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Abstract

The equations of three–dimensional plasma equilibrium based on the magnetic field representation with poloidal magnetic flux Ψ and plasma current F are formulated. As a result, the description of three–dimensional equilibrium configurations is obtained as a system of the following equations: an elliptic type equation for the poloidal flux, a magnetic differential equation (MDE) for the poloidal current and the equations for the "base" vector field **b**.

For the resolution of the difficulties with possible singular solutions of the MDE on the rational toroidal magnetic surfaces small regularizing terms are introduced into the proposed system of equations. Second order differential terms with a small parameter are added to the MDE transforming it to an elliptic type equation. Several variants of such a regularization are proposed.

The system of equations formulated can serve as a basis for a numerical code development of three-dimensional equilibrium calculations with island structures.

1 Introduction

For the description of the magnetic confinement systems with a coordinate of symmetry, the following "poloidal" representation of the magnetic field is convenient:

$$2\pi \mathbf{B} = [\nabla\Psi \mathbf{b}_0] + F\mathbf{b}_0, \tag{1}$$

where Ψ and $F(\Psi)$ are the external poloidal flux and plasma current, and the base vector field $\mathbf{b}_0$ is known. Particularly, $\mathbf{b}_0 = \nabla\phi$ corresponds to axial symmetry. The general case of the configurations with a coordinate of symmetry is described by the following formula

$$\mathbf{b}_0 = \frac{\hbar \mathbf{e}_z + r\mathbf{e}_\phi}{\hbar^2 + r^2}, \tag{2}$$

with $\mathbf{e}_z = \nabla z$, $\mathbf{e}_\phi = r\nabla\phi$, $2\pi\hbar$ is a helicity pitch parameter. The magnetic field representation (1) with vector $\mathbf{b}$ from (2) reduces the system of MHD equilibrium equations

$$[\mathbf{jB}] = p'(\Psi)\nabla\Psi, \tag{3}$$

$$\mathbf{j} = \operatorname{rot}\mathbf{B}, \tag{4}$$

$$\operatorname{div}\mathbf{B} = 0, \tag{5}$$

to the following two–dimensional elliptic equation for the flux function $\Psi(r,z)$ describing plasma equilibrium in symmetric confinement systems:

$$\frac{1}{b_0^2}\operatorname{div}(b_0^2\nabla\Psi) = \frac{2\hbar}{\hbar^2 + r^2}F - \frac{4\pi^2}{b_0^2}p'(\Psi) - FF'(\Psi), \tag{6}$$

where $b_0^2 = \dfrac{1}{\hbar^2 + r^2}$. The equation (6) is used as a base for numerical codes in two–dimensional plasma equilibrium computations.

The difficulty of three-dimensional plasma equilibrium modeling concerns the compatibility of representation (1) with prescribed a vector field $\mathbf{b}$. Thus it is impossible to describe the equilibrium with one equation for the flux function Ψ, generalizing the two–dimensional equilibrium equation. Nevertheless the practical necessity of 3D equilibrium calculations stimulated the development of several numerical codes. As a rule the codes are based on the Clebsch representation of the magnetic field $2\pi\mathbf{B} = [\nabla\Psi\nabla\lambda]$. One group of codes (code VMEC [1], BETA [2], POLAR-3D [3]) uses an assumption of nested magnetic flux surfaces, other codes: PIES [4] and HINT [5] treat arbitrary 3D equilibria including possible magnetic islands. For practical calculations, the most widely used are the VMEC and HINT codes. The VMEC code is based on a preconditioned spectral energy minimization procedure that calculates equilibria with nested magnetic flux surfaces extremely efficiently when the total number of Fourier harmonics required to describe the equilibrium state is limited to 200 or less. With higher number of the harmonics the iterations might not converge. The codes that determine 3D equilibria with island structures such as HINT are based on time relaxation methods which is very time consuming and use relatively coarse meshes. So searching for the 3D equilibrium problem formulation and the development of the corresponding code which could allow to perform massive calculations remains a desirable and relevant task.

In the present paper, the possibility of generalization of the equation (6) to the 3D equilibrium problem is discussed. The further task should be to clarify whether such type of scheme can offer significant improvements in the efficiency and resolution of computations of arbitrary 3D structures with magnetic islands and stochastic regions.

The specific feature of the approach presented here is an invariant vector formulation of the equilibrium problem for the poloidal flux function. It allows to use for calculations an arbitrary coordinate systems connected with magnetic surfaces as well as Eulerian coordinates. In this respect, the approach differs from work that essentially uses coordinates related to magnetic configuration (see for instance the recent paper of M.Tessarotto et al. [6]).

As a rule for computations of stellarator type systems, the following contravariant representation of the magnetic field in flux coordinates [1] is used:

$$2\pi\mathbf{B} = [\nabla\Phi\nabla\theta] + [\nabla\Psi\nabla\zeta] + [\nabla\rho\nabla\eta]. \tag{7}$$

Here $\Phi = \Phi(\rho)$ and $\Psi = \Psi(\rho)$ are toroidal and external poloidal magnetic fluxes, θ and ζ are arbitrary poloidal and toroidal angle-like variables, ρ – the flux surface label. The representation (7) implies that the toroidal magnetic surfaces are nested.

The magnetic field vector $\mathbf{B}$ in the form (7) automatically satisfies the equation $\operatorname{div}\mathbf{B} = 0$ and one projection of equation (3): $(\mathbf{B}\cdot\nabla\rho) = 0$. Another projection: $(\mathbf{j}\cdot\nabla\rho) = \operatorname{div}[\mathbf{B}\nabla\rho] = 0$ should be satisfied by the appropriate choice of the function $\eta(\mathbf{r})$. It vanishes when special θ,ζ -coordinates are chosen (straight magnetic field lines coordinates, SMFL).

Alternatively the magnetic field can be also represented in the covariant form:

$$2\pi\mathbf{B} = J\nabla\theta + F\nabla\zeta - \nu\nabla\rho + \nabla\phi. \tag{8}$$

Here the presence of the toroidal current $J(\rho)$ suggests also (as well as in (7)) the imposition of nested magnetic surfaces. The first two terms in (8) reveal the relation of magnetic field $\mathbf{B}$ with its sources: toroidal current

$$J(\rho) = \oint \mathbf{B}\cdot d\mathbf{l}_p \tag{9}$$

and external poloidal current

$$F(\rho) = \oint \mathbf{B}\cdot d\mathbf{l}_t. \tag{10}$$

Here $d\mathbf{l}_p$ and $d\mathbf{l}_t$ are vector length elements along closed poloidal and toroidal contours at magnetic surfaces. The functions η, ϕ should provide the condition $\operatorname{div}\mathbf{B} = 0$ and $(\mathbf{B}\cdot\nabla\rho) = 0$ with arbitrary prescribed $\theta(\mathbf{r})$ and $\zeta(\mathbf{r})$ [7, 8]. If the representations (7),(8) are used simultaneously, then the special choice of the coordinates θ,ζ allows to prescribe one of three functions η,ν,ϕ. In particular, in SMFL coordinates the choice $\nu = 0$ corresponds to Hamada coordinates, and the choice $\phi = 0$ — to Boozer coordinates.

A more general representation of the magnetic field, which is not connected with the nested surfaces assumption, was considered in [9] where the following generalization of (1) was proposed:

$$2\pi\mathbf{B} = [\nabla\Psi\mathbf{b}_F] + F\mathbf{b}_\Psi. \tag{11}$$

Here two vector fields $\mathbf{b}_F$ and $\mathbf{b}_\Psi$ are introduced. By special choice of them both "poloidal" and "toroidal" terms in (11) satisfy independently the magnetostatic requirements:

$$\operatorname{div}\mathbf{B} = 0,$$

$$(\mathbf{B}\cdot\nabla\Psi) = 0, \tag{12}$$

$$(\mathbf{j}\cdot\nabla\Psi) = \operatorname{div}[\mathbf{B}\nabla\Psi] = 0.$$

The advantage of the representation (11) is that the transition matrix between covariant and contravariant representations does not depend on the coordinates θ,ζ [8, 9]. Using the "poloidal" representation of the magnetic field without the introduction of the toroidal flux Φ looks very attractive. It naturally generalizes the equilibrium equation (6) for axial ($\hbar = 0$) and helical ($\hbar \neq 0$) symmetry onto the 3D case and does not imply nested magnetic surfaces. It makes it possible to develop numerical codes based on Eulerian coordinates independently of magnetic surface topology. It allows to hope that such a way can be effective for the description of 3D equilibrium configurations with magnetic islands and regions of stochastic magnetic field lines.

These notes present the system of equations for three–dimensional plasma equilibrium based on the magnetic field representation (1), i.e. with one a priori unknown base vector field $\mathbf{b}$. This approach leads to the direct generalization of the equation (6) to the 3D problem and allows to describe so called quasi-symmetric configurations [10, 11].

Another essential feature of the approach concerns the extraction of the functions α or m from the toroidal current density $\mathbf{j}\cdot\mathbf{b}$ contributing to the right hand side of the equation for Ψ. The functions α, m are related to the secondary current from the following equalities: $\alpha = [\mathbf{B}\nabla p]\nabla m/B^2 = \mathbf{j}\cdot\mathbf{B}/B^2$. They can be singular at rational magnetic surfaces, so their extraction allows to introduce regularizing corrections in an appropriate place.

Of course such an extraction of the function α, m is possible also for the representation of the magnetic field with two base vectors $\mathbf{b}_F, \mathbf{b}_\Psi$. The corresponding formulas are given in Appendix A.

2 The three–dimensional base vector field b

Let us use the magnetic field representation (11) with $\mathbf{b}_F = \mathbf{b}_\Psi = \mathbf{b}$:

$$2\pi\mathbf{B} = F\mathbf{b} + [\nabla\Psi\mathbf{b}]. \tag{13}$$

Then the vector $\mathbf{B}$ length takes the form

$$4\pi^2 B^2 = (F^2 + |\nabla\Psi|^2)b^2. \tag{14}$$

Let us require that the vector $\mathbf{b}$ satisfies the closedness condition, and it is normalized such that

$$\oint \mathbf{b}\cdot d\mathbf{l} = 2\pi, \tag{15}$$

where the integration is performed along the vector $\mathbf{b}$. Then its poloidal flux vanishes

$$d\Psi_b = \oint \mathbf{b}\cdot[d\mathbf{l}d\mathbf{n}] = 0, \tag{16}$$

(here $d\mathbf{n}$ is a length element normal to the magnetic surface). The length element $d\mathbf{l}$ can be related to the toroidal angle variable ζ such that

$$dl = \mathbf{q}\, d\zeta, \quad \mathbf{q} = \frac{\mathbf{b}}{b^2}. \tag{17}$$

The requirements (15)-(17) identify the functions Ψ, F as the external poloidal flux and current through the toroidal contour formed by the vector $\mathbf{b}$.

The constraints (12) for the magnetic field $\mathbf{B}$ and (13) lead to the following equations for $\mathbf{b}$:

$$\operatorname{div}(F\mathbf{b} + [\nabla\Psi\mathbf{b}]) = 0,$$

$$(\mathbf{b}\cdot\nabla\Psi) = 0, \tag{18}$$

$$\operatorname{div}((F^2 + |\nabla\Psi|^2)\mathbf{b}) = 0.$$

The last of them is a consequence of the equation $\mathrm{div}(F\mathbf{B} + [\mathbf{B}\nabla\Psi]) = 0$, which is a combination of the second and third equations in (12) and the representation (13). One can see that here the poloidal and toroidal components of the field are connected.

The equations (18) must be solved simultaneously with the equation for the function Ψ.

To derive such a three-dimensional equation for the function Ψ let us present some features of the representation (13).

Taking the scalar and the vector product of (13) with $\mathbf{q} = \mathbf{b}/b^2$ leads to

$$2\pi(\mathbf{q}\cdot\mathbf{B}) = F, \tag{19}$$

$$2\pi[\mathbf{qB}] = \nabla\Psi. \tag{20}$$

Using the obvious equality

$$\mathbf{q} = \frac{(\mathbf{q}\cdot\mathbf{B})\mathbf{B} + [\mathbf{B}[\mathbf{qB}]]}{B^2} \tag{21}$$

and the equations (17), (18) we obtain the formula for vector $\mathbf{q}$:

$$\mathbf{q} = \frac{\mathbf{b}}{b^2} = \frac{F\mathbf{B} + [\mathbf{B}\nabla\Psi]}{2\pi B^2}, \tag{22}$$

or taking into account (14),

$$\mathbf{b} = 2\pi\frac{F\mathbf{B} + [\mathbf{B}\nabla\Psi]}{F^2 + |\nabla\Psi|^2}. \tag{23}$$

One can see the equivalence of the vector $\mathbf{q}$ with the vector of quasi–symmetry, introduced in [11]. The equations (22),(18) lead to the equality

$$\mathrm{div}(\mathbf{q}B^2) = 0. \tag{24}$$

For nested magnetic surfaces, the vector $\mathbf{q}$ coincides with the Boozer coordinate basis vector: $\mathbf{q} = \mathbf{e}_\zeta$

$$\mathbf{e}_\zeta = \sqrt{g}[\nabla\rho\nabla\theta] = V'(\rho)\frac{< B^2 >}{B^2}[\nabla\rho\nabla\theta]. \tag{25}$$

Here the brackets $< . >$ mean averaging along the magnetic field line

$$< f >= \int f\frac{dl}{B} \Big/ \int \frac{dl}{B}. \tag{26}$$

In closed magnetic confinement systems with nested irrational magnetic surfaces this averaging is equivalent to averaging the volume within the layer between closed magnetic surfaces ($\rho = const, \rho{+}\delta\rho = const$), then $< f >= f_0(\rho)$. Formally, on a rational magnetic surface (when the field line is closed) $< f >= f_0(\rho, \lambda)$, where λ is a magnetic line label. However invoking continuity conditions we assume that $< f >= f_0(\rho)$ everywhere. Note that the definition of the vector $\mathbf{q}$ (22) or $\mathbf{b} = b^2\mathbf{q}$ (23) does not imply nested flux surfaces.

3 The three-dimensional equation for poloidal flux function Ψ

In order to get the equation for Ψ, let us apply the div operator to the equation (20) multiplied by b^2:

$$\operatorname{div}(b^2\nabla\Psi) = 2\pi(\mathbf{B}\cdot \operatorname{rot}\mathbf{b}) - 2\pi(\mathbf{j}\cdot\mathbf{b}). \tag{27}$$

The first term in the right hand side of (27) can be transformed using the equation (13) to the form

$$2\pi\mathbf{B}\cdot \operatorname{rot}\mathbf{b} = (F\mathbf{b} + [\nabla\Psi\mathbf{b}])\cdot \operatorname{rot}\mathbf{b}. \tag{28}$$

The term $F\mathbf{b}\cdot \operatorname{rot}\mathbf{b}$ is a "source" of vacuum stellarator magnetic surfaces. The term $[\nabla\Psi\mathbf{b}]\cdot \operatorname{rot}\mathbf{b}$ is inherent only to 3D magnetic plasma confinement system geometry. It brings first derivatives of Ψ to the differential equilibrium equation (6).

The second term $2\pi(\mathbf{j}\cdot\mathbf{b})$ is connected with the "toroidal" component of the current density $\mathbf{j}_t = \mathbf{b}(\mathbf{j}\cdot\mathbf{b})/b^2$. It is defined by the equilibrium equation (3). Substituting the representation of $\mathbf{B}$ (13) into it, we obtain

$$F[\mathbf{j}\mathbf{b}] + (\mathbf{j}\cdot\mathbf{b})\nabla\Psi = 2\pi p'\nabla\Psi. \tag{29}$$

This gives

$$I.\quad 2\pi(\mathbf{j}\cdot\mathbf{b}) = 4\pi^2 p'(\Psi) - 2\pi F\frac{\mathbf{j}\cdot[\mathbf{b}\nabla\Psi]}{|\nabla\Psi|^2}. \tag{30}$$

Another expression for $2\pi(\mathbf{j}\cdot\mathbf{b})$ can be obtained starting from the conventional representation of the vector $\mathbf{j}$ as a sum of longitudinal and normal components with respect to the vector $\mathbf{B}$:

$$\mathbf{j} = \alpha\mathbf{B} + p'(\Psi)\frac{[\mathbf{B}\nabla\Psi]}{B^2}. \tag{31}$$

Here

$$\alpha = \frac{\mathbf{j}\cdot\mathbf{B}}{B^2}. \tag{32}$$

Excluding $\nabla\Psi$ from (31) with the help of (20) and using (22) we get the decomposition of the vector $\mathbf{j}$ onto the vectors $\mathbf{B}$ and $\mathbf{q}$:

$$\mathbf{j} = \left(\alpha - \frac{p'F}{B^2}\right)\mathbf{B} + 2\pi p'\mathbf{q}. \tag{33}$$

From here we find using the equality $(\mathbf{q}\cdot\mathbf{b}) = 1$ (17)

$$II.\quad 2\pi(\mathbf{j}\cdot\mathbf{b}) = 4\pi^2 p'(\Psi) + b^2 F\left(\alpha - \frac{p'F}{B^2}\right). \tag{34}$$

The third expression for the same value can be obtained using the "m – representation" [13] for the current density

$$\mathbf{j} = \alpha_m(\Psi)\mathbf{B} + p'(\Psi)C(\Psi)[\mathbf{B}\nabla\Psi] + p'[\nabla\Psi\nabla m] \tag{35}$$

which explicitly satisfies the $\operatorname{div}\mathbf{j} = 0$ condition. Here m is a "magnetic function", its definition is given below. On the closed toroidal magnetic surfaces, this function is an invariant combination of the functions η, ν, ϕ entering the contra- and covariant representations (see [11]), eq.60 and [13], eq.89):

$$2\pi m = \frac{\phi}{< B^2 >} + \frac{\nu}{p'} - \frac{\alpha_0}{p'}\eta, \tag{36}$$

where

$$\alpha_0 = \frac{< \mathbf{j} \cdot \mathbf{B} >}{< B^2 >}. \tag{37}$$

Below (if it is not mentioned specially) we do not imply magnetic surface closedness and do not use these equalities.

Rewriting in (35) the terms $[\mathbf{B}\nabla\Psi]$ and $[\nabla\Psi\nabla m]$ with the use of (19), (20) in the form

$$[\mathbf{B}\nabla\Psi] = 2\pi B^2\mathbf{q} - F\mathbf{B}, \tag{38}$$

$$[\nabla\Psi\nabla m] = 2\pi\left\{(\mathbf{q} \cdot \nabla m)\mathbf{B} - (\mathbf{B} \cdot \nabla m)\mathbf{q}\right\}, \tag{39}$$

we get the projections of the current density vector $\mathbf{j}$ (35) onto the vectors $\mathbf{B}$ and $\mathbf{q}$:

$$\mathbf{j} = \{\alpha_m - p'CF + 2\pi p'(\mathbf{q}\nabla m)\}\mathbf{B} + 2\pi p'\left(CB^2 - \mathbf{B}\nabla m\right)\mathbf{q}. \tag{40}$$

Comparing the representation with (33), we get the magnetic differential equation (MDE) for m and the relation between m, α and $\alpha_m(\Psi)$:

$$\mathbf{B}\nabla m = CB^2 - 1, \tag{41}$$

$$\alpha = \frac{[\mathbf{B}\nabla p]\nabla m}{B^2} = \alpha_m - p'F\left(C - \frac{1}{B^2}\right) + 2\pi p'(\mathbf{q}\nabla m). \tag{42}$$

The multiplier $C(\Psi)$ should be determined from the periodicity condition for the solution of the equation (41) the short and long ways around the torus. For closed magnetic surfaces multiplying the equations (41),(42) by B^2 and averaging over the volume of the magnetic layer we get the representation for $C(\Psi)$ and $\alpha_m(\Psi)$ [13]

$$\begin{aligned} C(\Psi) &= 1/ < B^2 >, \\ \alpha_m &= \alpha_0 - 2\pi p'\frac{< B^2(\mathbf{q}\nabla m) >}{< B^2 >} = \alpha_0. \end{aligned} \tag{43}$$

Here we take into account that the relations $\operatorname{div}(\mathbf{q}B^2) = 0$, $\mathbf{q}\nabla\Psi = 0$ and (24) lead to the equalities $< B^2(\mathbf{q}\nabla m) > = < \operatorname{div}(mB^2\mathbf{q}) > = 0$.

Finally from (40) we find using (19) and (43):

$$III. \quad 2\pi(\mathbf{j} \cdot \mathbf{b}) = Fb^2\left\{\alpha_0 - \frac{p'F}{< B^2 >} + 2\pi p'(\mathbf{q}\nabla m)\right\} + 4\pi^2 p'. \tag{44}$$

Three forms of the equation (27) for Ψ can be obtained corresponding to the three forms of $2\pi(\mathbf{j} \cdot \mathbf{b})$ representation: (30), (34) and (44).

3a. The $\nabla\Psi$ - equation for Ψ

Expressing the current density $\mathbf{j} = \operatorname{rot}\mathbf{B}$ in the right hand side of (30) with the use of the representation (13) for $\mathbf{B}$

$$2\pi\mathbf{j} = F \operatorname{rot}\mathbf{b} - F'[\mathbf{b}\nabla\Psi] - \operatorname{rot}[\mathbf{b}\nabla\Psi], \tag{45}$$

we get the equation (27) for Ψ in the form

$$\begin{aligned} \operatorname{div}(b^2\nabla\Psi) + \frac{[\mathbf{b}\nabla\Psi]}{|\nabla\Psi|^2}\cdot\left\{F\operatorname{rot}[\mathbf{b}\nabla\Psi] - (F^2 - |\nabla\Psi|^2)\operatorname{rot}\mathbf{b}\right\} = \\ F(\mathbf{b}\cdot\operatorname{rot}\mathbf{b}) - 4\pi^2 p' - b^2 F F'. \end{aligned} \tag{46}$$

It coincides with the equation obtained in [8, 9] with $\mathbf{b}_F = \mathbf{b}_\Psi = \mathbf{b}$. Except for the linear differential operator $\operatorname{div}(b^2\nabla\Psi)$ this equation contains the combination of second derivatives of Ψ with coefficients nonlinear in Ψ and its first derivatives and unknown components of the vector $\mathbf{b}$. It is hard to define its type which makes it difficult to set consistent boundary conditions and to choose the solution method. It seems that a clearer view on the problem setting can be reached when the set of equations consisting of equations (18) for the vector $\mathbf{b}$, the equation for Ψ and a separate equation for α or m is considered as a generalization of 2D equilibrium equation for Ψ (6) in the case of three dimensions. We will call the corresponding equations for Ψ the α–equation and the m–equation.

3b. The α–equation for Ψ

When the equation (34) is used for $2\pi(\mathbf{j}\cdot\mathbf{b})$, the equation for Ψ takes the form

$$\operatorname{div}(b^2\nabla\Psi) + \operatorname{rot}\mathbf{b}\cdot[\mathbf{b}\nabla\Psi] = F\mathbf{b}\cdot\operatorname{rot}\mathbf{b} - 4\pi^2 p'\frac{|\nabla\Psi|^2}{F^2 + |\nabla\Psi|^2} - b^2 F\alpha. \tag{47}$$

where α is determined by the expression (32).

The condition $\operatorname{div}\mathbf{j} = 0$ leads to the MDE for α:

$$\mathbf{B}\nabla\alpha = -p'[\mathbf{B}\nabla\Psi]\nabla\left(\frac{1}{B^2}\right). \tag{48}$$

This equation for α presents the main difficulty in the solution of equilibrium equations. The periodicity condition for the solution of the MDE

$$\mathbf{B}\nabla r = s \tag{49}$$

on each closed magnetic field line can be satisfied if and only if the right hand side satisfies the Newcomb condition

$$\oint s\frac{dl}{B} = 0. \tag{50}$$

In general this condition is not satisfied in 3D equilibria.

Using the representation (33) for the current density and the condition (24) for $\mathbf{q}B^2$, we get the modified equation for α:

$$\mathbf{B}\nabla\left(\alpha - \frac{p'F}{B^2}\right) = \frac{2\pi p'}{B^2}(\mathbf{q}\nabla)B^2. \tag{51}$$

Let us note that when the quasi–symmetry condition is satisfied [10, 11]

$$(\mathbf{q}\nabla)B^2 = 0, \tag{52}$$

the equation for α (51) is resolvable. On the closed surfaces the solution has the following form [11]:

$$\alpha - \frac{p'F}{B^2} = \alpha_0 - \frac{p'F}{< B^2 >}. \tag{53}$$

However with this additional condition, the equilibrium equations are overdefined and can be satisfied only approximately for selected magnetic configurations in a close vicinity of some magnetic surface [12]. To find such quasi–symmetric system one can try to solve the equation for Ψ with the function α defined by (53),

$$\mathrm{div}\,(b^2\nabla\Psi) + \mathrm{rot}\,\mathbf{b}\cdot[\mathbf{b}\nabla\Psi] = F(\mathbf{b}\cdot\,\mathrm{rot}\,\mathbf{b} - \alpha_0 b^2) - 4\pi^2 p'\left(1 - \frac{F^2 b^2}{4\pi^2 < B^2 >}\right). \tag{54}$$

The boundary conditions can be varied in order to be as close as possible to the condition $\mathbf{q}\nabla B^2 = 0$ or $\mathbf{b}\nabla B^2$ i.e.

$$(F\mathbf{B} + [\mathbf{B}\nabla\Psi])\,\nabla B^2 = 0. \tag{55}$$

If the condition (50) is not satisfied and therefore the equation for α has no solution, it implies that the smooth solution of the set of magnetostatic equations is nonexistent. According to conventional views, current sheets can arise at rational magnetic surfaces. The current density amplitude in the sheets can be nonuniform in the poloidal direction which leads to topology change in magnetic surfaces (island structures, stochastic regions) when a dissipation is present. It seems that in the frame of magnetostatics, the difficulties can be resolved by regularization of a diffusive type (see Section 4).

Concluding the subsection let us make two remarks.

1) By comparing the expressions for $2\pi(\mathbf{j}\cdot\mathbf{b})$ (30) and (34) one can get the expression for α in terms of $\nabla\Psi$

$$\alpha = \frac{p'F}{B^2} + F' + \frac{[\mathbf{b}\nabla\Psi]\cdot\,\mathrm{rot}\,[\mathbf{b}\nabla\Psi]}{b^2|\nabla\Psi|^2} - F\frac{\mathrm{rot}\,\mathbf{b}\cdot[\mathbf{b}\nabla\Psi]}{b^2|\nabla\Psi|^2}. \tag{56}$$

2) In force–free magnetic fields, where $\mathbf{B}\nabla\alpha = 0$, the 3D equilibrium problem is reduced to the solution of the equation (47) with prescribed $\alpha(\Psi)$ and the equations for the base vector field $\mathbf{b}$.

3c. The m–equation for Ψ

The equation for Ψ which follows from the representation (35) for $\mathbf{j}$ with the magnetic function m is based on the assumption of nested toroidal flux surfaces. In this case (see [11], equation (8))

$$\alpha_0 - \frac{p'F}{< B^2 >} = F'. \tag{57}$$

So the expression for $2\pi(\mathbf{j}\cdot\mathbf{b})$ (44) becomes simpler and the equation for Ψ takes the form:

$$\mathrm{div}\,(b^2\nabla\Psi) + \mathrm{rot}\,\mathbf{b}\cdot[\mathbf{b}\nabla\Psi] = F\mathbf{b}\cdot\mathrm{rot}\,\mathbf{b} - 4\pi^2 p' - FF'b^2 - 2\pi p' F b^2(\mathbf{q}\nabla m). \tag{58}$$

The function m is defined by the equation (41) bringing the same difficulties as the equation for α. Let us note that from the quasi–symmetry equation $\mathbf{q}\nabla B^2 = 0$ together with the equation (24), it follows that the requirement $\mathrm{div}\,\mathbf{j} = 0$ leads to $\mathbf{B}\cdot\nabla(\mathbf{q}\nabla m) = 0$ and further under periodic conditions to the equality: $\mathbf{q}\nabla m = 0$. In this case the equation (58) takes the simplest form:

$$\mathrm{div}\,(b^2\nabla\Psi) + \mathrm{rot}\,\mathbf{b}\cdot[\mathbf{b}\nabla\Psi] = F\mathbf{b}\cdot\mathrm{rot}\,\mathbf{b} - 4\pi^2 p' - FF'b^2. \tag{59}$$

Again the problem is overdefined under such condition and the solution can exist only for special choice of magnetic configuration (see for example [10]).

4 Regularization of the three–dimensional equilibrium equation

Three 3D equilibrium problem formulations were considered in the previous Section. In the second and third one, the functions α and m satisfying magnetic differential equations were introduced. These equations are hyperbolic and the conditions of their solvability are not satisfied at the rational magnetic surfaces in general. However, an introduction of a small diffusive terms into the equations can provide existence of the solution. From the point of view of the 3D equilibrium problem formulation, it corresponds to the substitution of the exact equilibrium equation (3) with a model one. Let us consider some possible approaches to such a regularization.

4a. The regularization of α-equation for Ψ (first variant)

Let us choose as model equilibrium equation the following equality

$$[\mathbf{j} - \epsilon F_0\nabla\hat{\alpha}, \mathbf{B}] = \nabla p. \tag{60}$$

Here F_0 is a dimensional constant (for example $F_0 = F_{axe}$), ϵ - a small dimensionless parameter. By definition of the magnetic flux, $\mathbf{B}\nabla\Psi = 0$. From (60) it follows that $\mathbf{B}\nabla p = 0$ i.e. $p = p(\Psi)$. However in this model $\mathbf{j}\cdot\nabla p \neq 0$:

$$\mathbf{j}\cdot\nabla\Psi = \epsilon F_0(\nabla\hat{\alpha}\cdot\nabla\Psi). \tag{61}$$

The additional term $\epsilon F_0\nabla\hat{\alpha}$ also changes the expression for toroidal current density. Now we have

$$\mathbf{j} = \hat{\alpha}\mathbf{B} + p'\frac{[\mathbf{B}\nabla\Psi]}{B^2} + \epsilon F_0\nabla\hat{\alpha}, \tag{62}$$

$$\alpha = \hat{\alpha} + \epsilon F_0\frac{\mathbf{B}\nabla\hat{\alpha}}{B^2}.$$

Instead of (33) we get

$$\mathbf{j} = \left(\hat{\alpha} - \frac{p'F}{B^2}\right)\mathbf{B} + 2\pi p'\mathbf{q} + \epsilon F_0 \nabla\hat{\alpha}. \tag{63}$$

For the "toroidal" component of the current density we get

$$2\pi\mathbf{j}\cdot\mathbf{b} = Fb^2\hat{\alpha} + p'\frac{b^2|\nabla\Psi|^2}{B^2} + 2\pi\epsilon F_0(\mathbf{b}\nabla\hat{\alpha}). \tag{64}$$

The substitution of (64) into (27) and the condition $\operatorname{div}\mathbf{j} = 0$ applied to (62) give the equations for Ψ and $\hat{\alpha}$:

$$\operatorname{div}(b^2\nabla\Psi) + \operatorname{rot}\mathbf{b}\cdot[\mathbf{b}\nabla\Psi] = F\mathbf{b}\cdot\operatorname{rot}\mathbf{b} - 4\pi^2 p'\frac{|\nabla\Psi|^2}{F^2+|\nabla\Psi|^2} - Fb^2\hat{\alpha} - 2\pi\epsilon F_0(\mathbf{b}\nabla\hat{\alpha}), \tag{65}$$

$$\epsilon F_0\nabla^2\hat{\alpha} + \mathbf{B}\nabla\hat{\alpha} = -p'[\mathbf{B}\nabla\Psi]\nabla\left(\frac{1}{B^2}\right). \tag{66}$$

The term with ϵ is critically important in equation (66). It can be omitted in equation (65).

4b. The regularization of α–equation for Ψ (second variant)

Let us choose as a model equation the following equilibrium equation

$$[\mathbf{jB}] = (1 + \epsilon\mathbf{t}\cdot\nabla\alpha)p'\nabla\Psi, \tag{67}$$

where $\mathbf{t}$ is some vector with a dimension of length squared. Hence the current density has the form

$$\mathbf{j} = \alpha\mathbf{B} + (1 + \epsilon\mathbf{t}\cdot\nabla\alpha)p'\frac{[\mathbf{B}\nabla\Psi]}{B^2}, \tag{68}$$

and the equilibrium equation:

$$\operatorname{div}(b^2\nabla\Psi) + \operatorname{rot}\mathbf{b}\cdot[\mathbf{b}\nabla\Psi] = F\mathbf{b}\cdot\operatorname{rot}\mathbf{b} - 4\pi^2 p'\frac{|\nabla\Psi|^2}{F^2+|\nabla\Psi|^2}(1+\epsilon\mathbf{t}\cdot\nabla\alpha) - Fb^2\alpha. \tag{69}$$

Such a choice of the equilibrium model evidently provides the conventional relations: $\mathbf{B}\nabla\Psi = 0, \mathbf{j}\nabla\Psi = 0$, $\alpha = \mathbf{jB}/B^2$. Choosing for $\mathbf{t}$ the following expression

$$\mathbf{t} = \frac{[\mathbf{B}\nabla\Psi]}{p'(\Psi)F}, \tag{70}$$

we get the equation for α with the regularizing term:

$$\frac{\epsilon}{F}[\mathbf{B}\nabla\Psi]\cdot\nabla\left(\frac{[\mathbf{B}\nabla\Psi]\cdot\nabla\alpha}{B^2}\right) + \mathbf{B}\nabla\alpha = -p'[\mathbf{B}\nabla\Psi]\nabla\left(\frac{1}{B^2}\right). \tag{71}$$

In equation (69), the correction term with ϵ can be omitted when $\epsilon \ll 1$.

4c. The regularization of m–equation for Ψ

For the regularization of the equations (41), (58) let us use the following model equilibrium equation

$$[\mathbf{jB}] = (1 + \epsilon F_0 \nabla^2 m) p' \nabla \Psi. \tag{72}$$

It follows that $\mathbf{B}\nabla\Psi = 0$, $\mathbf{j}\nabla\Psi = 0$ and

$$\mathbf{j} = \alpha \mathbf{B} + (1 + \epsilon F_0 \nabla^2 m) p' \frac{[\mathbf{B}\nabla\Psi]}{B^2}. \tag{73}$$

The function m which enters (72) is introduced with the following equation to satisfy $\operatorname{div}\mathbf{j} = 0$ condition:

$$\mathbf{j} = \alpha_m(\Psi)\mathbf{B} + C(\Psi) p' [\mathbf{B}\nabla\Psi] + p' [\nabla\Psi\nabla m]. \tag{74}$$

Decomposing both representations of the vector $\mathbf{j}$ (73) and (74) onto the vectors $\mathbf{B}$ and $\mathbf{q}$ and comparing corresponding projections we get the equation for m and the relation between α and m:

$$\epsilon F_0 \nabla^2 m + \mathbf{B}\nabla m = (1 + \epsilon F_0 < \nabla^2 m >) \frac{B^2}{< B^2 >} - 1, \tag{75}$$

$$\begin{aligned} \alpha = \alpha_0 + 2\pi p' \left(\mathbf{q}\nabla m - \frac{< B^2 (\mathbf{q}\nabla m) >}{< B^2 >} \right) + \\ p' F \left(\frac{1 + \epsilon_m F_0 \nabla^2 m}{B^2} - \frac{1 + \epsilon_m F_0 < \nabla^2 m >}{< B^2 >} \right). \end{aligned} \tag{76}$$

Taking into account the terms with ϵ_m in the expression (44) for $(\mathbf{j}\cdot\mathbf{b})$ we have here

$$\begin{aligned} 2\pi(\mathbf{j}\cdot\mathbf{b}) = \\ F b^2 \left\{ \alpha_0 - \frac{p' F}{< B^2 >} (1 + \epsilon_m F_0 < \nabla^2 m >) + 2\pi p' \left(\mathbf{q}\nabla m - \frac{< B^2 (\mathbf{q}\nabla m) >}{< B^2 >} \right) \right\} + \\ 4\pi^2 p' (1 + \epsilon_m F_0 \nabla^2 m). \end{aligned} \tag{77}$$

Substituting (77) into (27), we get the regularized form of the m-equation for Ψ with the terms keeping ϵ_m. The correction terms with small parameter ϵ_m are crucially important in the left hand side of the MDE (75) only. In addition if the magnetic surfaces are closed then the following term vanishes: $< B^2 (\mathbf{q}\nabla m) > = < \operatorname{div}(m B^2 \mathbf{q}) > = 0$, and for Ψ one can use the equation (58), and for m – the equation

$$\epsilon F_0 \nabla^2 m + \mathbf{B}\nabla m = \frac{B^2}{< B^2 >} - 1. \tag{78}$$

For closed magnetic surfaces the functions α and m are connected by the following relation

$$\alpha = \alpha_0 + 2\pi p' (\mathbf{q}\cdot\nabla m) + p' F \left(\frac{1}{B^2} - \frac{1}{< B^2 >} \right). \tag{79}$$

5 The problem of the base vector field b determination

Here we consider the problem to find the vector **b** from the equations (18). The second and third equations of the system can be satisfied by introducing one dimensionless function $\theta(\mathbf{r})$ such that

$$(F^2 + |\nabla\Psi|^2)\mathbf{b} = G(\Psi)[\nabla\Psi\nabla\theta]. \tag{80}$$

Taking the scalar product of this equality with $[\mathbf{B}\nabla\Psi]$, we get the connection of the normalizing coefficient G with the magnetic field:

$$2\pi B^2 = -G(\Psi)(\mathbf{B}\nabla\theta). \tag{81}$$

The dimensionless function θ serves as a label for the closed lines of the vector **b** on the magnetic surface $\Psi(\mathbf{r}) = const$. According to (15), it must satisfy the normalization

$$\oint \frac{G|[\nabla\Psi\nabla\theta]|}{F^2 + |\nabla\Psi|^2} dl = 2\pi, \tag{82}$$

where integration is performed along the vector **b** line.

When the vector **b** is chosen in the form (80), then the expression (13) for the magnetic field becomes:

$$2\pi\mathbf{B} = \frac{G}{F^2 + |\nabla\Psi|^2}(F[\nabla\Psi\nabla\theta] + [\nabla\Psi[\nabla\Psi\nabla\theta]]). \tag{83}$$

The condition $\operatorname{div}\mathbf{B} = 0$ leads to the elliptic type equation for θ on the magnetic surface $\Psi(\mathbf{r}) = const$:

$$\operatorname{div}\frac{[\nabla\Psi[\nabla\Psi\nabla\theta]] + F[\nabla\Psi\nabla\theta]}{F^2 + |\nabla\Psi|^2} = 0. \tag{84}$$

The function θ with the normalization (81), (82) must be periodic the long way around torus.

In a domain with closed magnetic surfaces, the function $\theta(\mathbf{r})$ corresponds to the poloidal angle variable of Boozer flux coordinates, it is monotone in the limits $0 \leq \theta \leq 2\pi$. One can check that in this case

$$G(\Psi) = <B>^2 V'(\Psi), \tag{85}$$

and instead of (84) we get the following equation for θ:

$$\operatorname{div}\nabla_\Psi\theta = \frac{2\pi F}{<B>^2 V'}[\mathbf{B}\nabla\Psi]\nabla\left(\frac{1}{|\nabla\Psi|^2}\right), \tag{86}$$

where

$$\nabla_\Psi = \nabla - \nabla\Psi\frac{(\nabla\Psi\nabla\ \ )}{|\nabla\Psi|^2}. \tag{87}$$

It certainly coincides with the equation corresponding to the Boozer coordinate θ presented in [8] (equation (2.62)).

The iterative process to solve the whole system of the 3D equilibrium equations presented should still be worked out.

6 Conclusions

We presented a new version of plasma equilibrium equations in toroidal systems based on the introduction of one "base" field **b**. The vector **b** lines are closed and lie on the magnetic surfaces. Their topology determines the poloidal magnetic flux Ψ and the plasma current F included into the magnetic field representation: $2\pi\mathbf{B} = F\mathbf{b} + [\nabla\Psi\mathbf{b}]$. In the systems with closed magnetic surfaces, the vector $\mathbf{q} = \mathbf{b}/b^2$ coincides with the toroidal basis vector of the Boozer flux coordinates, and it can be expressed through the functions Ψ and θ periodic the long way around torus. The equation for Ψ is rewritten in several forms (the equations (46), (47), (58)) using the base vector field **b** (80). For the label θ of this vector field, equation (84) is obtained which for nested magnetic surfaces coincides with the equation for the poloidal angle variable of Boozer coordinates.

For the resolution of the difficulties connected with the possibility of singular current sheets on rational magnetic surfaces, we propose model equilibrium equations with explicitly extracted parameters α or m that determine secondary currents. Including small terms with second derivatives allows to get model elliptic type equations for these functions (equation (66) or (71) for α and equation (75) for m).

In the general case, the system of equations presented does not imply closed toroidal surfaces and it is appropriate for the description of an island structure at least repeating the topology of the base vector **b** lines. More complicated topology likely needs the separation of the plasma volume onto discrete regions with different topology for the base vector **b**.

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A The secondary current extraction in the equilibrium equations with two base vector fields $\mathbf{b}_F$, $\mathbf{b}_\Psi$

The base vectors $\mathbf{b}_F$, $\mathbf{b}_\Psi$ are determined by the equations [8, 9]

$$\mathbf{b}_F \cdot \nabla\Psi = 0, \quad \operatorname{div}[\mathbf{b}_F \nabla\Psi] = 0, \quad \operatorname{div}(|\nabla\Psi|^2 \mathbf{b}_F) = 0, \tag{A.1}$$

$$\mathbf{b}_\Psi \cdot \nabla\Psi = 0, \quad \operatorname{div}[\mathbf{b}_\Psi \nabla\Psi] = 0, \quad \operatorname{div}\mathbf{b}_\Psi = 0, \tag{A.2}$$

that can be satisfied by means of two scalar functions λ_F, λ_Ψ such that $\mathbf{b}_F|\nabla\Psi|^2 = [\nabla\Psi\nabla\lambda_F]$, $\mathbf{b}_\Psi = [\nabla\Psi\nabla\lambda_\Psi]$. For these functions, the following equations are satisfied

$$\operatorname{div}\frac{[\nabla\Psi[\nabla\lambda_F\nabla\Psi]]}{|\nabla\Psi|^2} = 0, \quad \operatorname{div}[\nabla\Psi[\nabla\lambda_\Psi\nabla\Psi]] = 0. \tag{A.3}$$

Let us note that in contrast to the representation with one vector $\mathbf{b}$, these two vector fields $\mathbf{b}_\Psi, \mathbf{b}_F$ are determined only by the shape of the magnetic surfaces (when they are nested) and by the prescribed topology of their field lines. Moreover, the vectors $\mathbf{b}_\Psi, \mathbf{b}_F$ satisfy the normalization condition (15) for any contour with the same topology lying on the magnetic surface.

Expressions for $\nabla\Psi$ and $[\mathbf{B}\nabla\Psi]$

Let us use the notations

$$\mathbf{q}_\Psi = \frac{\mathbf{b}_\Psi}{\mathbf{b}_F \cdot \mathbf{b}_\Psi}, \quad \mathbf{q}_F = \frac{\mathbf{b}_F}{\mathbf{b}_F \cdot \mathbf{b}_\Psi}, \tag{A.4}$$

Then

$$\mathbf{q}_\Psi \cdot \mathbf{b}_F = 1. \tag{A.5}$$

From the formula (11) $2\pi\mathbf{B} = [\nabla\Psi\mathbf{b}_F] + F\mathbf{b}_\Psi$ it follows that

$$b_F^2\nabla\Psi = 2\pi[\mathbf{b}_F\mathbf{B}] - F[\mathbf{b}_F\mathbf{b}_\Psi], \tag{A.6}$$

$$\nabla\Psi = 2\pi[\mathbf{q}_\Psi\mathbf{B}]. \tag{A.7}$$

For $[\mathbf{B}\nabla\Psi], (\mathbf{B}\cdot\mathbf{b}_\Psi)$ and $(\mathbf{B}\cdot\mathbf{q}_F)$ we get using (A.7):

$$[\mathbf{B}\nabla\Psi] = 2\pi B^2\mathbf{q}_\Psi - 2\pi(\mathbf{B}\cdot\mathbf{q}_\Psi)\mathbf{B}, \tag{A.8}$$

$$2\pi(\mathbf{B}\cdot\mathbf{b}_\Psi) = [\mathbf{b}_\Psi\nabla\Psi]\mathbf{b}_F + Fb_\Psi^2, \quad 2\pi(\mathbf{B}\cdot\mathbf{q}_F) = F. \tag{A.9}$$

Current density and its toroidal projection

For the equilibrium current density $\mathbf{j} = \alpha\mathbf{B} + [\mathbf{B}\nabla p]/B^2$ using (A.8) and taking into account that $p = p(\Psi)$ we get:

$$\mathbf{j} = [\alpha - \frac{p'}{B^2}2\pi(\mathbf{B}\cdot\mathbf{q}_\Psi)]\mathbf{B} + 2\pi p'\mathbf{q}_\Psi, \tag{A.10}$$

and further:

$$2\pi(\mathbf{j}\mathbf{b}_F) = F(\mathbf{b}_F \cdot \mathbf{b}_\Psi)[\alpha - \frac{p'}{B^2}2\pi(\mathbf{B}\mathbf{q}_\Psi)] + 4\pi^2 p'. \qquad \text{(A.11)}$$

For the m–representation of the current density (35) with the equation (A.7), we get the following projections of the vector $\mathbf{j}$ onto the vectors $\mathbf{B}$ and $\mathbf{q}_\Psi$:

$$\mathbf{j} = \{\alpha_m(\Psi) - 2\pi p' C(\mathbf{B}\mathbf{q}_\Psi) + 2\pi p'(\mathbf{q}\cdot\nabla m)\}\,\mathbf{B} + 2\pi p'(CB^2 - \mathbf{B}\nabla m)\mathbf{q}_\Psi. \qquad \text{(A.12)}$$

Comparison of this representation with (A.10) leads to the equations analogous to (41), (42)

$$\mathbf{B}\nabla m = CB^2 - 1, \qquad \text{(A.13)}$$

$$\alpha = \alpha_m - 2\pi p'(\mathbf{B}\cdot\mathbf{q}_\Psi)\left(C - \frac{1}{B^2}\right) + 2\pi p'(\mathbf{q}_\Psi \cdot \nabla m). \qquad \text{(A.14)}$$

The analogue of the formula (44) takes the form

$$2\pi(\mathbf{j}\cdot\mathbf{b}_F) = F(\mathbf{b}_F\cdot\mathbf{b}_\Psi)\left[\alpha_0 - \frac{p'}{< B^2 >}2\pi(\mathbf{B}\cdot\mathbf{q}_\Psi) - 2\pi p'(\mathbf{q}_\Psi\cdot\nabla m)\right] + 4\pi^2 p'. \qquad \text{(A.15)}$$

Equations for Ψ and α

Applying the operator div to the equality (A.6) we get the equation for Ψ:

$$\mathrm{div}\,(b_F^2\nabla\Psi) = 2\pi(\mathbf{B}\cdot\ \mathrm{rot}\,\mathbf{b}_F) - 2\pi(\mathbf{j}\cdot\mathbf{b}_F) - \ \mathrm{div}\,F[\mathbf{b}_F\mathbf{b}_\Psi]. \qquad \text{(A.16)}$$

Substituting $\mathbf{B}$ from (11) and the term $2\pi(\mathbf{j}\cdot\mathbf{b}_F)$ from (A.11) we get the analogue of the equation (47):

$$\mathrm{div}\,(b_F^2\nabla\Psi) + \ \mathrm{rot}\,\mathbf{b}_F\cdot[\mathbf{b}_F\nabla\Psi] = F(\mathbf{b}_F\cdot\ \mathrm{rot}\,\mathbf{b}_\Psi) - F'(\Psi)[\mathbf{b}_\Psi\nabla\Psi]\cdot\mathbf{b}_F -$$

$$4\pi^2 p' \frac{b_F^2|\nabla\Psi|^2 + F[\mathbf{b}_\Psi\nabla\Psi]\cdot\mathbf{b}_F}{F^2 b_\Psi^2 + 2F[\mathbf{b}_\Psi\nabla\Psi]\cdot\mathbf{b}_F + b_F^2|\nabla\Psi|^2} - (\mathbf{b}_\Psi\cdot\mathbf{b}_F)F\alpha. \qquad \text{(A.17)}$$

The condition $\mathrm{div}\,\mathbf{j} = 0$ with $\mathbf{j}$ from the formula (A.10) leads, taking into account $\mathrm{div}\,\mathbf{b}_\Psi = 0$, to the following equation for α:

$$\mathbf{B}\nabla\left\{\alpha - \frac{2\pi p'}{B^2}(\mathbf{B}\mathbf{q}_\Psi)\right\} = 2\pi p'\mathbf{q}_\Psi\cdot\nabla(\mathbf{b}_\Psi\cdot\mathbf{b}_F). \qquad \text{(A.18)}$$

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