Effect of Parallel Magnetic Field Fluctuations (Finite Beta) in Linear Gyro-Kinetic Stability Analysis of Drift Waves

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Abstract

It is believed that electromagnetic aspects play a crucial role in turbulent transport of finite beta tokamaks driven by microinstabilities. Recent studies have revealed the existence of an electromagnetic mode called Alfvén-ITG (AITG). Nevertheless in this study, the magnetic field fluctuations were modelled by taking into account only the perpendicular component of the fluctuating B-field.

The present work is an attempt to formulate the problem to include the parallel component of the fluctuating B-field and extend the existing code accordingly. It is expected that this would make possible the study of much larger plasma beta effects and to study higher perturbation frequencies and their mode structure. The growth rates thus obtained may serve as estimates of linear transport coefficients and for future bench marking of the (then) global electromagnetic, gyrokinetic, time evolution codes (PIC or otherwise).

1 Introduction

Finite β, electromagnetic effects are considered as one of the fundamental issues in transport ensuing in hot toroidal plasmas. The paradigm of electromagnetic drift wave turbulence is a saturated state of linear mode evolution and their nonlinear coupling. Presence of pressure gradients, drifting orbits (which couple neighbouring flux surfaces), trapped particles (banana width) and their resonances demand that a simplest nontrivial model will have to be necessarily global (radially extended) and kinetic. To this end, a technique of *gyrokinetic* change of variables was employed by P.J.Catto *et al*[1] with *eikonal or spectral ansatz* whereas a self-consistent and energy conserving theoretical framework was provided by T. S. Hahm *et al*[2] based on Hamiltonian & Lie transformations, among others, resulting in gyro-kinetic equations and gyro-averaged Maxwell's equations for finite-β plasmas.

While on one hand, a combination of above said gyro-kinetic formalism, present day particle simulation techniques and rapidly growing computing capabilities have made it possible

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to develop time evolving linear & non-linear electrostatic (low β) & electromagnetic (finite β) codes, on the other hand, the need to develop a global, spectral approach to the problem of electromagnetic drift waves has also become necessary. There are two main reasons (i) the ballooning representation along the field lines is truely "local" and valid for only large $n-$numbers, whereas, large wavelength (low n) modes and radial mode coupling effects could result in strong transport (ii) to gain confidence on their nonlinear predictability, it becomes necessary to benchmark the time evolution codes with a linear model which is global, fully electromagnetic (both parallel and perpendicular $B-$field fluctuations), with finite larmor radius (FLR) effects to all orders and banana-width physics.

In the past there have been a few attempts [3, 4] to develop (electrostatic) global, spectral code in toroidal geometry. Unfortunately, those formulations are only valid up to second order in banana-width and do not take into account any FLR effects. One such model which overcomes the above said problems and computes gyrokinetic growth rate and global eigenmode structures for electrostatic drift waves for both cylinder [5] and torus [6] was developed by Brunner *et al.* Later, this code was extended to include low but finite β effects by incorporating parallel vector potential (perpendicular magnetic field) fluctuations in the gyro-kinetic equations and parallel component of Ampere's law for current fluctuations. In this model, ions were fully gyro-kinetic and electrons were modelled as drift-kinetic [7].

The aim of the present work is to attempt to generalize the formulation used in Refs.[5, 6, 7] to incorporate perpendicular vector potential (parallel magnetic field) fluctuations in the gyro-kinetic equations and close the system by invoking perpendicular component of Ampere's law. In the present work, while only fully passing ions and electrons are considered, both species are modelled as gyro-kinetic. Hence the model should be valid for relatively large β values and wide range of frequencies well above ion temperature gradient modes. Here, it is presented in a form suitable to implement in the already existing code EM-GLOGYSTO[5, 6, 7].

2 Starting Equations

To describe hot toroidal plasmas, the collionless Vlasov equations and Maxwell equations are used. Since our approach is spectral, the full distribution function $f_j(\mathbf{r}, \mathbf{v}, t)$ of species j is linearized about an suitable equilibrium $f_{0j} = f_{0j}(\mathbf{r}, \mathbf{v})$ such that $f_j(\mathbf{r}, \mathbf{v}, t) = f_{0j}(\mathbf{r}, \mathbf{v}) + \tilde{f}_j(\mathbf{r}, \mathbf{v}, t)$ with the assumption that $\tilde{f}_j / f_{0j} \ll 1$. Retaining terms up to first order, we get ;

$$\left.\frac{D}{Dt}\right|_{u.t.p.} f_{0j}(\mathbf{r}, \mathbf{v}) = 0 \quad \text{where} \quad \left.\frac{D}{Dt}\right|_{u.t.p.} \equiv \frac{\partial}{\partial t} + \mathbf{r} \cdot \nabla + \frac{q_j}{m_j}(\mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} \tag{1}$$

and

$$\left.\frac{D}{Dt}\right|_{u.t.p.} \tilde{f}_j(\mathbf{r}, \mathbf{v}, t) = -\frac{q_j}{m_j}(\tilde{\mathbf{E}} + \mathbf{v} \times \tilde{\mathbf{B}}) \cdot \nabla_{\mathbf{v}} f_{0j} \tag{2}$$

Here $u.t.p$ implies *unperturbed trajectories of particles*, $\mathbf{B} = B\ \hat{e}_{||}$ is the equilibrium toroidal magnetic field, $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$ are the perturbed electric and magnetic fields, q_j and m_j are the electric charge and mass of the species respectively. Expressing $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$ in terms of $\tilde{\varphi}$ and $\tilde{\mathbf{A}}$ and defining the following change of variables $(\mathbf{r}, \mathbf{v}) \to (\mathbf{r}, \xi = v^2/2, \mu = v_\perp^2/2B)$ and using particle canonical angular momentum for species j, i.e., $\psi_{0j} = |\mathbf{r} \times (\mathbf{A} + m_j \mathbf{v}/q_j)| = \psi + m_j r v_\phi / q_j$, one can write $f_{0j}(\mathbf{r}, \mathbf{v}) = f_{0j}(\mathbf{r}, \xi, \mu, \psi_{0j})$. Here cylindrical co-ordinate $\mathbf{r} \equiv (r, \phi, z)$ has been introduced and $\psi = rA_\phi$ is the poloidal flux function per unit radian. Such a transformation would enable one to express f_{0j} in terms of single particle constants of motion. Thus $\nabla_{\mathbf{v}} f_{0j}$

term on $r.h.s$ of Eq.(2) becomes

$$\nabla_{\mathbf{v}} f_{0j}(\mathbf{r}, \xi, \mu, \psi_{0j}) = \mathbf{v}\left(1 + \frac{m_j r v_\phi}{q_j}\frac{\partial}{\partial \psi_{0j}}\right)\frac{\partial f_{0j\psi}}{\partial \xi} + \frac{\mathbf{v}_\perp}{B}\frac{\partial f_{0j\psi}}{\partial \mu} + \frac{m_j r \hat{e}_\phi}{q_j}\frac{\partial f_{0j}}{\partial \psi_{0j}}\Bigg|_{\psi_0=\psi} \quad (3)$$

where $f_{0j\psi} \equiv f_{0j}(\psi_{0j} = \psi)$ and $\hat{e}_\phi$ is the toroidal unit vector. To obtain Eq.(3), f_{0j} is Taylor expanded to first order in $\{m_j r v_\phi / q_j\}$ around $\psi_{0j} = \psi$. Then, the following ordering is used :

$$\begin{aligned} &\underline{\text{gyro} - \text{ordering :}} \quad && \frac{\omega}{w_{cj}} \ll 1, \quad k_\perp \varrho_{Lj} \simeq O(1), \quad k_{||}\varrho_{Lj} \simeq \frac{\varrho_{Lj}}{L_{eq}}; \\ &\underline{\text{transport} - \text{ordering :}} \quad && \frac{\partial}{\partial \mu} \ll \frac{\partial}{\partial \xi} \end{aligned} \quad (4)$$

where $k_\perp^{-1}, k_{||}^{-1}, \varrho_{Lj}$ are perpendicular and parallel perturbation scales and Larmor radius of the species j respectively and L_{eq} is a typical equilibrium scale length. Using large aspect ratio equilibrium, rewriting $\tilde{f}_j$ in Eqs.(2), using the change of variables defined by:

$$\tilde{f}_j = h_j^{(0)} + \tilde{\varphi}\frac{q_j}{m_j}\left[\left(1 - \frac{v_\varphi}{\Omega_{pj}}\nabla_n\right)\frac{\partial f_{0j\psi}}{\partial \xi}\right] + \frac{q_j}{m_j B}\frac{\partial f_{0j\psi}}{\partial \mu}\left(\tilde{\varphi} - v_{||}A_{||}\right) - \frac{q_j}{m_j}\frac{A_\phi}{\Omega_{pj}}\nabla_n f_{0j}\Bigg|_{\psi_{0j}=\psi} \quad (5)$$

invoking gyro-ordering & transport-ordering and finally using some standard vector algebra, we arrive at

$$\frac{D}{Dt}\Bigg|_{u.t.p} h_j^{(0)}(\mathbf{r}, \mathbf{v}, t) = -\frac{q_j}{m_j}\left[\frac{\partial f_{0j\psi}}{\partial \xi}\frac{\partial}{\partial t} + \frac{v_{||}}{B}\frac{\partial f_{0j\psi}}{\partial \mu}\hat{e}_{||}\cdot\nabla + \frac{1}{\Omega_{pj}}\nabla_n f_{0j}\Bigg|_\psi \hat{e}_\phi \cdot \nabla\right]\left(\tilde{\varphi} - \mathbf{v}\cdot\tilde{\mathbf{A}}\right) + O(\epsilon) \quad (6)$$

In Eqs.(5-6), we have introduced the following definitions: $\Omega_{pj} = w_{cj} B_p / B$, $w_{cj} = q_j B / m_j$, $B_p = |\nabla \psi| / r$ and $h_j^{(0)}$ is the zeroth order term of the perturbative series in the "inverse gyro-frequency expansion" of the nonadiabatic part $h_j = h_j^{(0)} + \frac{1}{w_{cj}} h_j^{(1)} + \frac{1}{w_{cj}^2} h_j^{(2)}$..... Note that since $D/Dt \simeq O(w_{cj})$, only $h_j^{(0)}$ is retained which is independent of w_{cj} and hence the gyro-angle (defined below). In the rest of this presentation $h_j^{(0)}$ is referred simply as h_j. Eq.(6) is our starting equation.

3 Gyro-Averaging & Gauge

For a large aspect ratio tokamak geometry, $\mathbf{v} = v_\perp(\hat{e}_\varrho cos\alpha + \hat{e}_\theta sin\alpha) + v_{||}\hat{e}_{||}$, where unit vectors $(\hat{e}_\varrho, \hat{e}_\theta, \hat{e}_\phi)$ define the toroidal co-ordinates and α is the gyro-angle. We define gyro-averaging a quantity "Q" as

$$< Q >= \frac{1}{2\pi}\int_0^{2\pi} d\alpha Q(\alpha; ..)$$

In Eq.(6), the terms in square brackets ([..]) on the *r.h.s.* are all *equilibrium quantities* and are independent of α. Thus only the potentials are to be averaged. Similarly, on the *l.h.s*, h_j is independent of α, hence, only $D/Dt|_{u.t.p}$ is to be gyro-averaged. Therefore,

$$\frac{D}{Dt}\Bigg|_{u.t.p} \overset{gyro-averaging}{\Longrightarrow} \frac{D}{Dt}\Bigg|_{u.t.g} \equiv \frac{\partial}{\partial t} + (v_{||}\hat{e}_{||} + \mathbf{v_{dj}}) \cdot \frac{\partial}{\partial \mathbf{R}}$$

where $\mathbf{v_{dj}} = (v_\perp^2/2 + v_{||}^2)\hat{e}_z/(rw_{cj})$, *u.t.g.* implies *unperturbed trajectory of guiding centers* $\mathbf{R}$ defined by $\mathbf{R} = \mathbf{r} + \mathbf{v} \times \hat{e}_{||}/w_{cj}$. Therefore,

$$< \tilde{\varphi} - \mathbf{v} \cdot \tilde{\mathbf{A}} >= \frac{1}{2\pi} \int_0^{2\pi} d\alpha \left[\tilde{\varphi}(\mathbf{r}[\alpha], t) - \mathbf{v} \cdot \tilde{\mathbf{A}}(\mathbf{r}[\alpha], t) \right] \Bigg|_{\mathbf{r}=\mathbf{R}-\mathbf{v}\times\hat{e}_{||}/w_{cj}}$$

Since $\tilde{\varphi}(\mathbf{r}[\alpha], t)$ and $\tilde{\mathbf{A}}(\mathbf{r}[\alpha], t)$ are unknown functions, the gyro-averaging is performed by first Fourier decomposing these functions, then representing the particle co-ordinate $\mathbf{r}$ by gyro-center $\mathbf{R}$ and remembering that

$$J_p(x) = \frac{1}{2\pi} \int_0^{2\pi} d\alpha \exp[\iota(x sin\alpha - p\alpha)]$$

We choose the following gauge for $\tilde{\mathbf{A}} \equiv (\tilde{A}_{||}, \tilde{A}_\perp) \equiv (\tilde{A}_{||}, \tilde{A}_\theta)$ With the above said procedure, one obtains the following *gyro-kinetic equation*:

$$\frac{D}{Dt}\Bigg|_{u.t.g} h_j(\mathbf{R}, \mathbf{v}, t) = -\left(\frac{q_j}{m_j}\right) \left[\frac{\partial f_{0j\psi}}{\partial \xi} \frac{\partial}{\partial t} + \frac{v_{||}}{B} \frac{\partial f_{0j\psi}}{\partial \mu} \hat{e}_{||} \cdot \nabla + \frac{1}{\Omega_{pj}} \nabla_n f_{0j} \Bigg|_\psi \hat{e}_\phi \cdot \nabla \right] \times$$

$$\left(\tilde{\varphi}(\mathbf{k};) J_0(k_\perp \varrho_{Lj}) - v_{||} \tilde{A}_{||}(\mathbf{k};) J_0(k_\perp \varrho_{Lj}) + \iota \frac{k}{k_\perp} v_\perp \tilde{A}_\theta(\mathbf{k};) J_1(k_\perp \varrho_{Lj}) \right) + O(\epsilon) \qquad (7)$$

Solution to Eq.(7) is obtained by *Green function technique* (unit source solution *say* $\mathcal{P}$) [8]. An explicit form of $\mathcal{P}$ is obtained analytically by the method of characteristics of $u.t.g$ and then using a perturbative technique for the guiding center velocity [9]. Morever, the unit source solution, $\mathcal{P}$, to Eq.(7) is independent of the type of perturbation (electrostaic or electromagnetic) and solely depends on the considered *equilibrium*. We assume for equilibrium f_{0j}, a local Maxwellian of the form

$$f_{0j}(\xi, \mu, \psi) = f_{Mj}(\xi, \psi) = \frac{N(\psi)}{\left(\frac{2\pi T_j(\psi)}{m_j}\right)^{3/2}} \exp - \frac{\xi}{(T_j(\psi)/m_j)}$$

so that $\partial f_{0j}/\partial \mu \equiv 0$ by choice and density profile $N(\psi)$ is independent of the species type j. Thus, the solution to Eq.(7) is :

$$h_j(\mathbf{R}, \mathbf{v}, \omega) = -\left(\frac{q_j F_{Mj}}{T_j}\right) \int d\mathbf{k} \exp \iota \mathbf{k} \cdot \mathbf{R} \left(\omega - \omega_j^*\right) (\iota\, \mathcal{P}_j) \times$$

$$\left(\tilde{\varphi}(\mathbf{k};) J_0(k_\perp \varrho_{Lj}) - v_{||} \tilde{A}_{||}(\mathbf{k};) J_0(k_\perp \varrho_{Lj}) + \iota \frac{k}{k_\perp} v_\perp \tilde{A}_\theta(\mathbf{k};) J_1(k_\perp \varrho_{Lj}) \right) + O(\epsilon) \qquad (8)$$

Here, $\mathbf{k} = \kappa\, \hat{e}_\rho + k_\theta\, \hat{e}_\theta + k_\phi\, \hat{e}_\phi$ and $\kappa = (2\pi/\Delta\rho)\; k_\rho$, with $\Delta\rho = \rho_u - \rho_l$ which defines the radial domain, $k_\phi = n/r$ and $k_\theta = m/\rho$; ω is the *eigenvalue* and $\omega_j^* = \omega_{nj} \left[1 + \frac{\eta_j}{2} \left(\frac{v_{||}^2}{v_{thj}^2} - 3 \right) + \frac{\eta_j v_\perp^2}{2\, v_{thj}^2} \right]$ with $\omega_{nj} = (T_j \nabla_n \ln N k_\theta)/(q_j B)$ is the *diamagnetic drift frequency*; $\eta_j = (d \ln T_j)/(d \ln N)$. Note also that since the large aspect ratio equilibria considered are axisymmetric, toroidal mode numer n can be fixed and the problem is effectively two dimensional in (ρ, θ) or (κ, k_θ).

To obtain density fluctuations $\tilde{n}_j(\mathbf{r}; \omega)$ and current densities fluctuations $\tilde{j}_{||}(\mathbf{r}; \omega)$ and $\tilde{j}_\theta(\mathbf{r}; \omega)$, one needs to go from *g.c.* co-ordinate $\mathbf{R}$ to *particle co-ordinate* $\mathbf{r}$ using $\mathbf{R} = \mathbf{r} + \mathbf{v} \times \hat{e}_{||}/w_{cj}$, replace h_j using Eq.(5) and the integrate over $\mathbf{v}$ keeping in mind the *gyro-angle* integration over α. This last integration yields additional Bessel functions.

Thus, in real space $\mathbf{r}$, for species j, we finally have:

$$\tilde{n}_j(\mathbf{r};\omega) = -\left(\frac{q_j N}{T_j}\right)\left[\tilde{\varphi} + \int d\mathbf{k} \exp \iota\mathbf{k}\cdot\mathbf{r} \int d\mathbf{v}\frac{f_{Mj}}{N}\left(\omega-\omega_j^*\right)(\iota\mathcal{P}_j)\times\right.$$
$$\left.\left\{\left[\tilde{\varphi}(\mathbf{k};)-v_{||}\tilde{A}_{||}(\mathbf{k};)\right]J_0^2(x_{Lj}) + \iota\frac{k}{k_\perp}v_\perp\tilde{A}_\theta(\mathbf{k};)J_0(x_{Lj})J_1(x_{Lj})\right\}\right]$$
$$\tilde{j}_{||j}(\mathbf{r};\omega) = -\left(\frac{q_j^2}{T_j}\right)\left[\int d\mathbf{k} \exp \iota\mathbf{k}\cdot\mathbf{r} \int v_{||}d\mathbf{v}f_{Mj}\left(\omega-\omega_j^*\right)(\iota\mathcal{P}_j)\times\right.$$
$$\left.\left\{\left[\tilde{\varphi}(\mathbf{k};)-v_{||}\tilde{A}_{||}(\mathbf{k};)\right]J_0^2(x_{Lj}) + \iota\frac{k}{k_\perp}v_\perp\tilde{A}_\theta(\mathbf{k};)J_0(x_{Lj})J_1(x_{Lj})\right\}\right] \quad (9)$$
$$\tilde{j}_{\theta j}(\mathbf{r};\omega) = -\left(\frac{q_j^2}{T_j}\right)\left[\int d\mathbf{k} \exp \iota\mathbf{k}\cdot\mathbf{r} \int v_\perp d\mathbf{v}f_{Mj}\left(\omega-\omega_j^*\right)(\iota\mathcal{P}_j)\times\right.$$
$$\left.\left\{\iota\frac{k}{k_\perp}\left[\tilde{\varphi}(\mathbf{k};)-v_{||}\tilde{A}_{||}(\mathbf{k};)\right]J_0(x_{Lj})J_1(x_{Lj}) - \frac{k^2}{k_\perp^2}v_\perp\tilde{A}_\theta(\mathbf{k};)J_1^2(x_{Lj})\right\}\right] \quad \text{where } \; x_{Lj}=k_\perp\varrho_{Lj}$$

4 Closure and Eigen Value Matrix

Equations are finally closed by invoking *quasineutrality condition* and Ampere's law.

$$\sum_j \tilde{n}_j(\mathbf{r};\omega)\simeq 0; \qquad \frac{1}{\mu_0}\nabla_\perp^2\tilde{A}_{||} = -\sum_j \tilde{j}_{||j}; \qquad \frac{1}{\mu_0}\nabla_\perp^2\tilde{A}_\theta = -\sum_j \tilde{j}_{\theta j} \quad (10)$$

Thus we have 4 (four) unknowns $\omega, \tilde{\varphi}(\mathbf{r}), \tilde{A}_{||}(\mathbf{r}), \tilde{A}_\perp(\mathbf{r})$ and 3 (three) equations Eqs.(10). This eigenvalue problem is conveniently solved in Fourier space. We adopt the following two dimensional Fourier convention:

$$f(\mathbf{k}) = \frac{1}{2\pi}\frac{1}{\Delta\rho}\int_{\rho_l}^{\rho_u} d\rho \int_0^{2\pi} d\theta f(\mathbf{r})\exp(-\iota\kappa\rho-\iota m\theta) \quad (\mathrm{Fourier-Transform})$$

$$f(\mathbf{r}) = \int d\mathbf{k}\exp(\iota\mathbf{k}\cdot\mathbf{r})f(\mathbf{k}) = \sum_{\kappa,m}\exp(\iota\kappa\rho+\iota m\theta)f_{\kappa,m} \quad (\mathrm{Fourier-Decomposition})$$

where $\Delta\rho = \rho_u - \rho_l$. By Fourier decomposing the potentials in Eq.(10) and then taking Fourier transform,we obtain a convolution matrix in fourier space. If we assume a hydrogen-like plasma with 2 species (ions and electrons):

$$\sum_{\mathbf{k}'}\begin{pmatrix} \sum_{j=i,e}\hat{\mathcal{M}}^j_{\tilde{\varphi}\tilde{\varphi},\mathbf{k},\mathbf{k}'} & \sum_{j=i,e}\hat{\mathcal{M}}^j_{\tilde{\varphi}\tilde{A}_{||},\mathbf{k},\mathbf{k}'} & \sum_{j=i,e}\hat{\mathcal{M}}^j_{\tilde{\varphi}\tilde{A}_\theta,\mathbf{k},\mathbf{k}'} \\ \sum_{j=i,e}\hat{\mathcal{M}}^j_{\tilde{A}_{||}\tilde{\varphi},\mathbf{k},\mathbf{k}'} & \sum_{j=i,e}\hat{\mathcal{M}}^{j,\nabla^2}_{\tilde{A}_{||}\tilde{A}_{||},\mathbf{k},\mathbf{k}'} & \sum_{j=i,e}\hat{\mathcal{M}}^j_{\tilde{A}_{||}\tilde{A}_\theta,\mathbf{k},\mathbf{k}'} \\ \sum_{j=i,e}\hat{\mathcal{M}}^j_{\tilde{A}_\theta\tilde{\varphi},\mathbf{k},\mathbf{k}'} & \sum_{j=i,e}\hat{\mathcal{M}}^j_{\tilde{A}_\theta\tilde{A}_{||},\mathbf{k},\mathbf{k}'} & \sum_{j=i,e}\hat{\mathcal{M}}^{j,\nabla^2}_{\tilde{A}_\theta\tilde{A}_\theta,\mathbf{k},\mathbf{k}'} \end{pmatrix}\begin{pmatrix}\tilde{\varphi}_{\mathbf{k}'} \\ \tilde{A}_{||,\mathbf{k}'} \\ \tilde{A}_{\theta,\mathbf{k}'}\end{pmatrix} = 0$$

where $\mathbf{k} = (\kappa, m)$ and $\mathbf{k}' = (\kappa', m')$. Numerically, for each species, each of the of these submatrices shown above, is a 2D-band matrix with sub and super diagonals. Note also that the

Laplacian for the parallel and perpendicular Ampere's law is added to the appropriate matrix elements above. This matrix is *symmetric* about the diagonal. Hence we write down below the diagonal and upper diaogonal elements only. Matrix elements are as follows:

$$\hat{\mathcal{M}}^i_{\tilde{\varphi}\tilde{\varphi},\mathbf{k},\mathbf{k}'} = \frac{1}{\Delta\rho}\int_{\rho_l}^{\rho_u} d\rho \exp(-\iota(\kappa-\kappa')\rho) \times \left[\alpha_p \delta_{mm'} + \exp(\iota(m-m')\bar{\theta}) \sum_p \hat{I}^0_{p,i}\right]$$

$$\hat{\mathcal{M}}^i_{\tilde{\varphi}\tilde{A}_{||},\mathbf{k},\mathbf{k}'} = -\frac{1}{\Delta\rho}\int_{\rho_l}^{\rho_u} d\rho \exp(-\iota(\kappa-\kappa')\rho) \times \left[\exp(\iota(m-m')\bar{\theta}) \sum_p \hat{I}^1_{p,i}\right]$$

$$\hat{\mathcal{M}}^i_{\tilde{\varphi}\tilde{A}_{\theta},\mathbf{k},\mathbf{k}'} = \frac{1}{\Delta\rho}\int_{\rho_l}^{\rho_u} d\rho\iota \exp(-\iota(\kappa-\kappa')\rho) \times \frac{\kappa'}{k'_\perp}\left[\exp(\iota(m-m')\bar{\theta}) \sum_p \hat{V}^0_{p,i}\right]$$

$$\hat{\mathcal{M}}^e_{\tilde{\varphi}\tilde{\varphi},\mathbf{k},\mathbf{k}'} = \frac{1}{\Delta\rho}\int_{\rho_l}^{\rho_u} d\rho \exp(-\iota(\kappa-\kappa')\rho) \times \left[\frac{\alpha_p}{\tau(\rho)}\delta_{mm'} + \frac{\exp(\iota(m-m')\bar{\theta})}{\tau(\rho)} \sum_p \hat{I}^0_{p,e}\right]$$

$$\hat{\mathcal{M}}^e_{\tilde{\varphi}\tilde{A}_{||},\mathbf{k},\mathbf{k}'} = -\frac{1}{\Delta\rho}\int_{\rho_l}^{\rho_u} d\rho \exp(-\iota(\kappa-\kappa')\rho) \times \left[\frac{\exp(\iota(m-m')\bar{\theta})}{\tau(\rho)} \sum_p \hat{I}^1_{p,e}\right]$$

$$\hat{\mathcal{M}}^e_{\tilde{\varphi}\tilde{A}_{\theta},\mathbf{k},\mathbf{k}'} = \frac{1}{\Delta\rho}\int_{\rho_l}^{\rho_u} d\rho \exp(-\iota(\kappa-\kappa')\rho) \times \frac{\kappa}{k_\perp}\left[\frac{\iota\exp(\iota(m-m')\bar{\theta})}{\tau(\rho)} \sum_p \hat{V}^0_{p,e}\right]$$

Since $\hat{\mathcal{M}}^{i,\nabla^2}_{\tilde{A}_{||}\tilde{A}_{||},\mathbf{k},\mathbf{k}'} = \hat{\mathcal{M}}^i_{\tilde{A}_{||}\tilde{A}_{||},\mathbf{k},\mathbf{k}'} + \hat{\mathcal{M}}^{\nabla^2}_{\tilde{A}_{||}\tilde{A}_{||},\mathbf{k},\mathbf{k}'}$ we have

$$\hat{\mathcal{M}}^i_{\tilde{A}_{||}\tilde{A}_{||},\mathbf{k},\mathbf{k}'} = \frac{1}{\Delta\rho}\int_{\rho_l}^{\rho_u} d\rho \exp(-\iota(\kappa-\kappa')\rho) \times \left[\exp(\iota(m-m')\bar{\theta}) \sum_p \hat{I}^2_{p,i}\right] \tag{11}$$

$$\hat{\mathcal{M}}^{\nabla^2}_{\tilde{A}_{||}\tilde{A}_{||},\mathbf{k},\mathbf{k}'} = -\frac{1}{\Delta\rho}\int_{\rho_l}^{\rho_u} d\rho \exp(-\iota(\kappa-\kappa')\rho) \left(\kappa'^2 + \frac{m'^2}{\rho^2}\right)\left(\frac{T_i(\rho)}{q_i^2 N \mu_0}\right)$$

$$\hat{\mathcal{M}}^e_{\tilde{A}_{||}\tilde{A}_{||},\mathbf{k},\mathbf{k}'} = \frac{1}{\Delta\rho}\int_{\rho_l}^{\rho_u} d\rho \frac{\exp(-\iota(\kappa-\kappa')\rho)}{\tau(\rho)} \times \left[\exp(\iota(m-m')\bar{\theta}) \sum_p \hat{I}^2_{p,e}\right]$$

$$\hat{\mathcal{M}}^i_{\tilde{A}_{||}\tilde{A}_{\theta},\mathbf{k},\mathbf{k}'} = -\frac{1}{\Delta\rho}\int_{\rho_l}^{\rho_u} d\rho\iota \exp(-\iota(\kappa-\kappa')\rho) \times \frac{\kappa'}{k'_\perp}\left[\exp(\iota(m-m')\bar{\theta}) \sum_p \hat{V}^1_{p,i}\right]$$

$$\hat{\mathcal{M}}^e_{\tilde{A}_{||}\tilde{A}_{\theta},\mathbf{k},\mathbf{k}'} = -\frac{1}{\Delta\rho}\int_{\rho_l}^{\rho_u} d\rho \frac{\iota}{\tau(\rho)} \exp(-\iota(\kappa-\kappa')\rho) \times \frac{\kappa'}{k_{\perp'}}\left[\exp(\iota(m-m')\bar{\theta}) \sum_p \hat{V}^1_{p,e}\right]$$

Since $\hat{\mathcal{M}}^{i,\nabla^2}_{\tilde{A}_{\theta}\tilde{A}_{\theta},\mathbf{k},\mathbf{k}'} = \hat{\mathcal{M}}^i_{\tilde{A}_{\theta}\tilde{A}_{\theta},\mathbf{k},\mathbf{k}'} + \hat{\mathcal{M}}^{\nabla^2}_{\tilde{A}_{\theta}\tilde{A}_{\theta},\mathbf{k},\mathbf{k}'}$ we have

$$\hat{\mathcal{M}}^i_{\tilde{A}_{\theta}\tilde{A}_{\theta},\mathbf{k},\mathbf{k}'} = -\frac{1}{\Delta\rho}\int_{\rho_l}^{\rho_u} d\rho \exp(-\iota(\kappa-\kappa')\rho) \times \frac{\kappa'^2}{k'^2_\perp}\left[\exp(\iota(m-m')\bar{\theta}) \sum_p \hat{W}^0_{p,i}\right]$$

$$\hat{\mathcal{M}}^e_{\tilde{A}_{\theta}\tilde{A}_{\theta},\mathbf{k},\mathbf{k}'} = -\frac{1}{\Delta\rho}\int_{\rho_l}^{\rho_u} d\rho \exp(-\iota(\kappa-\kappa')\rho) \times \frac{\kappa'^2}{k'^2_\perp}\left[\frac{\exp(\iota(m-m')\bar{\theta})}{\tau(\rho)} \sum_p \hat{W}^0_{p,e}\right] \quad \text{where}$$

$$\hat{I}^l_{p,j} = \frac{1}{\sqrt{2\pi}v^3_{th,j}(\rho)}\int_{-vmax_j(\rho)}^{vmax_j(\rho)} v^l_{||} dv_{||} \exp-\left(\frac{v^2_{||}}{v^2_{th,j}(\rho)}\right)\left\{\frac{N^j_1 \hat{I}_{0,j} - N^j_2 \hat{I}_{1,j}}{D^j_1}\right\}_{p'=p-(m-m')}$$

$$\hat{V}^l_{p,j} = \frac{1}{\sqrt{2\pi}v^3_{th,j}(\rho)}\int_{-vmax_j(\rho)}^{vmax_j(\rho)} v^l_{||} dv_{||} \exp-\left(\frac{v^2_{||}}{v^2_{th,j}(\rho)}\right)\left\{\frac{N^j_1 \hat{V}_{0,j} - N^j_2 \hat{V}_{1,j}}{D^j_1}\right\}_{p'=p-(m-m')} \tag{12}$$

$$\hat{W}^l_{p,j} = \frac{1}{\sqrt{2\pi} v^3_{th,j}(\rho)} \int_{-vmax_j(\rho)}^{vmax_j(\rho)} v^l_{||} dv_{||} \exp - \left(\frac{v^2_{||}}{v^2_{th,j}(\rho)} \right) \left\{ \frac{N^j_1 W_{0,j} - N^j_2 W_{1,j}}{D^j_1} \right\}_{p'=p-(m-m')}$$

and

$$\hat{I}_{n,j} = \int_{-v_{\perp max,j(\rho)}}^{v_{\perp max,j(\rho)}} v_\perp^{2n+1} dv_\perp \exp - \left(\frac{v^2_\perp}{2v^2_{th,j}(\rho)} \right) J^2_0(x_{Lj}) J_p(x'_{tj}) J_{p'}(x'_{tj})$$

$$\hat{V}_{n,j} = \int_{-v_{\perp max,j(\rho)}}^{v_{\perp max,j(\rho)}} v_\perp^{2n+2} dv_\perp \exp - \left(\frac{v^2_\perp}{2v^2_{th,j}(\rho)} \right) J_0(x_{Lj}) J_1(x_{Lj}) J_p(x'_{tj}) J_{p'}(x'_{tj}) \quad (13)$$

$$\hat{W}_{n,j} = \int_{-v_{\perp max,j(\rho)}}^{v_{\perp max,j(\rho)}} v_\perp^{2n+3} dv_\perp \exp - \left(\frac{v^2_\perp}{2v^2_{th,j}(\rho)} \right) J^2_1(x_{Lj}) J_p(x'_{tj}) J_{p'}(x'_{tj})$$

We have introduced the following definitions: $v_{\perp max,j(\rho)} = min(v_{||}/\sqrt{\epsilon}, v_{max,j})$ which is "trapped particle exclusion" from ω independent perpendicular velocity integrals namely, $\hat{I}_{n,j}$, $\hat{V}_{n,j}$, $\hat{V}_{n,j}$; $\alpha_p = 1 - \sqrt{\epsilon/(1+\epsilon)}$ is the fraction of passing particles; $\hat{I}^l_{p,j}$, $\hat{V}^l_{p,j}$, $\hat{V}^l_{p,j}$ are $\omega-$ dependent parallel integrals; $x_{tj} = k_\perp q_s (v^2_\perp/2 + v^2_{||})/(w_{cj} v_{||})$, $N^j_1 = \omega - w_{n,j} \left[1 + (\eta_j/2)(v^2_{||}/v^2_{th,j}) - 3) \right]$; $N^j_2 = w_{n,j}\eta_j/(2v^2_{th,j})$ and $D^j_1 =< w_{t,j}(\rho) > (nq_s - m' - p)(v_{||}/v_{th,j}) - \omega$ where $< w_{t,j}(\rho) >= v_{th,j}(\rho)/(rq_s)$ is the average *transit frequency* of the species j.

5 Conclusion

The matrix equation above is a global, electromagnetic, gyro-kinetic, 2D formulation which should be able to through some light into the *radial structure* of the eigenmodes, apart from predicting the growth rates of modes with frequencies much higher than regular toroidal ITG modes or Alfvén-ITG's.

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