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Continuous Spectra of a Cylindrical Magnetohydrodynamic Equilibrium: the Derivation

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Continuous spectra of a cylindrical magnetohydrodynamic equilibrium: the derivation

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Abstract

The system of two first-order spatial differential equations describing small-amplitude harmonic motions in a cylindrical equilibrium of a perfectly conducting plasma is derived from the basic set of the linear ideal magnetohydrodynamic equations. Enough details of the derivation are given to prove explicitly that the system earlier derived by the authors [Phys. Fluids, 17, 1471 (1974)] is complete and that the frequency spectrum of the motions contains at most two continua, thereby contradicting a recent publication of Lashmore-Davies, Thyagaraja and Cairns [Physics of Plasmas, 4, 3243 (1997)].

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I. INTRODUCTION

Lashmore-Davies, Thyagaraja and Cairns have recently stoked up [1,2] an old controversy between Harold Grad and a part of the MHD community concerning the number of continua contained in the spectrum of ideal magnetohydrodynamics in a cylindrical screw pinch. From an inspection of the Hain-Lüst second-order differential equation [3] governing the radial plasma displacement, Grad had conjectured [4] that there were four continua, one associated with the Alfven wave, two with the slow magnetosonic wave (termed "slow wave" and "cusp" continuum) and one associated with the fast magnetosonic wave. This conjecture was at variance with numerical results which here and there were beginning to be obtained based on the strong variational formulation of the MHD problem. A little later [5], the present authors were able to derive from the basic linear MHD equations a system of two first-order equations which manifestly were only singular for frequencies in the Alfven and the slow-wave continuum. It was therefore concluded that there were only two continua.

In their papers [1,2] concerning this topic, Lashmore-Davies, Thyagaraja and Cairns claim that the existing theory is incomplete and that there are four continua as conjectured by Grad. They also claim that their findings have important physical consequences. The reason for their astonishing claims can be traced to an additional term [2] in the aforementioned first-order system which they have obtained with sophisticated mathematics. This is in striking contrast with our 1974 work for which we had used simple transparent mathematics to derive the first-order system and where no such term had been found. Considering our derivation to be cumbersome but elementary, we did not at that time find it necessary to publish the details of it. This assessment has changed with the publications of Lashmore-Davies *et al.* We now think that it is important that the community can see without effort how our result had been obtained. The goal of this paper therefore is to sketch the derivation of our old result [5] (referred to as I from now on) and by doing so to prove that the result and the conclusions of Lashmore-Davies, Thyagaraja and Cairns are wrong.

II. DERIVATION

As in the original paper, our starting point will be the linearized equations of ideal single-fluid magnetohydrodynamics. Let ρ_0, p_0, $\mathbf{B}_0$ represent the equilibrium values of the mass density, pressure, and magnetic field. The pertinent basic equations can then be written in the form [6]

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = -\nabla p - \frac{1}{\mu_0}\left[\mathbf{B} \times (\nabla \times \mathbf{B}_0) + \mathbf{B}_0 \times (\nabla \times \mathbf{B})\right], \tag{1}$$

$$\mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0), \tag{2}$$

$$p + \boldsymbol{\xi} \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \boldsymbol{\xi} = 0, \tag{3}$$

where $\boldsymbol{\xi}$ is the displacement from an equilibrium position, $\mathbf{B}$ and p the magnetic field and the pressure perturbations, and γ is the adiabaticity index.

In a cylindrical equilibrium, the field components $B_{0\theta}$, B_{0z} and the pressure p_0 are functions of radius r, and are constrained by the pressure balance

$$\frac{d}{dr}\left(p_0 + \frac{{B_0}^2}{2\mu_0}\right) + \frac{{B_{0\theta}}^2}{\mu_0 r} = 0. \tag{4}$$

The mass density ρ_0 is an arbitrary function of radius.

Now turning to the Fourier analysis of eqs.(1)-(3), we take the time and space dependence as $\exp[i(kz + m\theta - \omega t)]$ and introduce the total perturbed pressure

$$P = p + \frac{\mathbf{B}_0 \cdot \mathbf{B}}{\mu_0} \tag{5}$$

as a new variable. Equation (1) then takes the explicit form

$$-\rho_0 \omega^2 \xi_r = -P' + \frac{i}{\mu_0} B_r F - \frac{2}{r\mu_0} B_\theta B_{0\theta}, \tag{6}$$

$$-\rho_0 \omega^2 \xi_\theta = -i\frac{m}{r} P + \frac{i}{\mu_0} B_\theta F + \frac{1}{\mu_0} B_r (\nabla \times \mathbf{B}_0)_z, \tag{7}$$

$$-\rho_0 \omega^2 \xi_z = -ikP + \frac{i}{\mu_0} B_z F - \frac{1}{\mu_0} B_r (\nabla \times \mathbf{B}_0)_\theta, \tag{8}$$

where the prime $'$ stands for d/dr and $F = (m/r)B_{0\theta} + kB_{0z}$. Likewise, eqs.(2) and (3) become

$$B_r = iF\xi_r, \tag{9}$$

$$B_\theta = -B_{0\theta}\nabla\cdot\boldsymbol{\xi} - r\left(\frac{B_{0\theta}}{r}\right)'\xi_r + iF\xi_\theta, \tag{10}$$

$$B_z = -B_{0z}\nabla\cdot\boldsymbol{\xi} - {B_{0z}}'\xi_r + iF\xi_z, \tag{11}$$

and

$$p + \gamma p_0\nabla\cdot\boldsymbol{\xi} + {p_0}'\xi_r = 0. \tag{12}$$

The eight equations (5)-(12) determine the eight unknowns P, $\boldsymbol{\xi}$, $\mathbf{B}$ and p in cylindrical coordinates. Four equations, namely (5) and (7)-(9) are algebraic, eq.(6) contains a first-order derivative d/dr on the unknown P and the remaining three equations (10)-(12) contain the expression $\nabla\cdot\boldsymbol{\xi}$. The most transparent way of deriving the first-order system presented in I is to introduce $\nabla\cdot\boldsymbol{\xi}$ as a new unknown,

$$\epsilon = \frac{1}{r}(r\xi_r\,)' + \frac{im}{r}\xi_\theta + ik\xi_z\,, \tag{13}$$

herewith turning eqs.(10)-(12) into algebraic relations.

We now have a system of nine equations for the nine unknowns P, $\boldsymbol{\xi}$, $\mathbf{B}$, p and ϵ with one single derivative acting on P, eq.(6), and one on ξ_r, eq.(13). The whole system can therefore be reduced to 2 first-order differential equations for ξ_r and P, eqs.(5) and (6) in I, or to a single second-order differential equation for either ξ_r (the Hain-Lüst equation) or for P. The elimination procedure is elementary but cumbersome. In order to show that Thyagaraja's equation (51) in [2] is wrong a mere sketch of the procedure is sufficient. We have just to show that apart from trivial divisions by r or μ_0 the only denominator ever occuring is that giving rise to the two continua mentioned in I.

We first rewrite eqs.(6) using eqs.(9), (10) and (13) and restate eq.(13):

$$P' = \left[\rho_0\omega^2 - \frac{F^2}{\mu_0} + \frac{2B_{0\theta}}{\mu_0}\left(\frac{B_{0\theta}}{r}\right)'\right]\xi_r + \frac{2{B_{0\theta}}^2}{\mu_0 r}\epsilon - \frac{2B_{0\theta}F}{\mu_0 r}i\xi_\theta, \tag{14}$$

$$(r\xi_r\,)' = r\epsilon - mi\xi_\theta - kri\xi_z. \tag{15}$$

Now, the perturbed pressure p can be eliminated from eq.(12) with the help of eq.(5). Then, the resulting equation together with eqs.(7) and (8) can be freed from B_r, B_θ and B_z using eqs.(9)-(11) and (13) with the result,

$$-\left(\gamma p_0+\frac{{B_0}^2}{\mu_0}\right)\epsilon+\frac{B_{0\theta}F}{\mu_0}i\xi_\theta+\frac{B_{0z}F}{\mu_0}i\xi_z=P-\frac{2}{r}\frac{{B_{0\theta}}^2}{\mu_0}\xi_r, \tag{16}$$

$$-\frac{B_{0\theta}F}{\mu_0}\epsilon-\left(\rho_0\omega^2-\frac{F^2}{\mu_0}\right)i\xi_\theta=\frac{m}{r}P-\frac{2}{r}\frac{B_{0\theta}F}{\mu_0}\xi_r, \tag{17}$$

$$-\frac{B_{0z}F}{\mu_0}\epsilon-\left(\rho_0\omega^2-\frac{F^2}{\mu_0}\right)i\xi_z=kP. \tag{18}$$

Note that in obtaining eq.(16) from eq.(12) the equilibrium pressure balance, eq.(4) has been used. Equations (16)-(18) constitute a linear system for the unknowns ϵ, $i\xi_\theta$ and $i\xi_z$ which can be computed in terms of P and ξ_r as long as the determinant,

$$D=\left(\rho_0\omega^2-\frac{F^2}{\mu_0}\right)\left[\rho_0\omega^2\left(\gamma p_0+\frac{{B_0}^2}{\mu_0}\right)-\gamma p_0\frac{F^2}{\mu_0}\right], \tag{19}$$

is non-vanishing. For $D\neq 0$ we therefore have

$$-D\epsilon=\rho_0\omega^2\left(\rho_0\omega^2-\frac{F^2}{\mu_0}\right)\left(P-\frac{2}{r\mu_0}{B_{0\theta}}^2\xi_r\right), \tag{20}$$

$$-Di\xi_\theta=\left[\frac{m}{r}\gamma p_0\left(\rho_0\omega^2-\frac{F^2}{\mu_0}\right)+\rho_0\omega^2\frac{B_{0z}V}{\mu_0}\right]P-\frac{2B_{0\theta}F}{r\mu_0}\left[\gamma p_0\left(\rho_0\omega^2-\frac{F^2}{\mu_0}\right)+\rho_0\omega^2\frac{{B_{0z}}^2}{\mu_0}\right]\xi_r, \tag{21}$$

$$-Di\xi_z=\left[k\gamma p_0\left(\rho_0\omega^2-\frac{F^2}{\mu_0}\right)-\rho_0\omega^2\frac{B_{0\theta}V}{\mu_0}\right]P+\frac{2B_{0z}F}{r\mu_0}\rho_0\omega^2\frac{{B_{0\theta}}^2}{\mu_0}\xi_r, \tag{22}$$

where $V=(m/r)B_{0z}-kB_{0\theta}$.

The final system of two first-order differential equations, eqs.(5) and (6) in I, are obtained by multiplying eqs. (14) and (15) by $D\neq 0$ and substituting $D\epsilon$, $Di\xi_\theta$ and $Di\xi_z$ by their expressions given in eqs.(20)-(22):

$$D(r\xi_r)'=C_1(r\xi_r)-rC_2P, \tag{23}$$

$$DP'=\frac{1}{r}C_3(r\xi_r)-C_1P, \tag{24}$$

where C_1, C_2, C_3 are defined in I and are complicated expressions in terms of the quantities r, ω, m, k, ρ_0, $B_{0\theta}$, B_{0z} and γp_0 without any denominators other than r and μ_0. This means that there is no way other than by error that an additional term, and in particular a singular term like in Thyagaraja's equation (51) in ref.2, could appear in these final equations.

III. DISCUSSION AND CONCLUSIONS

We conclude therefore that the mathematics of Thyagaraja *et al* has gone out of control somewhere along their apparently unsuitable elimination path.

Our choice of systematically working towards the equations for ξ_r and P had originally been motivated by their usefulness in other problems in cylindrical coordinates and in particular in the derivation by Kadomtsev [7] of approximate solutions for the instabilities of a surface-layer pinch. In this problem, other choices for the two independent variables can lead to heavy inconsistencies. It is also interesting to note that together ξ_r and P make up the radial energy flux which, in a cold plasma, can be interpreted as the Poynting flux. The choice of ξ_r and P as the master variables in cylindrical MHD is therefore physically appealing and perhaps even unavoidable.

An additional remark concerning numerical solutions of the ideal MHD spectral problem is in order because Thyagaraja *et al* also pretend that the continua had never been obtained with mathematical rigor in general cases. This is another incomprehensible statement. Finite-element discretizations of the strong variational form of the spectral MHD problem are based on rigorous mathematical theory and have never, general cases included, shown anything more than 2 continua. Numerical analysis has substantially contributed to the progress of MHD theory in the last 25 years and it is inexcusable that Thyagaraja *et al* ignore this fact.

We restate here that the pertinent system of equations (23) and (24) has the only singularity at $D = 0$, and that this singularity can merely give rise to two continuous spectra.

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