \int_{\mathbb{R}^N} u^2\,dx - 1\Big).$$

Notice that $\Phi(z_\alpha, \alpha, \Lambda(\alpha)) = 0$ for $\ker A^\alpha_{\Lambda(\alpha)} = \operatorname{span}\{z_\alpha\}$ with $|z_\alpha|_2 = 1$, and that $A^\alpha_{\Lambda(\alpha)} := -\Delta - \alpha + \Lambda(\alpha)g : H^2(\mathbb{R}^N) \longrightarrow L^2(\mathbb{R}^N)$ is a Fredholm operator of index zero, since $\inf \sigma_e(A^\alpha_{\Lambda(\alpha)}) = \Lambda(\alpha) - \alpha > 0$. Furthermore,

$$D_{(u,\lambda)}\Phi(z_\alpha,\alpha,\Lambda(\alpha))(v,\mu) = \Big(A^\alpha_{\Lambda(\alpha)}v + \mu g z_\alpha, 2\int_{\mathbb{R}^N} z_\alpha v\,dx\Big),$$

and, as above, we have $\int_{\mathbb{R}^N} g z_\alpha^2\,dx > 0$, since otherwise z_α would be an L^2-eigenfunction of $-\Delta$ on $\mathbb{R}^N$. It is now straightforward to show that

$$D_{(u,\lambda)}\Phi(z_\alpha,\alpha,\Lambda(\alpha)) : H^2(\mathbb{R}^N)\times\mathbb{R} \longrightarrow L^2(\mathbb{R}^N)\times\mathbb{R}$$

is an isomorphism.

(iv) This follows from Lemma 3.4.

(v) Suppose that u satisfies (1.1) with $\lambda > \alpha$. Then $\int_{\mathbb{R}^N} gu^2\,dx \neq 0$, since otherwise we have $gu \equiv 0$ on $\mathbb{R}^N$ and u would be an L^2-eigenfunction of Δ on $\mathbb{R}^N$, and, as we have already remarked several times, this is false. However, now (1.1) now yields

$$\int_{\mathbb{R}^N} |\nabla u|^2 - \alpha(1-g)u^2\,dx = (\alpha - \lambda)\int_{\mathbb{R}^N} gu^2\,dx < 0,$$

from which it follows that $\int_{\mathbb{R}^N}(1-g)u^2\,dx \neq 0$ and that $\alpha > \Gamma$. □

REMARK 4.1. As a by-product of the proof of the smoothness of $\Lambda(\alpha)$, we obtain the formula

$$\frac{d}{d\alpha}\Lambda(\alpha) = \frac{\int_{\mathbb{R}^N} z_\alpha^2\,dx}{\int_{\mathbb{R}^N} g z_\alpha^2\,dx} = \frac{1}{\int_{\mathbb{R}^N} g z_\alpha^2\,dx} > 0,$$

confirming the strict monotonicity of Λ that was established directly in Lemma 3.4.

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