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FITTING OF TRANSFER FUNCTIONS TO FREQUENCY RESPONSE MEASUREMENTS

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ABSTRACT

An algorithm for approximating a given complex frequency response with a rational function of two polynomials with real coefficients is presented, together with its extension to distributed parameter systems, the corresponding error analysis and its application to a real case.

1. Introduction

In many situations it is possible to characterise a real system by measuring its frequency response, using for example homodyne synchronous detection or spectral analysis. The measured complex transfer function may consist of a large number of data points and it may be useful to express it in a more compact form, so that the system may be characterised by a reduced set of parameters. The simplest case is that of a single-input single-output linear and stationary system described by an ordinary differential equation

$$\sum_{n=0}^{N} a_n \frac{d^n y(t)}{dt^n} = \sum_{m=0}^{M} b_m \frac{d^m u(t)}{dt^m} , \qquad (1)$$

where u and y are respectively the input and the output, and $\{a_0,\ldots,a_N,b_0,\ldots,b_M\}$ are real coefficients which characterise the system. The transfer function of such a system is obtained by taking the Fourier transform of equation (1) and takes the form of a rational function of two polynomials in $j\omega$ with real coefficients:

$$H(\omega) \equiv \frac{y(j\omega)}{u(j\omega)} = \frac{\sum_{m=0}^{M} b_m (j\omega)^m}{\sum_{n=0}^{N} a_n (j\omega)^n} . \qquad (2)$$

The data reduction task then consists of adjusting these coefficients so that this rational function approximates the measured transfer function.

An algorithm has been specifically developed to perform this reduction and is presented in the next section. Its extension to distributed parameter systems is then derived (section 3) followed by an appropriate error analysis (section 4). The application to the study of the JET plasma response in the Alfvén eigenmode frequency range illustrates the power of the method (section 5).

2. Algorithm

The core problem is that of approximating an experimental complex function $H(\omega_k)$ measured for a set of discrete values ω_k with a rational function of polynomials in $j\omega$ with real coefficients.

In the field of automatic control this approximation task is referred to as the H_∞ method; H_∞ denotes here the class of the proper analytic functions which are real on the real axis and have no poles in the right hand plane. They aim at setting up a controller for a plant whose performances have been specified in the frequency domain. Calculation methods have been developed for this purpose [1]. Two difficulties arise when applying the previously proposed methods to the present problem: firstly, the calculation requires at some step the evaluation of the Fourier coefficients of the function to be approximated, which may prove cumbersome if this function is known only in a restricted frequency range. Secondly, the method is not readily extensible to multi-variable systems. Thus the quest for a better suited algorithm seems inescapable.

The problem may be posed in these terms: which rational function in $j\omega$ with real coefficients and with a given numerator and denominator degrees minimises an adequate cost function that is chosen to quantify the deviation of this function from the measurements. The function to be found will be denoted by

$$\widehat{H}(\omega) = \frac{\sum_{m=0}^{M} b_m (j\omega)^m}{\sum_{n=0}^{N} a_n (j\omega)^n} = \frac{B(j\omega)}{A(j\omega)}, \qquad (3)$$

$A(j\omega)$ being normalised so that $a_N = 1$. The differences with the measurements are

$$e(\omega_k) = \widehat{H}(\omega_k) - H(\omega_k), \qquad (4)$$

where k runs over the samples. If these are all independent, one obvious choice for the cost function is the mean square of the error modulus:

$$\sigma^2 = \frac{1}{K} \sum_{k=1}^{K} e^*(\omega_k)\, e(\omega_k) \,. \qquad (5)$$

The asterisk denotes the conjugate complex. Since the error is not a linear combination of the coefficients to be adjusted, the minimisation of this cost function is not straightforward and would require undesirable non-linear programming. Instead a slightly modified definition of the error can be used:

$$e_0 = A\, e = B - A\, H. \qquad (6)$$

This is now a linear combination of the sought parameters and the minimisation of the corresponding cost function is a simple linear operation, namely the resolution of the overdetermined system of linear equations

$$(j\omega_k)^N H(\omega_k) = - \sum_{n=0}^{N-1} a_n (j\omega_k)^n H(\omega_k) + \sum_{m=0}^{M} b_m (j\omega_k)^m \,, \quad k = 1,\ldots,K \,, \qquad (7)$$

with respect to the parameter set $\{a_0,\ldots,a_{N-1},b_0,\ldots,b_M\}$, under the constraint that these parameters be real (see Appendix 1).

Modifying the definition of the error can be seen as a weighting of the individual measurements with $A(j\omega_k)$. This weighting is not favorable since the modulus of A is small where the transfer function is large and where important information is to be found. This undesirable effect may by attenuated if an approximate value of A, A_0 is known, in which case a less biased error may be built:

$$e_1 = \frac{A}{A_0}\, e = \frac{B}{A_0} - \frac{A}{A_0}\, H \,. \qquad (8)$$

At this point the following iterative procedure may be set up: the cost function corresponding to the error e_1 is minimum for the two polynomials A_1 and B_1; these may in turn be used to define the next error:

$$e_2 = \frac{B}{A_1} - \frac{A}{A_1}\, H \,. \qquad (9)$$

The sequence of errors that this procedure defines in fact converges toward the original error e:

$$e_i = \frac{B_i}{A_{i-1}} - \frac{A_i}{A_{i-1}} H \rightarrow e \,, \qquad (10)$$

in the sense that A_i/A_{i-1} tends toward 1 and B_i/A_{i-1} towards the transfer function to be identified, B/A, thus removing the unfavorable weighting near the resonances. The most natural choice for the seed value of A_i is $A_0 \equiv 1$.

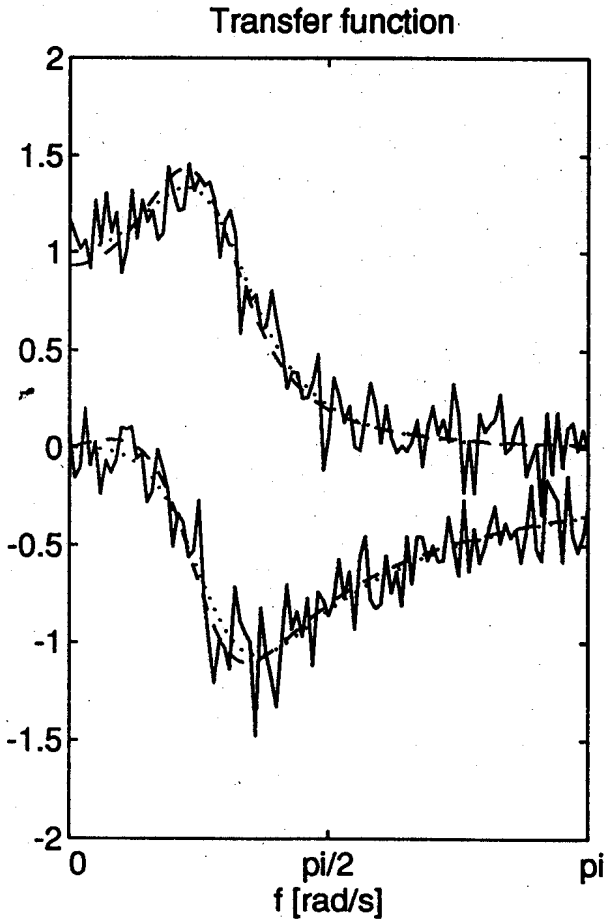

Fig. 1 Real and imaginary parts of the test transfer function given by equation (11). Solid line: raw signal, dotted line: exact signal, dashed line: best approximation.

The algorithm is illustrated with a simple second order transfer function,

$$H(\omega) = \frac{1 + j\omega}{1 + j\omega - \omega^2}, \qquad (11)$$

to which a gaussian noise of standard deviation 0.2 has been added (Fig. 1). Convergence is reached after a few iterations (Fig. 2). The difference with the unperturbed transfer function is due to the presence of noise.

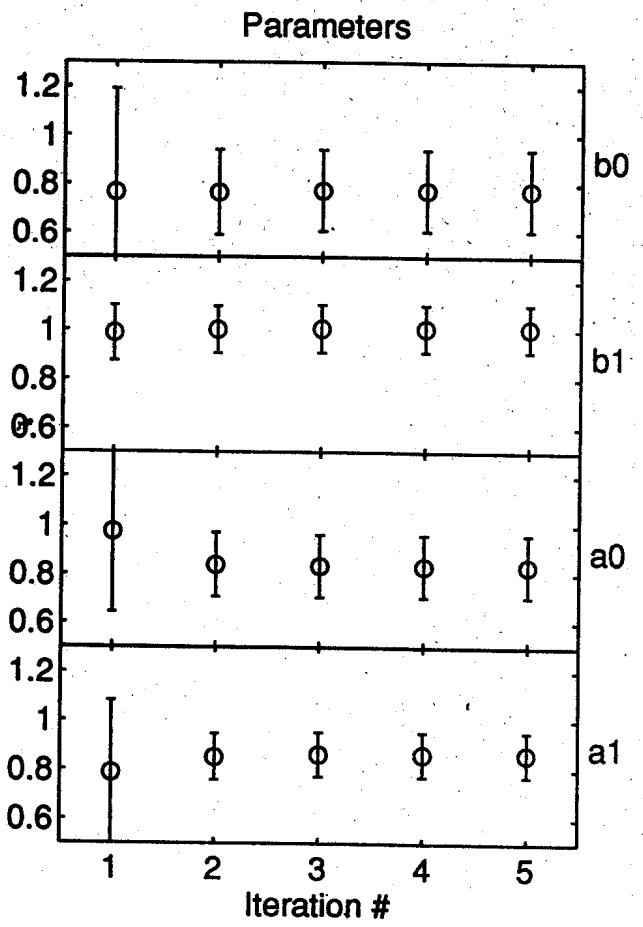

Fig. 2 Convergence of the transfer function parameters for the test case together with their error bars.

In this procedure the transfer function structure, i.e. the numerator and denominator degrees, must be chosen. This may be determined by a priori knowledge of the system dynamic. If not, inspection of the cost function when the degrees of the numerator and the denominator are

increased will show a clear saturation in the improvement of the cost function (Fig. 3), when the adequate structure is encountered [2]. Thus the transfer function structure can safely be chosen as the one with the minimum number of free parameters which yields an acceptable cost function.

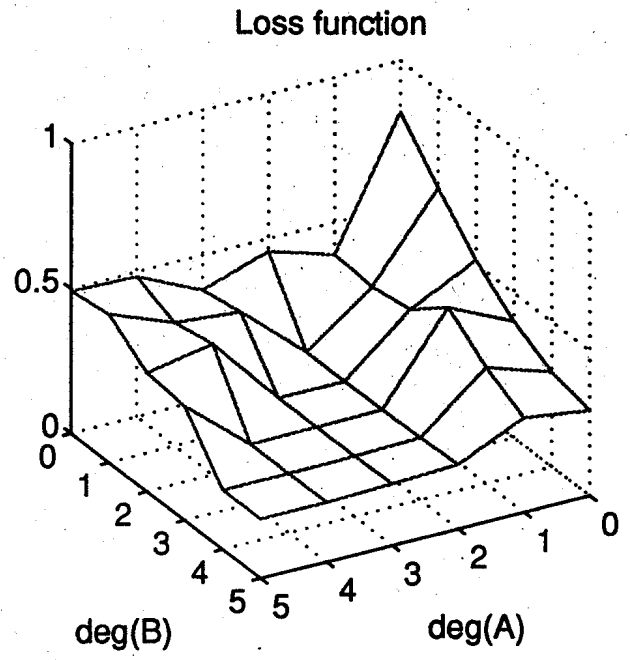

Fig. 3 Cost function for different transfer function structures.

3. Distributed parameter systems

It can be shown [3] that the transfer function of a distributed parameter system at different positions is characterised by a common denominator but distinct numerators. The proposed algorithm is easily extended to such a system. The overdetermined system to solve may be rewritten keeping the previous statement in mind:

$$(j\omega_k)^N H(\omega_k,x_l) = -\sum_{n=0}^{N-1} a_n (j\omega_k)^n H(\omega_k,x_l) + \sum_{m=0}^{M} b_m(x_l) (j\omega_k)^m \qquad (12)$$
$$k = 1,\ldots,K,\ l = 1,\ldots,L$$

and the definition of the corresponding cost function adapted:

$$\sigma^2 = \frac{1}{K}\frac{1}{L}\sum_{k=1}^{K}\sum_{l=1}^{L}\left|\frac{B(\omega_k,x_l)}{A(\omega_k)} - H(\omega_k,x_l)\right|^2 . \qquad (13)$$

Here x_l denotes the measurement positions.

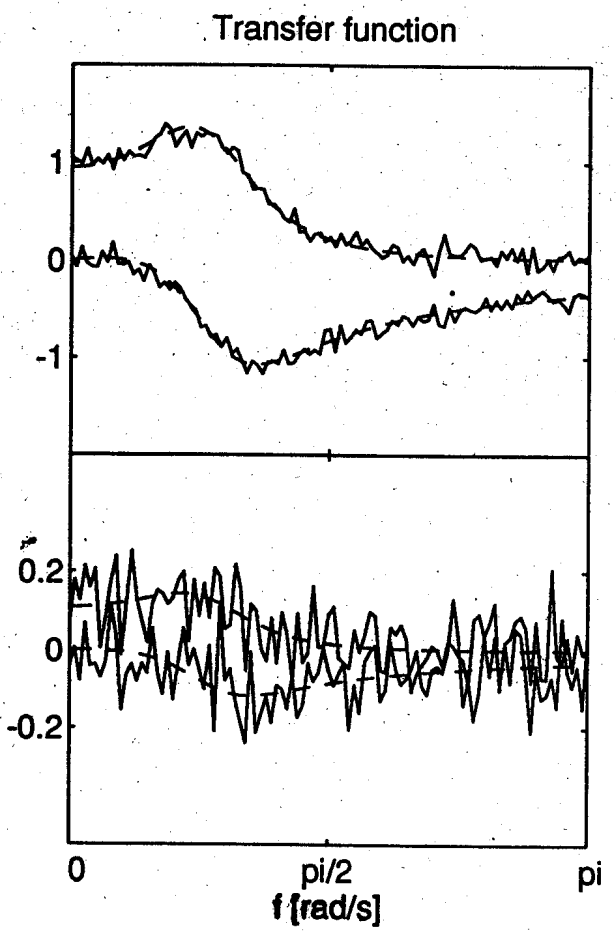

Fig. 4 Real and imaginary parts of a two valued transfer function with a small amplitude second value. Solid line: raw signals, dashed line: best approximation.

There are two situations in which the simultaneous analysis of many measurements may be advantageous. The statistical error decreases with an increasing number of treated signals, thus making possible the analysis of poor signal to noise measurements without reducing the noise level or

increasing the number of data points. In a similar way, combined analysis of good signals together with noisy signals or signals with a small amplitude may improve the estimation of the transfer function of the latter. The reliability of these estimations must then be tested against an appropriate error analysis, developed in the next section.

This is illustrated in the following example: the transfer function of the previous example with a noise level of 0.1 is analysed together with a similar transfer function, 10 times smaller in amplitude but with the same noise level, reaching then 100% (Fig. 4). It is, however, still possible to extract the transfer function of the smaller signal with a 15% standard deviation.

In coding the algorithm for multiple transfer functions, it is worth taking advantage of the fact that the independent variable matrix in the linear equation set (12) is sparse with a density decreasing with the number of analysed signals. Thus the algorithm will gain both in execution time and in memory usage if an appropriate solving method is chosen.

4. Error analysis

Any experimental data analysis should include an estimation of the error of the extracted parameters, giving some insight into the reliability of the results. A substantial amount of this information is contained in the covariance matrix of the estimated parameters.

This covariance matrix for the coefficients in the fitted rational function may be calculated using standard linear regression techniques, assuming that all samples of the measured transfer functions are independent and have the same probability distribution. For high transfer function orders however the interpretation of the polynomial coefficients is not straighforward and it may be easier to work with the parameters of the partial fraction decomposition:

$$H(\omega) = \frac{\sum_{m=0}^{M} b_m (j\omega)^m}{\sum_{n=0}^{N} a_n (j\omega)^n} = \frac{1}{2}\sum_{n=0}^{N} \frac{r_n}{j\omega - p_n} + \sum_{m=0}^{M-N} d_m (j\omega)^m . \qquad (14)$$

The covariance matrix of the parameter set $\{a_N,...,a_0,b_M,...,b_0\}$ will be denoted by C_{AB} and that of $\{r_1,...,r_N,p_1,...,p_N,d_0,...,d_{M-N}\}$ by C_{rpd}. The calculation to convert the two covariance matrices is as follows.
Differentiating the first form of equation (14) gives

$$dH = \frac{-B \sum_{n=0}^{N} da_n (j\omega)^n + A \sum_{m=0}^{M} db_m (j\omega)^m}{A^2} \qquad (15)$$

and the second form:

$$dH = \frac{\frac{a_N}{2} A \sum_{n=1}^{N} dr_n Q_n + \frac{a_N^2}{2} \sum_{n=1}^{N} r_n\, dp_n\, Q_n^2 + A^2 \sum_{m=0}^{M-N} dd_m (j\omega)^m}{A^2}, \qquad (16)$$

where Q_n denotes

$$Q_n = \frac{A}{a_N (j\omega - p_n)} . \qquad (17)$$

Equating the coefficients of each power of $j\omega$ of both forms of the differential together with some linear algebra (Appendix 2) yields the Jacobian of the mapping, J. Conversion of the covariance matrix then follows the usual rule:

$$C_{rpd} = J\, C_{AB}\, J^t . \qquad (18)$$

5. A specific application

The Alfvén Eigenmode Active Diagnostic (AEAD) of JET [4] aims at characterising the various Alfvén eigenmodes in different plasma conditions. Alfvén waves are externally driven by AC currents in the

antennae. The frequency range covered runs from 30 kHz to 500 kHz. The plasma response is recorded via parameters with appropriate temporal resolution, mainly the poloidal magnetic field and the voltage on the antennae used in reception. The synchronous homodyne detection electronics mixes the diagnostic signals with two reference signals in quadrature to produce the real and imaginary components of the response. Applying the same processing to the exciting AC current allows to build the transfer function.

It is necessary to specify exactly the expected form of the transfer function, thus avoiding an inadequate set of free parameters both in size or in their definition.

In the presence of a single resonance with resonance frequency ω_0 and damping rate γ, the transfer function will take the form

$$H(\omega) = \frac{c + d\,\frac{j\omega}{\omega_0}}{1 + \frac{2\gamma}{\omega_0}\frac{j\omega}{\omega_0} - \frac{\omega^2}{\omega_0^2}} . \qquad (19)$$

For small damping rate ($\gamma << \omega_0$), c and d are proportional respectively to the in-phase and in-quadrature components of the signal. Both coefficients are real for real signals.

The partial fraction decomposition will also be used, giving a representation in terms of complex conjugate poles and residues:

$$H(\omega) = \frac{1}{2}\left(\frac{r}{j\omega - p} + \frac{r^*}{j\omega - p^*}\right) . \qquad (20)$$

The following relations link the two parameter sets, writing $p = p_r + j\,p_i$ and $r = r_r + j\,r_i$:

$$\begin{aligned} \omega_0 &= |p| \\ \gamma &= -p_r \\ c &= \frac{(r_r p_r + r_i p_i)}{|p|^2} \\ d &= \frac{r_r}{|p|} \end{aligned} \qquad (21)$$

Again for small damping rate ($p_r << p_i$), the real and imaginary parts of the residue are proportional respectively to the in-quadrature and in-phase components of the signal.

For a local quantity which varies with position, the transfer function depends on some spatial coordinate x; this dependance only enters through its numerator; its denominator is the same at any position (Section 3); this implies that the poles are the same for all positions while the residues depend on the spatial coordinates:

$$H(\omega,x) = \frac{c(x) + d(x)\frac{j\omega}{\omega_0}}{1 + \frac{2\gamma}{\omega_0}\frac{j\omega}{\omega_0} - \frac{\omega^2}{\omega_0^2}} = \frac{1}{2}\left(\frac{r(x)}{j\omega - p} + \frac{r^*(x)}{j\omega - p^*}\right). \qquad (22)$$

If more than one resonance falls in the measured frequency range, the global transfer function is the superposition of all individual transfer functions:

$$H(\omega,x) = \frac{1}{2}\sum_{n=1}^{N/2}\left(\frac{r_n(x)}{j\omega - p_n} + \frac{r^*_n(x)}{j\omega - p^*_n}\right) = \frac{\sum_{m=0}^{N-1} b_m(x)\,(j\omega)^m}{\sum_{n=0}^{N} a_n\,(j\omega)^n} = \frac{B(j\omega,x)}{A(j\omega)}, \qquad (23)$$

N/2 being the number of resonances. Numerator and denominator degrees are N-1 and N respectively.

Some measured quantities may be sensitive to direct pick-up from the exciting current. This is in particular the case for the magnetic measurements. This adds a term equal to the intensity of this direct coupling to the transfer function:

$$H(\omega,x) = \frac{B(j\omega,x)}{A(j\omega)} + D(x) = \frac{B'(j\omega,x)}{A(j\omega)}. \qquad (24)$$

Here D is a real scalar depending on the measurement location. This transfer function can also be reduced to a ratio of two real coefficient polynomials whose numerator and denominator have equal degrees.

To summarize in all cases the expected transfer function takes the form of a rational polynomial function in $j\omega$ with real coefficients. These coefficients depend on the measurement location for the numerator but are fixed for the denominator.

The proposed algorithm is routinely used to extract form the frequency response measurements the characteristics of the resonances. Fig. 5 shows a typical response recorded by 15 magnetic probes located at different positions.

The selected frequency range contains a single resonance but the numerator degree of the fitted transfer function has been increased to 5 to account for a more complicated background signal than a simple contribution proportional to the exciting current. Note that the signal of probe #9 is a good example of a low amplitude signal whose analysis is made possible by a simultaneous processing with other good signal to noise ratio signals.

6. Conclusion

The problem of approximating a complex transfer function measured for discrete values of the frequency by a rational function of $j\omega$ has been addressed. A dedicated iterative algorithm has been developed to avoid non-linear minimisation programming. The method is readily extensible to distributed parameter systems. It has been successfully applied to the characterisation of Alfvén eigenmodes on JET and one can find numerous applications in other fields.

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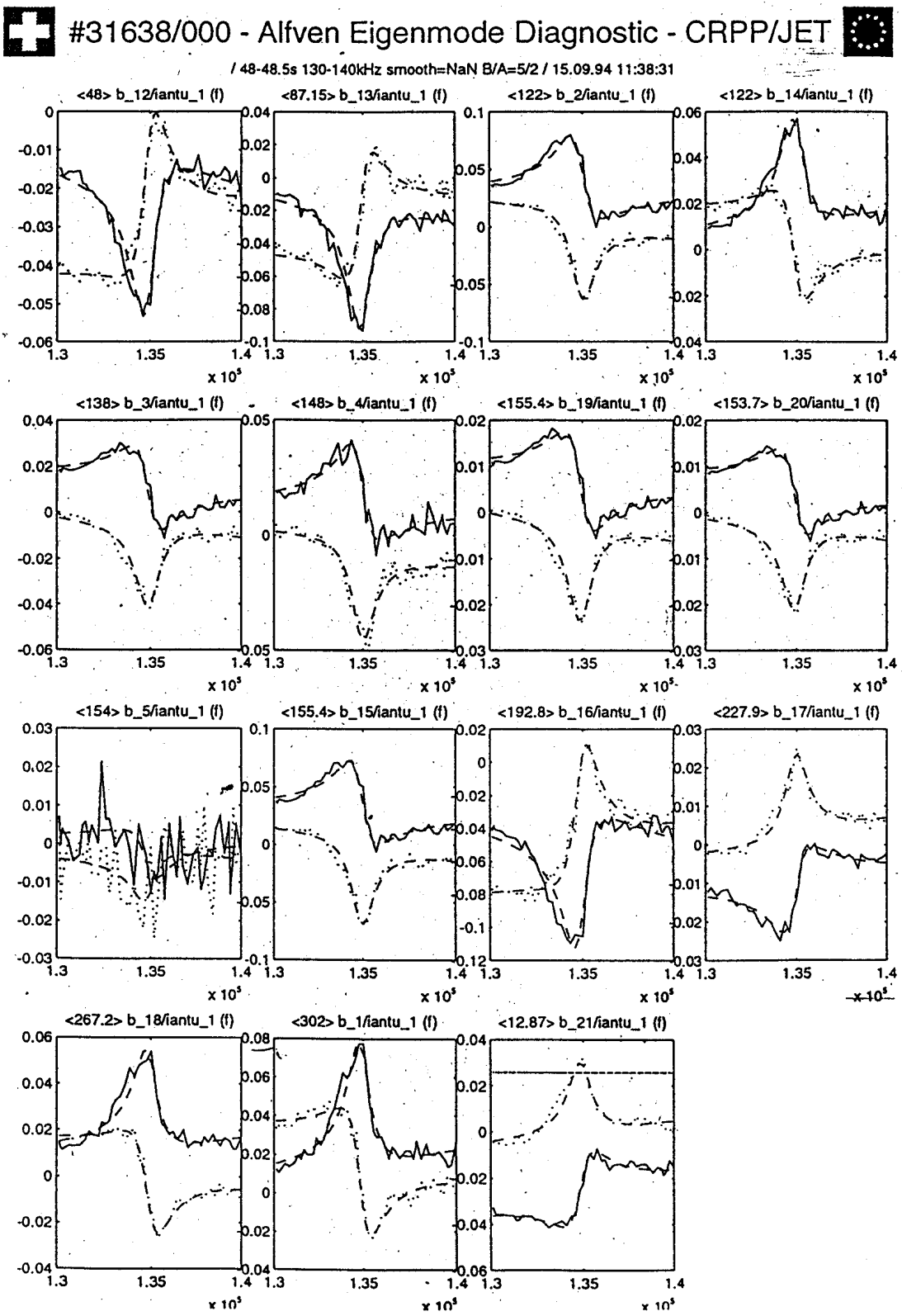

Fig. 5 An Alfvén eigenmode resonance measured by 15 magnetic probes. Solid line: real part, dotted line: imaginary part, dashed line: best fit approximation.

Appendix 1. Mixed real and complex linear regression

The problem is to solve a linear regression of complex variables with real coefficients

$$y \cong x \cdot a \,, \tag{25}$$

with $y \in \mathbb{C}^{Nx1}$, $x \in \mathbb{C}^{NxK}$, $a \in \mathbb{R}^{Kx1}$. A general formulation of the cost function is

$$\sigma^2 = (w \cdot e)^{*t} \cdot (w \cdot e) \,, \tag{26}$$

with $e = x \cdot a - y$ and w a given complex weighting matrix, that may be diagonal for simple weights or even the identity matrix for uniform weighting. This cost function reaches its minimum under the constraint that a is real, for

$$a = Re\left((x^{*t} \cdot w^{*t}) \cdot (w \cdot x)\right)^{-1} Re\left((x^{*t} \cdot w^{*t}) \cdot (w \cdot y)\right) . \tag{27}$$

Calculation of these coefficients may take advantage of the symmetry of the matrix to be inverted and use Choleski decomposition.

Appendix 2. Partial fraction decomposition Jacobian

The linear algebra hidden in Section 5 to compute the Jacobian of the mapping between the denominator and numerator coefficients $\{a_N,\ldots,a_0, b_M,\ldots,b_0\}$ to the residue and pole representation $\{r_1,\ldots,r_N,p_1\ldots,p_N,d_0,\ldots,d_{M-N}\}$ is derived here. The first step is to show that polynomial multiplication is a matrix operation if polynomials are written as scalar products of the form:

$$A = \left((j\omega)^N,\ldots,1\right) \cdot (a_N,\ldots,a_0)^t \,. \tag{28}$$

Explicitly the product of two polynomials A and B writes:

$$B\,A = \left((j\omega)^{N+M},\ldots,1\right)\cdot\begin{pmatrix} b_M & 0 & \cdots & 0 \\ b_{M-1} & b_M & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ b_0 & b_1 & & \vdots \\ 0 & b_0 & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & b_0 \end{pmatrix}\cdot\left(a_N,\ldots,a_0\right)^t \qquad (29)$$

The implied matrix is an M+N+1 x N+1 asymmetric Toeplitz matrix generated from the B coefficients that will be symbolised by T(B,N).

The two forms of the differential given in Section 5 may then be written as

$$A^2\,dH = \left((j\omega)^{M+N},\ldots,1\right)\cdot\left(-T(B,N)\ \ T(A,M)\right)\cdot\left(da_N,\ldots,da_0,db_M,\ldots,db_0\right)^t \qquad (30)$$

$$A^2\,dH = \left((j\omega)^{M+N},\ldots,1\right)\cdot \left(\begin{pmatrix} \frac{a_N}{2}\,T(A,N-1)\cdot\begin{pmatrix} Q_1 \\ \vdots \\ Q_N \end{pmatrix} & \frac{a_N^2}{2}\begin{pmatrix} r_1Q_1 \\ \vdots \\ r_NQ_N \end{pmatrix} \\ 0 & 0 \end{pmatrix} T(A^2,M-N)\right)\cdot \qquad (31)$$

$$\left(dr_1,\ldots,dr_N,dp_1,\ldots,dp_N,dd_0,\ldots,dd_{M-N}\right)^t$$

Denoting the two central matrices respectively G_{AB} and G_{rpd} and equating both expressions yield

$$\begin{aligned}&\left(dr_1,\ldots,dr_N,dp_1,\ldots,dp_N,dd_0,\ldots,dd_{M-N}\right)^t = \\ &G_{rpd}^{-1}\cdot G_{AB}\cdot\left(da_N,\ldots,da_0,db_M,\ldots,db_0\right)^t\end{aligned} \qquad (32)$$

and thus to the sought Jacobian

$$J = G_{rpd}^{-1}\cdot G_{AB}\,. \qquad (33)$$

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