POSITIVE EIGENFUNCTIONS OF A SCHRÖDINGER OPERATOR

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Abstract

The paper considers the eigenvalue problem

\[-\Delta u - \alpha u + \lambda g(x)u = 0 \text{ with } u \in H^1(\mathbb{R}^N), \ u \neq 0,\]

where \(\alpha, \lambda \in \mathbb{R}\) and

\[g(x) \equiv 0 \text{ on } \Omega, \ g(x) \in (0, 1] \text{ on } \mathbb{R}^N \setminus \Omega \quad \text{and} \quad \lim_{|x| \to +\infty} g(x) = 1\]

for some bounded open set \(\Omega \subset \mathbb{R}^N\).

Given \(\alpha > 0\), does there exist a value of \(\lambda > 0\) for which the problem has a positive solution? It is shown that this occurs if and only if \(\alpha\) lies in a certain interval \((\Gamma, \xi_1)\) and that in this case the value of \(\lambda\) is unique, \(\lambda = \Lambda(\alpha)\). The properties of the function \(\Lambda(\alpha)\) are also discussed.

1. Introduction

In this paper we discuss the eigenvalue problem

\[
\begin{cases}
-\Delta u - \alpha u + \lambda gu = 0 & \text{in } \mathbb{R}^N \\
u \in H^1(\mathbb{R}^N), & u \neq 0,
\end{cases}
\]  

(1.1)

where the function \(g\) has the following properties.

\[g \in L^\infty(\mathbb{R}^N, \mathbb{R}), \text{ and there exists a non-empty bounded open set } \Omega \subset \mathbb{R}^N \]

with Lipschitz boundary such that \(g(x) \equiv 0 \text{ on } \Omega, \ g(x) \in (0, 1] \text{ on } \mathbb{R}^N \setminus \Omega \)

and \(\lim_{|x| \to +\infty} g(x) = 1\).

Thus \(g\) represents a potential well that deepens as \(\lambda > 0\) increases. In (1.1), both \(\alpha\) and \(\lambda\) are real numbers and we are concerned with the following question. Given \(\alpha > 0\), does there exist a value of \(\lambda\) for which the problem has a positive solution? More precisely, a number \(\lambda\) is said to be an eigenvalue of (1.1) whenever there exists \(u \in H^1(\mathbb{R}^N) \setminus \{0\}\) such that

\[
\int_{\mathbb{R}^N} [\nabla u \cdot \nabla v - \alpha uv + \lambda g uv] \, dx = 0 \quad \text{for all } v \in H^1(\mathbb{R}^N).
\]

In our discussion we take advantage of the additional regularity of eigenfunctions that follows from our assumptions.

Proposition 1.1. If \(g\) satisfies (G1) and \(v \in H^1(\mathbb{R}^N)\) is an eigenfunction of (1.1), then \(v \in W^{2,p}(\mathbb{R}^N)\) for all \(p \in [2, \infty)\). Hence \(v \in C^1(\mathbb{R}^N)\).

There are values of $\alpha$ for which (1.1) has no eigenvalues and the following quantities enable us to clarify the situation. Let $\xi_1$ be the first eigenvalue of the Dirichlet problem
\begin{equation}
\begin{cases}
-\Delta \varphi = \xi \varphi & \text{in } \Omega \\
\varphi \in H^1_0(\Omega), & \Omega \text{ is given by (G1)}.
\end{cases}
\end{equation}
As is well known, $\xi_1 > 0$, and there is a unique eigenfunction satisfying the conditions
\begin{equation}
\int_\Omega \varphi^2 \, dx = 1 \quad \text{and} \quad \varphi > 0 \text{ on } \Omega.
\end{equation}
Next set
\begin{equation}
\Gamma = \inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 \, dx : u \in H^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} (1 - g) u^2 \, dx = 1 \right\}.
\end{equation}
We begin by establishing the following result concerning the quantity $\Gamma$.

**Lemma 1.2.** Let (G1) be satisfied.
(i) $0 \leq \Gamma < \xi_1$.
(ii) If $N = 1, 2$, then $\Gamma = 0$.
(iii) If $N \geq 3$ and
\[ \ell = \liminf_{|x| \to +\infty} |1 - g(x)| |x|^2 > 0, \]
then $\Gamma \leq ((N - 2)/2)^2/\ell$. In particular, $\Gamma = 0$ if $\ell = \infty$.
(iv) If $N \geq 3$ and $\|1 - g\|_{L^{N/2}(\mathbb{R}^N)} < \infty$, then $\Gamma \geq S_N/\|1 - g\|_{L^{N/2}(\mathbb{R}^N)}$, where $S_N := \inf \{ \int_{\mathbb{R}^N} |\nabla u|^2 \, dx : u \in H^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} |u|^2 \, dx = 1 \}$ and $2^* = 2N/(N - 2)$.

**Remark 1.3.** Observe that, if there exists $\gamma > 2$ such that
\[ \lim_{|x| \to +\infty} \sup |1 - g(x)| |x|^\gamma < \infty, \]
then $\|1 - g\|_{L^{N/2}(\mathbb{R}^N)} < \infty$, whereas if
\[ \ell = \liminf_{|x| \to +\infty} |1 - g(x)| |x|^2 > 0, \]
then $\|1 - g\|_{L^{N/2}(\mathbb{R}^N)} = \infty$.

Furthermore, the value of $S_N$ can be found in [6], for example.

Problem (1.1) may have no eigenvalues $\lambda$ in the interval $(-\infty, \alpha)$. In order to formulate a precise result of this kind, we introduce the following condition.

\[ \int_{-\infty}^{\infty} |1 - g(x)| \, dx < \infty \quad N = 1 \]
\[ \lim_{|x| \to +\infty} |x| |1 - g(x)| = 0 \quad N \geq 2. \]

We use this condition in the next result to ensure that the Schrödinger operator $-\Delta - \lambda (1 - g)$ has no $L^2$-eigenvalues in the interval $(0, \infty)$. It can be replaced by any other hypothesis that yields the same conclusion, such as [8, Theorem XIII.58].

**Lemma 1.4.** Under the hypotheses (G1) and (G2), problem (1.1) has no eigenvalues $\lambda$ in the interval $(-\infty, \alpha)$.
Proof. If $u$ satisfies (1.1), then
\[-\Delta u - \lambda(1 - g)u = (\alpha - \lambda)u,\]
and so $\alpha - \lambda$ is an $L^2$-eigenvalue of the Schrödinger operator $-\Delta - \lambda(1 - g)$. Using (G2) and [2, Proposition 10.10], this implies that $\lambda > \alpha$ if $N \geq 2$. For $N = 1$, the same conclusion follows from the asymptotic form of all solutions of the differential equation; see the proof of [8, Theorem XIII.56] for example.

Henceforth, we concentrate on the existence of eigenvalues of (1.1) in the interval $(\alpha, \infty)$. Our main results concerning problem (1.1) can be summarized as follows.

**Theorem 1.5.** Let the condition (G1) be satisfied.

(i) If $\alpha \geq \xi_1$, then there is no eigenvalue of (1.1) in $[\alpha, \infty)$ with a non-negative eigenfunction.

(ii) If $\Gamma < \alpha < \xi_1$, then there exists a unique eigenvalue $\lambda = \Lambda(\alpha)$ of (1.1) having a positive eigenfunction. Furthermore, $\Lambda(\alpha) > \alpha$, and it is simple in the sense that $\ker(-\Delta - \alpha + \Lambda(\alpha)g) = \text{span}\{u_{\Lambda(\alpha)}\}$, where $u_{\Lambda(\alpha)} > 0$ on $\mathbb{R}^N$. All other eigenvalues of (1.1) are less than $\Lambda(\alpha)$, and their eigenfunctions change sign.

(iii) The function $\Lambda \in C^\infty((\Gamma, \xi_1))$ and is strictly increasing with
\[
\lim_{\alpha \to \Gamma^+} \Lambda(\alpha) = \Gamma \quad \text{and} \quad \lim_{\alpha \to \xi_1^-} \Lambda(\alpha) = +\infty.
\]

(iv) For $\Gamma < \alpha < \xi_1$, $\Lambda(\alpha)$ is characterized as the unique value of $\lambda$ for which (1.1) has no eigenvalues $\lambda$ in the interval $(\alpha, \infty)$.

(v) If $\alpha \leq \Gamma$, then problem (1.1) has no eigenvalues $\lambda$ in the interval $(\alpha, \infty)$.

**Remark 1.6.** Of course the alternative point of view in which $\lambda$ is fixed and we seek values of $\alpha$ for which (1.1) has a solution is the standard eigenvalue for the Schrödinger operator $-\Delta + \lambda g(x)$, and it is well understood. However, even for this problem, our work yields the following non-trivial conclusion. If $\alpha(\lambda)$ denotes the lowest eigenvalue of $-\Delta + \lambda g(x)$, then $\alpha(\lambda)$ increases from $\Gamma$ to $\xi_1$ as $\lambda$ increases from $\Gamma$ to $\infty$. A more intuitive form of this result is obtained by shifting the top of the potential well to the level zero. In this case, (1.1) can be written as
\[-\Delta u + \lambda(g - 1)u = \rho u,
\]
where $\rho = \alpha - \lambda$, and we have
\[
\rho(\lambda) = -\lambda + \xi_1 + \mathcal{O}\left(\frac{1}{\lambda}\right) \quad \text{as} \ \lambda \to \infty,
\]
where $\rho(\lambda)$ is the lowest eigenvalue of this problem.
Our work involves describing the eigenvalue $\lambda$ as a function of the parameter $\alpha$ rather than the eigenvalue $\alpha$ as a function of the parameter $\lambda$ in the traditional treatment. We were confronted by this form of the problem in our work [10] on the following nonlinear eigenvalue problem, which has (1.1) as its asymptotic linearization.

$$\begin{align*}
-\Delta u + u + \lambda g(x)u &= f(u) \quad \text{in } \mathbb{R}^N \\
u &\in H^1(\mathbb{R}^N) \quad \text{with } u \neq 0, N \geq 1,
\end{align*}$$

(1.7)

where $g$ satisfies (G1) and $f$ has the following properties.

(F1) $f \in C^1(\mathbb{R}, \mathbb{R})$ and $f(s)/s \to 0$ as $s \to 0$.  
(F2) There exists $\alpha > 0$ such that $f(s)/s \to \alpha + 1$ as $|s| \to +\infty$ and $0 \leq f(s)/s \leq \alpha + 1$ for all $s \neq 0$.

Replacing $f(u)$ by its asymptotic linearization $(\alpha + 1)u$ leads to (1.1) with $\alpha > 0$.

2. Proof of Lemma 1.2

(i) Let $\varphi \in H^1_0(\Omega)$ be an eigenfunction of (1.2) corresponding to $\xi_1$ with $\int_\Omega \varphi^2 \, dx = 1$. Extending $\varphi$ by zero outside $\Omega$, we construct a function $\tilde{\varphi} \in H^1(\mathbb{R}^N)$ such that $g\tilde{\varphi} \equiv 0$, and hence $\int_{\mathbb{R}^N} (1 - g)\tilde{\varphi}^2 \, dx = 1$. Thus

$$\int_{\mathbb{R}^N} |\nabla \tilde{\varphi}|^2 \, dx = \int_{\Omega} |\nabla \varphi|^2 \, dx = \xi_1 \int_{\Omega} \varphi^2 \, dx = \xi_1 \int_{\mathbb{R}^N} (1 - g)\tilde{\varphi}^2 \, dx,$$

showing that $\Gamma \leq \xi_1$. However, if $\Gamma = \xi_1$, it follows that $\tilde{\varphi} \in H^1(\mathbb{R}^N)$ minimizes $\int_{\mathbb{R}^N} |\nabla u|^2 \, dx$ under the constraint $\int_{\mathbb{R}^N} (1 - g)u^2 \, dx = 1$ and consequently

$$\int_{\mathbb{R}^N} \nabla \tilde{\varphi} \cdot \nabla v \, dx = \xi_1 \int_{\mathbb{R}^N} (1 - g)\tilde{\varphi}v \, dx \quad \text{for all } v \in H^1(\mathbb{R}^N).$$

Since $g\tilde{\varphi} \equiv 0$, on $\mathbb{R}^N$, this implies that $\tilde{\varphi}$ is an $L^2$-eigenfunction of $-\Delta$ on $\mathbb{R}^N$. However, as is well known (see [9, Theorem 3.8] for example), $-\Delta$ has no such eigenfunctions and hence $\Gamma < \xi_1$.

(ii) By (G1), there exists a function $\psi \in C_0^\infty(\mathbb{R}^N)$ such that $\psi \neq 0$ and $g - 1 \leq \psi \leq 0$ on $\mathbb{R}^N$. Given any $\varepsilon > 0$, it follows from [8, Theorem XIII.11] that there exists $v_\varepsilon \in H^2(\mathbb{R}^N) \setminus \{0\}$ and $\mu_\varepsilon < 0$ such that $(-\Delta + \varepsilon \psi)v_\varepsilon = \mu_\varepsilon v_\varepsilon$. Hence

$$\int_{\mathbb{R}^N} [\nabla v_\varepsilon|^2 + \varepsilon(g - 1)v_\varepsilon^2] \, dx \leq \int_{\mathbb{R}^N} (\nabla \tilde{v}_\varepsilon|^2 + \varepsilon \psi \tilde{v}_\varepsilon^2) \, dx = \mu_\varepsilon \int_{\mathbb{R}^N} \tilde{v}_\varepsilon^2 \, dx < 0,$$

showing that $\Gamma \leq \varepsilon$.

(iii) Consider any $T > ((N - 2)/2)^2/\ell$. We can choose $\varepsilon \in (0, 1)$ and $C = C(\varepsilon) \in (0, \ell)$ such that

$$\left[\frac{N - 2}{2} + \varepsilon\right]^2 < TC.$$

There exists $R = R(C) > 0$ such that

$$(1 - g(x))|x|^2 \geq C \quad \text{for all } |x| \geq R.$$

Then we set

$$\psi(x) = \begin{cases} 1 & |x| \leq R \\ \frac{1}{(|x|/R)^{(N-2)/2}} & |x| > R \end{cases}$$
Now $\psi \notin H^1(\mathbb{R}^N)$, but $\nabla \psi$ and $\psi/|x| \in L^2(\mathbb{R}^N)$ with
\[
\int_{|x| \geq R} |x|^{-2} \psi(x)^2 \, dx = \omega_N R^{N-2+\varepsilon} \int_{R}^\infty r^{-1-2\varepsilon} \, dr
\]
\[
\int_{\mathbb{R}^N} |\nabla \psi(x)|^2 \, dx = \omega_N R^{N-2+\varepsilon} \left[ \frac{N-2}{2} + \varepsilon \right] \int_{R}^\infty r^{-1-2\varepsilon} \, dr,
\]
where $\omega_N$ denotes the $(N-1)$-dimensional measure of the unit sphere in $\mathbb{R}^N$. Hence
\[
\int_{\mathbb{R}^N} |\nabla \psi(x)|^2 \, dx - TC \int_{|x| \geq R} |x|^{-2} \psi(x)^2 \, dx
\]
\[
= \omega_N R^{N-2+\varepsilon} \left\{ \left( \frac{N-2}{2} + \varepsilon \right)^2 - TC \right\} \int_{R}^\infty r^{-1-2\varepsilon} \, dr < 0.
\]
Let $\zeta \in C^1(\mathbb{R}^N)$ be such that $\zeta(x) \equiv 1$ for $|x| \leq 1$ and $\zeta(x) \equiv 0$ for $|x| \geq 2$, and set $\psi_k(x) = \zeta(x/k) \psi(x)$. It follows that $\psi_k \in H^1(\mathbb{R}^N)$ for any fixed $k \in \mathbb{N}$ with
\[
\int_{|x| \geq R} |x|^{-2} \psi_k(x)^2 \, dx \to \int_{|x| \geq R} |x|^{-2} \psi(x)^2 \, dx
\]
as $k \to \infty$. Furthermore,
\[
\nabla \psi_k(x) = \frac{1}{k} \psi(x) \nabla \zeta \left( \frac{x}{k} \right) + \zeta \left( \frac{x}{k} \right) \nabla \psi,
\]
where
\[
\int_{\mathbb{R}^N} \zeta \left( \frac{x}{k} \right)^2 |\nabla \psi(x)|^2 \, dx \to k \int_{\mathbb{R}^N} |\nabla \psi(x)|^2 \, dx
\]
by dominated convergence, and
\[
\int_{\mathbb{R}^N} \left[ \frac{1}{k} \psi(x) \nabla \zeta \left( \frac{x}{k} \right) \right]^2 \, dx \to 0,
\]
since
\[
\int_{\mathbb{R}^N} \left[ \frac{1}{k} \psi(x) \nabla \zeta \left( \frac{x}{k} \right) \right]^2 \, dx
\]
\[
= \left( \int_{|x| \leq R} + \int_{|x| \geq R} \right) \left[ \frac{1}{k} \psi(x) \nabla \zeta \left( \frac{x}{k} \right) \right]^2 \, dx
\]
\[
\leq C^2 \frac{k^2}{k^2} \int_{|x| \leq R} \, dx + \frac{k^2}{k^2} k^N \int_{R/k \leq |y| \leq 2} \frac{|\nabla \zeta(y)|^2 (k|y|/R)}{R}^{-N+2-2\varepsilon} \, dy
\]
\[
\leq C^2 \frac{k^2}{k^2} \int_{|x| \leq R} \, dx + k^{-2\varepsilon} R^{N-2+2\varepsilon} \int_{1 \leq |y| \leq 2} \frac{|\nabla \zeta(y)|^2 |y|^{-N+2-2\varepsilon}}{y} \, dy \to 0.
\]
Hence
\[
\int_{\mathbb{R}^N} |\nabla \psi_k|^2 \, dx \to \int_{\mathbb{R}^N} |\nabla \psi|^2 \, dx.
\]
Therefore there exists $k_0$ such that
\[
\int_{\mathbb{R}^N} |\nabla \psi_k|^2 \, dx - TC \int_{|x| \geq R} |x|^{-2} \psi_k^2 \, dx < 0 \quad \text{for all } k \geq k_0.
\]
It follows that
\[
\int_{\mathbb{R}^N} |\nabla \psi_k|^2 \, dx - T \int_{|x| \geq R} (1 - g) \psi_k^2 \, dx \\
\leq \int_{\mathbb{R}^N} |\nabla \psi_k|^2 \, dx - T \int_{|x| \geq R} |x|^{-2} \psi_k^2 \, dx < 0
\]
for all \( k \geq k_0 \), showing that \( \Gamma \leq T \). Hence \( \Gamma \leq ((N - 2)/2)^2/\ell \). Clearly \( \Gamma = 0 \) if \( \ell = +\infty \).

(iv) For all \( u \in H^1(\mathbb{R}^N) \),
\[
0 \leq \int_{\mathbb{R}^N} (1 - g) u^2 \, dx \leq \left( \int_{\mathbb{R}^N} |1 - g|^{N/2} \, dx \right)^{2/N} \left( \int_{\mathbb{R}^N} |u|^{2^*} \, dx \right)^{(N - 2)/N} \\
\leq \|1 - g\|_{L^{N/2}(\mathbb{R}^N)} \|u\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx,
\]
and the proof of (iv) is complete.

3. Existence and properties of \( \Lambda(\alpha) \)

It follows from Proposition 1.1 that any eigenfunction \( u \) of problem (1.1) belongs to \( C(\mathbb{R}^N) \cap H^2(\mathbb{R}^N) \), and this leads us to introduce a Schrödinger operator having \( u \) as an eigenfunction. Define
\[
A_\lambda : D(A_\lambda) = H^2(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \longrightarrow L^2(\mathbb{R}^N)
\]
by
\[
A_\lambda u = -\Delta u - \alpha u + \lambda g u = -\Delta u - (\alpha - \lambda g) u. \tag{3.1}
\]
Then \( A_\lambda \) is a self-adjoint operator in \( L^2(\mathbb{R}^N) \) with spectrum \( \sigma(A_\lambda) \) and essential spectrum \( \sigma_e(A_\lambda) = [\lambda - \alpha, \infty) \) (see [9, Section 3] for example). Furthermore, setting
\[
\Sigma(\lambda) = \inf \sigma(A_\lambda),
\]
we have
\[
\Sigma(\lambda) = \inf \left\{ a_\lambda(u) : u \in H^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} u^2 \, dx = 1 \right\} > -\infty, \tag{3.2}
\]
where
\[
a_\lambda(u) = \int_{\mathbb{R}^N} [||\nabla u|^2 - \alpha u^2 + \lambda g u^2] \, dx
\]
(see [9, Theorem 3.10] for example). In fact, all the quantities just mentioned depend on \( \alpha \) as well as \( \lambda \). In most of the discussion, the value of \( \alpha \) is fixed and it is the variation with respect to \( \lambda \) that is of interest. However, when the dependence on \( \alpha \) is relevant, we use the more explicit notation
\[
A_\lambda^\alpha, \ a_\lambda^\alpha(u) \text{ and } \Sigma^\alpha(\lambda).
\]
If we set
\[
S_\alpha := \{ \lambda \geq \alpha : \Sigma^\alpha(\lambda) < 0 \} \quad \text{and} \quad T_\alpha := \{ \lambda \geq \alpha : \Sigma^\alpha(\lambda) > 0 \},
\]
it is clear from (3.2) that $S_{\alpha}$ and $T_{\alpha}$ are intervals since $\Sigma^{\alpha}(\lambda)$ is non-decreasing in $\lambda$.

**Lemma 3.1.** If (G1) holds and $\lambda > \alpha$, we have $\Sigma(\lambda) = 0$ if and only if $\lambda$ is an eigenvalue of (1.1) with a non-negative eigenfunction $u_{\lambda}$. In this case, 0 is a simple eigenvalue of $A_{\lambda}$, ker $A_{\lambda} = \text{span}\{u_{\lambda}\}$ and $u_{\lambda} > 0$ on $\mathbb{R}^N$.

**Proof.** Suppose first that $\Sigma(\lambda) = 0$. Then $0 = \inf \sigma(A_{\lambda})$ by (3.2) and $0 < \lambda - \alpha = \inf \sigma_e(A_{\lambda})$. Hence 0 is an eigenvalue of $A_{\lambda}$ and there exists $u_{\lambda} \in C(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ such that ker $A_{\lambda} = \text{span}\{u_{\lambda}\}$ and $u_{\lambda} > 0$ on $\mathbb{R}^N$ (see [9, Theorem 3.20] for example). Thus $\lambda$ is an eigenvalue of (1.1) with eigenfunction $u_{\lambda}$.

Conversely, if $\lambda$ is an eigenvalue of (1.1) with an eigenfunction $u_{\lambda} \geq 0$, then we have already observed that $u_{\lambda} \in C(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ and $A_{\lambda}u_{\lambda} = 0$. Thus $0 \in \sigma(A_{\lambda})$, and so $\Sigma(\lambda) \leq 0 < \inf \sigma_e(A_{\lambda})$. By [9, Theorem 3.20], this implies that $\Sigma(\lambda)$ is a simple eigenvalue of $A_{\lambda}$ with a positive eigenfunction $v \in H^2(\mathbb{R}^N)$. Thus

$$\Sigma(\lambda)(u_{\lambda}, v) = \langle u_{\lambda}, A_{\lambda}v \rangle = \langle A_{\lambda}u_{\lambda}, v \rangle = 0 \quad \text{and} \quad \langle u_{\lambda}, v \rangle > 0,$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product in $L^2(\mathbb{R}^N)$, showing that $\Sigma(\lambda) = 0$. \qed

**Lemma 3.2.** If (G1) holds, then $\alpha \in S_{\alpha}$ if and only if $\Gamma < \alpha$.

**Proof.** If $\Sigma^{\alpha}(\alpha) < 0$, then

$$\inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 - \alpha (1 - g)u^2 \, dx : u \in H^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} u^2 \, dx = 1 \right\} = \Sigma^{\alpha}(\alpha) < 0,$$

and so there exists $u \in H^1(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} u^2 \, dx = 1 \quad \text{and} \quad \int_{\mathbb{R}^N} [|\nabla u|^2 - \alpha (1 - g)u^2] \, dx < 0.$$

It follows that $\int_{\mathbb{R}^N} (1 - g)u^2 \, dx > 0$ and that $\Gamma < \alpha$.

On the other hand, if $\Gamma < \alpha$, then there exists $u \in H^1(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} |\nabla u|^2 \, dx < \alpha \int_{\mathbb{R}^N} (1 - g)u^2 \, dx$, and hence $\Sigma^{\alpha}(\alpha) < 0$.

**Lemma 3.3.** Let (G1) hold.

(i) $S_{\alpha}$ and $T_{\alpha}$ are open subsets of $[\alpha, +\infty)$.

(ii) If $\alpha \geq \xi_1$, then $S_{\alpha} = [\alpha, \infty)$.

(iii) If $\Gamma < \alpha < \xi_1$, then there exists $\Lambda(\alpha) \in (\alpha, +\infty)$ such that $S_{\alpha} = [\alpha, \Lambda(\alpha))$, where $\alpha < \Lambda(\alpha) < \infty$.

**Proof.** (i) By the definition of $a_{\alpha}$, we see that, for all $\lambda, \mu \in \mathbb{R}$ and $u \in H^1(\mathbb{R}^N)$,

$$a_{\lambda}(u) - a_{\mu}(u) = (\lambda - \mu) \int_{\mathbb{R}^N} g(x)u^2 \, dx. \quad (3.3)$$

Suppose that $\lambda \in S_{\alpha}$. Then there exists $u \in H^1(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} u(x)^2 \, dx = 1 \quad \text{and} \quad a_{\lambda}(u) < 0.$$

Since

$$a_{\mu}(u) \leq a_{\lambda}(u) + |\lambda - \mu| \int_{\mathbb{R}^N} g(x)u^2 \, dx \leq a_{\lambda}(u) + |\lambda - \mu|,$$
it follows that $\Sigma(\mu) < 0$ for all $\mu \geq \alpha$ such that $|\lambda - \mu| \leq \frac{1}{2} |a_\alpha(u)|$, showing that $S_\alpha$ is open.

Suppose now that $\lambda \in T_\alpha$. Then for all $u \in H^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} u(x)^2 \, dx = 1$, we have

$$a_\mu(u) \geq a_\alpha(u) - |\lambda - \mu| \geq \Sigma(\lambda) - |\lambda - \mu| \geq \frac{1}{2} \Sigma(\lambda) > 0$$

for all $\mu$ such that $|\lambda - \mu| \leq \frac{1}{2} \Sigma(\lambda)$. Thus $\Sigma(\mu) \geq \frac{1}{2} \Sigma(\lambda) > 0$ for all $\mu$ such that $|\lambda - \mu| \leq \frac{1}{2} \Sigma(\lambda)$, showing that $T_\alpha$ is open.

(ii) Let $\varphi_1 \in H_0^1(\Omega)$ be the eigenfunction of (1.2) satisfying (1.3), and set $\varphi = \varphi_1$ in $\Omega$, $\varphi \equiv 0$ in $\mathbb{R}^N \setminus \Omega$.

We now have $\varphi \in H^1(\mathbb{R}^N)$ and

$$a_\alpha(\varphi) = \int_{\Omega} (|\nabla \varphi_1|^2 - \alpha \varphi_1^2) \, dx = \xi_1 - \alpha$$

and $\int_{\mathbb{R}^N} \varphi^2 \, dx = 1$,

showing that $\Sigma(\lambda) < 0$ if $\alpha > \xi_1$. Furthermore, if $\alpha = \xi_1$ and $\Sigma(\lambda) = 0$, then

$$0 = a_\alpha(\varphi) = \min \left\{ \int_{\mathbb{R}^N} a_\alpha(u) \, dx : u \in H^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} u^2 \, dx = 1 \right\}.$$

Hence there is a Lagrange multiplier $\xi \in \mathbb{R}$ such that

$$\int_{\mathbb{R}^N} \{ \nabla \varphi \cdot \nabla v - [\alpha - \lambda g]\varphi v \} \, dx = \xi \int_{\mathbb{R}^N} \varphi v \, dx \quad \text{for all } v \in H^1(\mathbb{R}^N).$$

Putting $v = \varphi$, we see that $\xi = \xi_1 - \alpha = 0$, and then

$$\int_{\mathbb{R}^N} (\nabla \varphi \cdot \nabla v - \xi_1 \varphi v) \, dx = 0 \quad \text{for all } v \in H^1(\mathbb{R}^N)$$

since $g\varphi \equiv 0$ in $\mathbb{R}^N$. As in the proof of Lemma 1.2(iv), this is in contradiction to the fact that $-\Delta$ has no eigenfunctions in $L^2(\mathbb{R}^N)$. Hence $\Sigma(\lambda) < 0$ if $\alpha = \xi_1$ too.

(iii) Suppose now that $\Gamma < \alpha < \xi_1$. Then $\alpha \in S_\alpha$ by Lemma 3.2, and there exists $\Lambda(\alpha) > \alpha$ such that $S_\alpha = [\alpha, \Lambda(\alpha))$ since $S_\alpha$ is an open subset (interval) of $[\alpha, \infty)$. If $\Lambda(\alpha) = \infty$, then $S_\alpha = [\alpha, +\infty)$, and for any integer $n \geq \alpha$, there exists $u_n \in H^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} u_n^2 \, dx = 1$ such that

$$a_\alpha(u_n) = \int_{\mathbb{R}^N} (|\nabla u_n|^2 - [\alpha - ng]u_n^2) \, dx < 0.$$  \hspace{1cm} (3.4)

Since $g(x) \geq 0$, this implies that

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx \leq \alpha \int_{\mathbb{R}^N} u_n^2 \, dx = \alpha,$$

and so $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Passing to a subsequence, still denoted by $u_n$, we may assume that, for some $u \in H^1(\mathbb{R}^N)$,

$$u_n \rightharpoonup u \text{ weakly in } H^1(\mathbb{R}^N), \quad u_n \to u \text{ strongly in } L^2_{\text{loc}}(\mathbb{R}^N).$$  \hspace{1cm} (3.5)

By (3.4),

$$n \int_{\mathbb{R}^N} gu_n^2 \, dx < \alpha \int_{\mathbb{R}^N} u_n^2 \, dx = \alpha.$$  \hspace{1cm} (3.6)
Since \( \lim_{|x| \to +\infty} g(x) = 1 \), there exists a compact set \( K \subset \mathbb{R}^N \) such that \( g(x) \geq \frac{1}{2} \) for almost all \( x \notin K \). By (3.6), we have

\[
\frac{n}{2} \int_{\mathbb{R}^N \setminus K} u_n^2 \, dx \leq n \int_{\mathbb{R}^N \setminus K} g u_n^2 \, dx \leq n \int_{\mathbb{R}^N} g u_n^2 \, dx < \alpha,
\]

that is,

\[
\int_{\mathbb{R}^N \setminus K} u_n^2 \, dx < \frac{2\alpha}{n},
\]

and so

\[
1 = \int_{\mathbb{R}^N} u_n^2 \, dx = \int_{K} u_n^2 \, dx + \int_{\mathbb{R}^N \setminus K} u_n^2 \, dx < \int_{K} u_n^2 \, dx + \frac{2\alpha}{n}.
\]

Since \( K \) is compact, this implies that

\[
1 \leq \lim_{n \to \infty} \int_{K} u_n^2 \, dx = \int_{K} u^2 \, dx \leq \int_{\mathbb{R}^N} u^2 \, dx.
\]

However,

\[
\int_{\mathbb{R}^N} u^2 \, dx \leq \liminf_{n \to \infty} \int_{\mathbb{R}^N} u_n^2 \, dx = 1
\]

and hence

\[
\int_{\mathbb{R}^N} u^2 \, dx = \int_{K} u^2 \, dx = 1.
\]

However,

\[
a_n(u_n) = \int_{\mathbb{R}^N} (|\nabla u_n|^2 - [\alpha - ng] u_n^2) \, dx \geq \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx - \alpha \int_{\mathbb{R}^N} u_n^2 \, dx,
\]

and, by (3.4),

\[
0 \geq \liminf_{n \to +\infty} a_n(u_n) \geq \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \alpha. \tag{3.7}
\]

On the other hand, by (3.6),

\[
0 \leq \int_{\mathbb{R}^N} g u^2 \, dx \leq \liminf_{n \to \infty} \int_{\mathbb{R}^N} g u_n^2 \, dx \leq \liminf_{n \to \infty} \frac{\alpha}{n} = 0.
\]

However, \( g(x) \equiv 0 \) in \( \tilde{\Omega} \) and \( g(x) > 0 \) in \( \mathbb{R}^N \setminus \tilde{\Omega} \) by (G1). Hence this implies that

\[
u = 0 \text{ a.e. on } \mathbb{R}^N \setminus \tilde{\Omega} \quad \text{and} \quad u = 0 \text{ a.e. on } \mathbb{R}^N \setminus \Omega.
\]

Since \( \Omega \) has a Lipschitz boundary, we have \( \bar{u} \in H^1_0(\Omega) \), where \( \bar{u} \) is the restriction of \( u \) to \( \Omega \) (see [1, Lemma A 5.11] for example). By (1.2), \( \int_{\Omega} (|\nabla \bar{u}|^2 - \xi_1 \bar{u}^2) \, dx \geq 0 \).

Thus

\[
0 \leq \int_{\Omega} (|\nabla \bar{u}|^2 - \xi_1 \bar{u}^2) \, dx = \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \xi_1 < \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \alpha,
\]

since \( \int_{\mathbb{R}^N} u^2 \, dx = 1 \) and \( \alpha < \xi_1 \), which contradicts (3.7). Thus \( \Lambda(\alpha) = \sup S_\alpha < +\infty \). \( \square \)

**Lemma 3.4.** Let (G1) be satisfied with \( \Gamma < \alpha < \xi_1 \), and consider \( \lambda \geq \alpha \). Then \( \Sigma(\lambda) = 0 \) if and only if \( \lambda = \Lambda(\alpha) \), where \( \Lambda(\alpha) \) is given by Lemma 3.3(iii). Furthermore, \( \Lambda(\alpha) < \Lambda(\beta) \) for \( \Gamma < \alpha < \beta < \xi_1 \).
3.1 that there exists $z_\alpha$.

By Lemma 3.1, there exists $u_\lambda \in C(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ with

$$u_\lambda > 0, \quad A_\lambda u_\lambda = 0 \quad \text{and} \quad \int_{\mathbb{R}^N} u_\lambda^2 \, dx = 1. $$

Since $g(x) > 0$ on $\mathbb{R}^N \setminus \overline{\Omega}$,

$$\int_{\mathbb{R}^N} q u_\lambda^2 \, dx \neq 0.$$ 

For any $\varepsilon > 0$, it follows from (3.3) that

$$a_{\lambda - \varepsilon}(u_\lambda) = a_\lambda(u_\lambda) - \varepsilon \int_{\mathbb{R}^N} g u_\lambda^2 \, dx = -\varepsilon \int_{\mathbb{R}^N} g u_\lambda^2 \, dx < 0,$$

and this means that $\lambda - \varepsilon \in S_\alpha$ for any $\varepsilon > 0$. Therefore $\lambda = \sup S_\alpha = \Lambda(\alpha)$.

Conversely, if $\lambda = \Lambda(\alpha)$, it follows from Lemma 3.3 that $\lambda \not\in S_\alpha \cup T_\alpha$, and, since $\lambda \geq \alpha$, we must have $\Sigma(\lambda) = 0$.

Consider $\alpha, \beta \in (\Gamma, \xi_1)$ with $\alpha < \beta$. Since $\Sigma^\beta(\Lambda(\alpha)) = 0$, it follows from Lemma 3.1 that there exists $z_\alpha \in H^2(\mathbb{R}^N) \setminus \{0\}$ such that $\ker A^\beta_{\Lambda(\alpha)} = \text{span}\{z_\alpha\}$ and hence $a^\alpha_{\Lambda(\alpha)}(z_\alpha) = 0$. However,

$$a^\beta_{\Lambda(\alpha)}(z_\alpha) = a^\alpha_{\Lambda(\alpha)}(z_\alpha) + (\alpha - \beta) \int_{\mathbb{R}^N} z_\alpha^2 \, dx = (\alpha - \beta) \int_{\mathbb{R}^N} z_\alpha^2 \, dx < 0,$$

showing that $\Lambda(\alpha) \in S_\beta$ and consequently $\Lambda(\beta) > \Lambda(\alpha)$.

**Lemma 3.5.** Let $L : X = W^{2,p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$, where $p \in [2, \infty)$ is a Fredholm operator of index zero. Let $\{v_n\} \subset X$, $v_n \overset{A}{\to} v$ weakly in $X$, and let $\{Lv_n\}$ converge strongly in $L^p(\mathbb{R}^N)$. Then $v_n \overset{\text{strong}}{\to} v$ strongly in $X$.

**Proof.** Since $L : X \to L^p(\mathbb{R}^N)$ is a Fredholm operator of index zero, by [3, Chapter I, Theorem 3.15], there exists $T \in B(L^p(\mathbb{R}^N), X)$ such that

$$TL = I + K,$$

where $K : X \to X$ is a compact linear operator. Let $Lv_n \overset{n}{\to} w$ strongly in $L^p(\mathbb{R}^N)$ for some $w \in L^p(\mathbb{R}^N)$; then $(I + K)v_n = TLv_n \overset{n}{\to} Tw$ strongly in $X$. Since $K$ is compact, it follows that $Kv_n \overset{n}{\to} Kv$ strongly in $X$. Therefore, $v_n \overset{n}{\to} Tw - Kv$ strongly in $X$, and hence that $v_n \overset{n}{\to} v = Tw - Kv$ strongly in $X$. \qed

4. **Proof of Theorem 1.5**

(i) If $\alpha \geq \xi_1$, it follows from Lemma 3.3 that $\Sigma(\lambda) < 0$ for all $\lambda \geq \alpha$. Thus

$$\inf \sigma(A_\lambda) = \Sigma(\lambda) < 0 \quad \text{and} \quad \inf \sigma_+(A_\lambda) = \lambda - \alpha \geq 0 \quad \text{for} \quad \lambda \geq \alpha.$$ 

Hence there exists $v_\lambda \in C(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ such that $A_\lambda v_\lambda = \Sigma(\lambda)v_\lambda$ and $v_\lambda > 0$ on $\mathbb{R}^N$ (see [9, Theorem 3.20] for example). However, if $u \geq 0$ satisfies (1.1), it follows from Proposition 1.1 that $u \in C(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ and $A_\lambda u = 0$ on $\mathbb{R}^N$. As in the proof of Lemma 3.1, this leads to a contradiction. Hence (1.1) has no non-negative eigenfunction with $\lambda \geq \alpha$.

(ii) We now have $0 \leq \Gamma < \alpha < \xi_1$. It follows from Lemma 3.3(iii) and 3.4 that $S_\alpha = [\alpha, \Lambda(\alpha)]$, $T_\alpha = (\Lambda(\alpha), \infty)$ and $\lambda = \Lambda(\alpha) > \alpha$ is the unique point in $[\alpha, \infty)$.
such that $\Sigma(\lambda) = 0$. By Lemma 3.1, $\Lambda(\alpha)$ is an eigenvalue of (1.1) and 0 is a simple eigenvalue of $A_{\Lambda(\alpha)}$ with $\ker A_{\Lambda(\alpha)} = \text{span}\{z_\alpha\}$, where $z_\alpha = u_{\Lambda(\alpha)} > 0$ on $\mathbb{R}^N$. Suppose now that $\mu \neq \Lambda(\alpha)$ is also an eigenvalue of (1.1) with eigenfunction $w \in H^1(\mathbb{R}^N)$. Then, by Proposition 1.1, $w \in H^2(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ and so 0 is an eigenvalue of $A_\mu$. Since $\Sigma(\mu) = \inf \sigma(A_\mu)$, this shows that $\Sigma(\mu) \leq 0$ and hence $\mu \leq \sup S_\alpha = \Lambda(\alpha)$. Therefore $\Lambda(\alpha)$ is the largest eigenvalue of (1.1). Furthermore,

$$
0 = \int_{\mathbb{R}^N} \{\nabla z_\alpha \cdot \nabla w - \alpha z_\alpha w + \Lambda(\alpha)g(x)z_\alpha w\} \, dx
$$

$$
= \int_{\mathbb{R}^N} \{\nabla w \cdot \nabla z_\alpha - \alpha wz_\alpha + \mu g(x)wz_\alpha\} \, dx
$$

so that

$$
(\Lambda(\alpha) - \mu) \int_{\mathbb{R}^N} g(x)z_\alpha w \, dx = 0.
$$

For $\mu < \Lambda(\alpha)$, this implies that

$$
\int_{\mathbb{R}^N \setminus \Omega} g(x)z_\alpha w \, dx = 0.
$$

Since $z_\alpha > 0$ and $g(x) > 0$ on $\mathbb{R}^N \setminus \Omega$, it follows that either $w \equiv 0$ on $\mathbb{R}^N \setminus \Omega$ or $w$ must change sign. However, if $w \equiv 0$ on $\mathbb{R}^N \setminus \Omega$, then its restriction $\bar{w}$ to $\Omega$ belongs to $H^2(\Omega) \cap H_0^1(\Omega) \setminus \{0\}$, since $\partial \Omega$ is Lipschitz (see [1, Lemma A 5.11]) and satisfies $-\Delta \bar{w} - \alpha \bar{w} = 0$ on $\Omega$. However, $\alpha < \xi_1$, so this is impossible, and consequently $w$ must change sign on $\mathbb{R}^N \setminus \Omega$.

(iii) By part (ii), we know that for any $\alpha \in (\Gamma, \xi_1)$, there exists $\Lambda(\alpha) \in (\alpha, +\infty)$ such that $\Sigma^\alpha(\Lambda(\alpha)) = 0$, and it is a strictly increasing function of $\alpha$ by Lemma 3.4.

Suppose that $\{\alpha_n\} \subset (\Gamma, \xi_1)$ is an increasing sequence such that $\alpha_n \overset{n}{\to} \xi_1$. Then $\Lambda(\alpha_n) \overset{n}{\to} \Lambda$, where $\Lambda > \xi_1$, since $\Lambda(\alpha_n) > \alpha_n$. If $\Lambda < \infty$, for any $u \in H^1(\mathbb{R}^N)$, $a_{\Lambda(\alpha_n)}^\alpha(u) \overset{n}{\to} a_{\Lambda}^\xi(u)$. However, by Lemma 3.4, for all $n \in \mathbb{N}$, $0 = \Sigma^\alpha(\Lambda(\alpha_n)) = \inf\{a_{\Lambda(\alpha_n)}^\alpha(u) : u \in H^1(\mathbb{R}^N) \text{ and } |u|_2 = 1\}$, and so $a_{\Lambda(\alpha_n)}^\alpha(u) \geq 0$ for all $u \in H^1(\mathbb{R}^N)$. This implies that $a_{\Lambda}^\xi(u) \geq 0$ for all $u \in H^1(\mathbb{R}^N)$ and hence that $\Sigma^\xi(\Lambda) = \inf\{a_{\Lambda}^\xi(u) : u \in H^1(\mathbb{R}^N) \text{ and } |u|_2 = 1\} > 0$. This means that $\Lambda \in S_{\xi_1}$, contradicting the fact that $S_{\xi_1} = [\xi_1, \infty)$, which was established in Lemma 3.3. Thus $\lim_{n} \alpha_n = \Lambda(\alpha) = \infty$.

Let $\tau = \lim_{\alpha_n \to \Gamma} \Lambda(\alpha)$, and observe that since $\Lambda(\alpha) > \alpha$, we must have $\tau \geq \Gamma$. Let us suppose that $\tau > \Gamma$. Consider a decreasing sequence $\{\alpha_n\}$ such that $\alpha_n \overset{n}{\to} \Gamma$. As in part (ii), there exists $\{z_n\} \subset H^2(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ such that $|z_n|_2 = 1$ and

$$
-\Delta z_n - \alpha_n z_n + \Lambda(\alpha_n)g z_n = 0 \quad \text{on } \mathbb{R}^N.
$$

Hence $\{z_n\}$ is bounded in $L^2(\mathbb{R}^N)$, from which it follows that $\{z_n\}$ is bounded in $H^2(\mathbb{R}^N)$. Passing to a subsequence, we suppose henceforth that $z_n \overset{n}{\rightharpoonup} z$ weakly in $H^2(\mathbb{R}^N)$. However,

$$
-\Delta z_n - \Gamma z_n + \tau g z_n = (\alpha_n - \Gamma)z_n + (\tau - \Lambda(\alpha_n))g z_n \quad \text{on } \mathbb{R}^N,
$$

where $(\alpha_n - \Gamma)z_n + (\tau - \Lambda(\alpha_n))g z_n \rightharpoonup 0$ strongly in $L^2(\mathbb{R}^N)$ and $-\Delta - \Gamma + \tau g : H^2(\mathbb{R}^N) \rightharpoonup L^2(\mathbb{R}^N)$ is a Fredholm operator of index zero since $\lim_{|x| \to \infty} \{-\Gamma + \tau g(x)\} = -\Gamma + \tau > 0$ [5, Theorem 2.3]. Then Lemma 3.5 implies that $z_n \overset{n}{\to} z$ strongly in $H^2(\mathbb{R}^N)$, and hence $-\Delta z - \Gamma z + \tau g z = 0$ with $|z|_2 = 1$. Furthermore, $\int_{\mathbb{R}^N} g z^2 dx > 0$, since otherwise $z \equiv 0$ on $\mathbb{R}^N \setminus \Omega$, and we would then have $-\Delta u = \Gamma u$.
on \( \mathbb{R}^N \), contradicting the fact that \(-\Delta\) has no \(L^2\)-eigenfunctions on \( \mathbb{R}^N \). However, by the definition of \( \Gamma \), we have

\[
0 \leq \int_{\mathbb{R}^N} [\|\nabla z\|^2 - \Gamma (1 - g) z^2] \, dx = \int_{\mathbb{R}^N} [\Gamma z^2 - \tau g z^2 - \Gamma (1 - g) z^2] \, dx
\]

\[
= (\Gamma - \tau) \int_{\mathbb{R}^N} g z^2 \, dx < 0.
\]

This contradiction means that our assumption \( \tau > \Gamma \) must be rejected, and so \( \tau = \Gamma \). The smoothness of the function \( \Lambda : (\Gamma, \xi_1) \to \mathbb{R} \) follows by a standard application of the implicit function theorem to the mapping \( \Phi : H^2(\mathbb{R}^N) \times \mathbb{R} \times \mathbb{R} \to L^2(\mathbb{R}^N) \times \mathbb{R} \) defined by

\[
\Phi(u, \alpha, \lambda) = \left( -\Delta u - \alpha u + \lambda g u, \int_{\mathbb{R}^N} u^2 \, dx - 1 \right).
\]

Notice that \( \Phi(z_\alpha, \alpha, \Lambda(\alpha)) = 0 \) for \( \text{ker} \, A_{\Lambda(\alpha)}^\alpha = \text{span} \{ z_\alpha \} \) with \(|z_\alpha|^2 = 1\), and that \( A_{\Lambda(\alpha)}^\alpha := -\Delta - \alpha + \Lambda(\alpha) g : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N) \) is a Fredholm operator of index zero, since \( \inf \sigma_e(A_{\Lambda(\alpha)}^\alpha) = \Lambda(\alpha) - \alpha > 0 \). Furthermore,

\[
D_{(u, \lambda)} \Phi(z_\alpha, \alpha, \Lambda(\alpha))(v, \mu) = \left( A_{\Lambda(\alpha)}^\alpha v + \mu g z_\alpha, 2 \int_{\mathbb{R}^N} z_\alpha v \, dx \right),
\]

and, as above, we have \( \int_{\mathbb{R}^N} g z_\alpha^2 \, dx > 0 \), since otherwise \( z_\alpha \) would be an \( L^2 \)-eigenfunction of \(-\Delta\) on \( \mathbb{R}^N \). It is now straightforward to show that

\[
D_{(u, \lambda)} \Phi(z_\alpha, \alpha, \Lambda(\alpha)) : H^2(\mathbb{R}^N) \times \mathbb{R} \to L^2(\mathbb{R}^N) \times \mathbb{R}
\]

is an isomorphism.

(iv) This follows from Lemma 3.4.

(v) Suppose that \( u \) satisfies (1.1) with \( \lambda > \alpha \). Then \( \int_{\mathbb{R}^N} g u^2 \, dx \neq 0 \), since otherwise we have \( g u \equiv 0 \) on \( \mathbb{R}^N \) and \( u \) would be an \( L^2 \)-eigenfunction of \( \Delta \) on \( \mathbb{R}^N \), and, as we have already remarked several times, this is false. However, now (1.1) now yields

\[
\int_{\mathbb{R}^N} |\nabla u|^2 - \alpha (1 - g) u^2 \, dx = (\alpha - \lambda) \int_{\mathbb{R}^N} g u^2 \, dx < 0,
\]

from which it follows that \( \int_{\mathbb{R}^N} (1 - g) u^2 \, dx \neq 0 \) and that \( \alpha > \Gamma \).

**Remark 4.1.** As a by-product of the proof of the smoothness of \( \Lambda(\alpha) \), we obtain the formula

\[
\frac{d}{d\alpha} \Lambda(\alpha) = \frac{\int_{\mathbb{R}^N} z_\alpha^2 \, dx}{\int_{\mathbb{R}^N} g z_\alpha^2 \, dx} = \frac{1}{\int_{\mathbb{R}^N} g z_\alpha^2 \, dx} > 0,
\]

confirming the strict monotonicity of \( \Lambda \) that was established directly in Lemma 3.4.

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