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An energy functional given by $W = \int\int\int d^3x[B^2/(2\mu_0) + p_{\|}/(\Gamma - 1)]$ is proposed as a variational principle to determine three-dimensional ($3D$) magnetohydrodynamic (MHD) equilibria with anisotropic plasma pressure. It is demonstrated that the minimisation of W using an inverse coordinate spectral method reproduces the force balance relations that govern the MHD equilibrium properties of $3D$ plasmas with $p_{\|} \neq p_{\perp}$ that have nested magnetic flux surfaces. Numerical procedures already developed for the scalar pressure model can be easily extended to the anisotropic pressure model. Specifically, a steepest descent procedure coupled with the application of a preconditioning algorithm to improve the convergence behaviour has been employed to minimise the energy of the system. The numerical generation of $3D$ torsatron equilibria with highly localised anisotropic pressure distributions attests to the robustness of the method of solution considered.

1. Introduction

Auxiliary heating schemes required to drive magnetic plasma confinement configurations towards the regime that approaches the conditions imposed for sustained controlled thermonuclear fusion cause the plasma pressure to become anisotropic. In the Compact Helical System (CHS), for example, anisotropy factors (corresponding to the ratio of the pressure along the magnetic field lines $p_{\parallel}$ to the perpendicular pressure $p_{\perp}$) of the order of 3 have been measured with tangential neutral beam injection of 1.1 MW and energetic beam particle energy of 40 kev [1]. Furthermore, bootstrap currents have been measured in the $W7AS$ stellarator [2] and the ATF torsatron [3]. Full consistency between the bootstrap currents observed and the description of the MHD equilibrium state can only be reconciled when $p_{\parallel} \neq p_{\perp}$. Magnetic confinement configurations that rely on hot electron layers to provide the mechanism to stabilise the plasma such as the $ELMO$ Snakey Torus [4] can only be described with models in which the pressure is anisotropic. Thus, there is a motivation to develop numerical tools such as MHD equilibrium solvers with anisotropic plasma pressure. Until now, most anisotropic pressure MHD equilibrium solvers have been limited to axisymmetric [5] or helically symmetric geometry [6,7]. In stellarator geometry, an expansion approach has been applied to generate MHD equilibria with $p_{\parallel} \neq p_{\perp}$, but it is limited to cases in which the pressures remain exclusive functions of the radial variable [8].

Scalar pressure $3D$ MHD equilibrium codes based on finite difference schemes [9,10] and on spectral energy minimisation methods have been developed in the past [11-17]. In this article, we extend the formulation to allow for anisotropic pressure. Specifically, a positive-definite energy functional is devised whose variation yields the MHD equilibrium equations that describe $3D$ magnetic confinement systems with anisotropic pressure. Using an inverse coordinate spectral approach, the formulation is constrained to have nested magnetic flux surfaces with a single magnetic axis. Consequently, the determination of equilibria with magnetic islands, internal separatrices or X-points is excluded. A different approach is required to treat problems of this nature in a self-consistent manner [18-20]. Nevertheless, the constraint of flux surface nestedness does not appear to constitute a serious impediment for the applicability of codes such as $VMEC$ [17] to usefully model fully $3D$ configurations as has been demonstrated in the CHS compact torsatron [1]. The formulation we have adopted for the anisotropic pressure model permits a relatively straight forward adaptation of the methods previously developed to solve the scalar pressure problem. In particular, we have implemented the anisotropic pressure formulation of the $3D$ MHD equilibrium problem in a modified version of the $VMEC$ code [17]. We take advantage of the existing steepest descent procedure together with the preconditioning algorithm to improve the convergence properties to minimise the energy of the system [17]. This anisotropic pressure version of $VMEC$ is then applied to the ATF torsatron [3]. We numerically generate $3D$ equilibria with highly localised anisotropic pressure distributions that could serve to model an energetic particle layer induced by radio frequency or perpendicular neutral beam injection techniques.

2. The anisotropic pressure MHD equilibrium description

2.1. *The MHD equilibrium relations*

The basic relevant equations that describe MHD equilibrium properties in magnetically confined plasmas with anisotropic pressure are given by the Maxwell equation

$$\nabla \cdot \mathbf{B} = 0, \tag{1}$$

Ampère's Law

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} \tag{2}$$

and the MHD equilibrium force balance relation

$$\mathbf{F} = -\nabla \cdot \mathbf{P} + \mathbf{j} \times \mathbf{B}, \tag{3}$$

where $\mathbf{F}$ constitutes the residual MHD force and $\mathbf{P}$ denotes the pressure tensor that is expressed as [21]

$$\mathbf{P} = p_\perp \mathbf{I} + (p_{||} - p_\perp)\mathbf{bb}. \tag{4}$$

The magnetic field and the plasma current density field are $\mathbf{B}$ and $\mathbf{j}$, respectively. The permeability of free space is μ_0, $\mathbf{I}$ represents the identity matrix and $\mathbf{b} = \mathbf{B}/B$ corresponds to the unit vector along the magnetic field lines.

2.2. *The magnetic field representation*

We introduce a coordinate system (s, u, v) where the radial coordinate s labels the magnetic flux surfaces. Consequently, the condition $\mathbf{B} \cdot \nabla s = 0$ is imposed. The periodic poloidal and toroidal angle variables are denoted by u and v, respectively. A magnetic field that satisfies the conditions $\nabla \cdot \mathbf{B} = \mathbf{B} \cdot \nabla s = 0$ can be in general represented by the expression [11]

$$\mathbf{B} = \nabla v \times \nabla \Psi + \nabla \Phi \times \nabla \chi, \tag{5}$$

where χ is a generalised poloidal angle variable that makes the magnetic field lines straight. The function χ is related to the poloidal angle u through the expression

$$\chi = u + \lambda(s, u, v) \tag{6}$$

where λ is a periodic stream function that renormalises the poloidal angle in an iterative procedure to minimise the spectral width that is required to describe the MHD equilibrium state [11]. The toroidal magnetic flux is $2\pi\Phi(s)$ and the poloidal magnetic flux is $2\pi\Psi(s)$. Because Φ and Ψ are both functions of s, the magnetic flux surfaces are constrained to be nested with a single magnetic axis. Magnetic island structures are therefore excluded from consideration within this formulation of the problem.

The contravariant components of $\mathbf{B}$ are

$$B^u \equiv \mathbf{B} \cdot \nabla u = \frac{\Phi'(s)}{\sqrt{g}} \Big[\iota(s) - \frac{\partial \lambda}{\partial v} \Big], \tag{7}$$

$$B^v \equiv \mathbf{B} \cdot \nabla v = \frac{\Phi'(s)}{\sqrt{g}} \Big(1 + \frac{\partial \lambda}{\partial u} \Big), \tag{8}$$

$$B^s \equiv \mathbf{B} \cdot \nabla s = 0. \tag{9}$$

The B-field components within the covariant representation are related to the contravariant components through the metric elements, namely

$$B_\varsigma = g_{\varsigma u} B^u + g_{\varsigma v} B^v, \tag{10}$$

where ς denotes either s, u or v. The Jacobian has the relatively simple form

$$\sqrt{g} = R\left(\frac{\partial R}{\partial u}\frac{\partial Z}{\partial s} - \frac{\partial R}{\partial s}\frac{\partial Z}{\partial u}\right), \tag{11}$$

because v corresponds to the geometric toroidal angle [11]. The metric elements are expressed as

$$g_{\varsigma\varphi} \equiv \frac{\partial R}{\partial \varsigma}\frac{\partial R}{\partial \varphi} + R^2\frac{\partial v}{\partial \varsigma}\frac{\partial v}{\partial \varphi} + \frac{\partial Z}{\partial \varsigma}\frac{\partial Z}{\partial \varphi}, \tag{12}$$

where the labels ς and φ serve to identify s, u or v.

2.3. *Parallel force balance*

The scalar product of the magnetic field $\mathbf{B}$ with the force balance relation given by eq. (3) yields the expression

$$F_{\|} = -(\mathbf{B}\cdot\nabla)p_{\|} + \frac{p_{\|} - p_{\perp}}{B}(\mathbf{B}\cdot\nabla)B. \tag{13}$$

The conditions for an MHD equilibrium state are satisfied exactly when $\mathbf{F} = 0$. Thus requiring $F_{\|} = 0$ suggests the dependence $p_{\|} = p_{\|}(s, B)$ and $p_{\perp} = p_{\perp}(s, B)$ [22]. In principle, the pressures may also vary from field line to field line, but such dependence is ignored in this paper. The parallel force balance then reduces to [22]

$$\left.\frac{\partial p_{\|}}{\partial B}\right|_s = \frac{p_{\|} - p_{\perp}}{B}, \tag{14}$$

where the labelling indicates that the derivative of $p_{\|}$ with respect to B is to be evaluated at fixed s.

2.4. *The effective current density*

In the anisotropic pressure MHD equilibrium problem, it is convenient to define the effective plasma current density [22]

$$\mathbf{K} \equiv \nabla\times(\sigma\mathbf{B}), \tag{15}$$

which satisfies the same properties as $\mathbf{j}$ in the scalar pressure problem. In particular, $\mathbf{K}$ is divergence-free and at equilibrium $\mathbf{K}\cdot\nabla s = 0$. The anisotropy factor σ is written as [22]

$$\sigma = \frac{1}{\mu_0} - \frac{p_{\|} - p_{\perp}}{B^2} = \frac{1}{\mu_0} - \frac{1}{B}\frac{\partial p_{\|}}{\partial B}. \tag{16}$$

The poloidal and toroidal contravariant components of the effective current density field are, respectively

$$K^u \equiv \mathbf{K} \cdot \nabla u = \frac{1}{\sqrt{g}}\Big[\frac{\partial(\sigma B_s)}{\partial v} - \frac{\partial(\sigma B_v)}{\partial s}\Big], \tag{17}$$

$$K^v \equiv \mathbf{K} \cdot \nabla v = \frac{1}{\sqrt{g}}\Big[\frac{\partial(\sigma B_u)}{\partial s} - \frac{\partial(\sigma B_s)}{\partial u}\Big]. \tag{18}$$

2.5. *The rotational transform*

The effective toroidal plasma current is expressed as

$$2\pi J(s) = \frac{L}{2\pi}\int_0^{\frac{2\pi}{L}} dv \int_0^{2\pi} du \int_0^{s} ds\sqrt{g}\mathbf{K} \cdot \nabla v = 2\pi\langle\sigma B_u\rangle, \tag{19}$$

where we define the average as

$$\langle A\rangle \equiv \frac{L}{(2\pi)^2}\int_0^{\frac{2\pi}{L}} dv \int_0^{2\pi} du\sqrt{g}A(s,u,v). \tag{20}$$

Decomposing the poloidal magnetic field component B_u in the covariant representation according to eq. (10), we can obtain an expression for the rotational transform $\iota(s) \equiv d\Psi/d\Phi$ profile,

$$\iota(s) = \frac{\frac{J(s)}{\Phi'(s)} - \left\langle\sigma\left(1 + \frac{\partial\lambda}{\partial u}\right)\frac{g_{uv}}{\sqrt{g}}\right\rangle + \left\langle\sigma\frac{\partial\lambda}{\partial v}\frac{g_{uu}}{\sqrt{g}}\right\rangle}{\left\langle\sigma\frac{g_{uu}}{\sqrt{g}}\right\rangle}. \tag{21}$$

Thus, in the determination of MHD equilibria we can either prescribe the rotational transform profile or the effective toroidal plasma current profile.

3. The energy minimisation method for MHD equilibria

3.1. *The energy functional minimisation*

We define the total energy of the system as

$$W \equiv \int\int\int d^3x \Big[\frac{B^2}{2\mu_0} + \frac{p_\|}{\Gamma - 1}\Big], \tag{22}$$

where we express the parallel pressure as

$$p_\|(s,B) = M(s)[\Phi'(s)]^\Gamma \frac{1+p(s,B)}{\big\langle 1+p(s,B)\big\rangle^\Gamma}, \tag{23}$$

in which $M(s)$ corresponds to the plasma mass function previously introduced in the scalar pressure formulation of this problem [11] and corresponds to a flux surface quantity. The anisotropy in the pressure is modelled through the factor p in the expression for $p_\|$ because its magnitude can vary along a magnetic field line. The energy functional W is positive-definite when the adiabatic index $\Gamma > 1$. This condition guarantees that the minimum energy state corresponds to an MHD equilibrium. The energy functional W is varied with respect to an artificial time parameter t such that the magnetic flux functions Φ and Ψ, the plasma mass M and the coordinates s, u and v remain invariant [11]. The variation of the function p is performed through its dependence on B, namely

$$\frac{dp}{dt} = \frac{\partial p}{\partial B}\frac{\partial B}{\partial t}. \tag{24}$$

After simple algebraic manipulations of the integrand, we can write

$$\frac{dW}{dt} = \int\int\int dsdudv \Big[\frac{\sigma}{2\sqrt{g}}\frac{\partial(\sqrt{g}B)^2}{\partial t} - \Big(p_\perp + \frac{B^2}{2\mu_0}\Big)\frac{\partial\sqrt{g}}{\partial t}\Big]. \tag{25}$$

Decomposing $B^2 = B_u B^u + B_v B^v$, invoking the relations (7-11) and integrating by parts, the variation of the energy can be expressed as

$$\begin{aligned} \frac{dW}{dt} = &-\int\int\int dsdudv F_R \frac{\partial R}{\partial t} - \int\int\int dsdudv F_Z \frac{\partial Z}{\partial t} \\ &-\int\int\int dsdudv F_\lambda \frac{\partial \lambda}{\partial t} \\ &-\int\int_{s=1} R\Big(p_\perp + \frac{B^2}{2\mu_0}\Big)\Big(\frac{\partial R}{\partial u}\frac{\partial Z}{\partial t} - \frac{\partial Z}{\partial u}\frac{\partial R}{\partial t}\Big). \end{aligned} \tag{26}$$

The last term constitutes the deformation of the plasma-vacuum interface boundary. In a free boundary calculation, this surface contribution must be computed consistently with currents flowing in external coils [23]. In fixed boundary calculations, it vanishes (because by definition $\partial R/\partial t = \partial Z/\partial t = 0$ on the boundary). The coefficients of the internal plasma contribution to the variation of the total energy are

$$\begin{aligned} F_R = & \frac{\partial}{\partial u}\big[\sigma\sqrt{g}B^u(\mathbf{B}\cdot\nabla R)\big] + \frac{\partial}{\partial v}\big[\sigma\sqrt{g}B^v(\mathbf{B}\cdot\nabla R)\big] \\ & -\frac{\partial}{\partial u}\big[R\frac{\partial Z}{\partial s}\big(p_\perp + \frac{B^2}{2\mu_0}\big)\big] + \frac{\partial}{\partial s}\big[R\frac{\partial Z}{\partial u}\big(p_\perp + \frac{B^2}{2\mu_0}\big)\big] \\ & +\frac{\sqrt{g}}{R}\big[\big(p_\perp + \frac{B^2}{2\mu_0}\big) - \sigma R^2(B^v)^2\big], \end{aligned} \tag{27}$$

$$\begin{aligned} F_Z = & \frac{\partial}{\partial u}\big[\sigma\sqrt{g}B^u(\mathbf{B}\cdot\nabla Z)\big] + \frac{\partial}{\partial v}\big[\sigma\sqrt{g}B^v(\mathbf{B}\cdot\nabla Z)\big] \\ & +\frac{\partial}{\partial u}\big[R\frac{\partial R}{\partial s}\big(p_\perp + \frac{B^2}{2\mu_0}\big)\big] - \frac{\partial}{\partial s}\big[R\frac{\partial R}{\partial u}\big(p_\perp + \frac{B^2}{2\mu_0}\big)\big], \end{aligned} \tag{28}$$

and

$$F_\lambda = \Phi'(s)\big[\frac{\partial}{\partial u}(\sigma B_v) - \frac{\partial}{\partial v}(\sigma B_u)\big]. \tag{29}$$

3.2. *The MHD force balance relations*

We shall now demonstrate that the coefficients F_R, F_Z and F_λ shown in eqs. (27-29) actually correspond to different components of the MHD equilibrium force balance relation. For that purpose, it is convenient to present the alternative form of $\mathbf{F}$,

$$\mathbf{F} = -\nabla\big(p_\perp + \frac{B^2}{2\mu_0}\big) + (\mathbf{B}\cdot\nabla)(\sigma\mathbf{B}). \tag{30}$$

Then the cylindrical MHD force component

$$F_R = \sqrt{g}R\nabla v\times\nabla Z\cdot\mathbf{F} \tag{31}$$

corresponds to the expression given in eq. (27) and the cylindrical MHD force component

$$F_Z = \sqrt{g} R \nabla R \times \nabla v \cdot \mathbf{F} \tag{32}$$

corresponds to the expression given in eq. (28). The perpendicular component of the force balance is calculated by taking the scalar product of $\mathbf{F}$ with $\sqrt{g}(\mathbf{B} \times \nabla s)/B^2$ to obtain

$$F_\perp = -\sqrt{g}\mathbf{K} \cdot \nabla s = \frac{\partial(\sigma B_u)}{\partial v} - \frac{\partial(\sigma B_v)}{\partial u}. \tag{33}$$

Note that the vanishing of $F_\perp$ corresponds to the condition that the effective plasma current density lines lie on flux surfaces. We also identify that

$$F_\lambda = -\Phi'(s) F_\perp. \tag{34}$$

An MHD equilibrium state with $p_\| \neq p_\perp$ is achieved when dW/dt vanishes. This condition is realized when F_R, F_Z and F_λ vanish simultaneously.

3.3. *The method of solution*

The formulation of the MHD equilibrium problem presented here lends itself to a relatively easy adaptation of the techniques developed for the numerical computation of scalar pressure $3D$ equilibria as described, for example, in the $VMEC$ code [16-17]. Specifically, the anisotropy factor causes the dependence of the preconditioning matrix elements on the total pressure within $VMEC$ to be replaced with $p_\perp + B^2/2\mu_0$. The preconditioning algorithm is designed to improve the convergence properties of the steepest descent energy minimisation procedure invoked in $VMEC$ to compute $3D$ equilibria. The internal MHD forces that are required to calculate the variation of the energy of the system are modified according to eqs. (26-28). These have been implemented in $VMEC$ in order to compute anisotropic pressure equilibria. Further details of the numerical approach can be found in ref. [17].

3.4. *Diagnostics*

The reduced MHD force balance relation after applying the conditions of parallel force balance becomes

$$\mathbf{F} = -\frac{\partial p_{\|}}{\partial s}\bigg|_B \nabla s + \mathbf{K} \times \mathbf{B}. \tag{35}$$

In order to verify that the minimal energy state computed through the vanishing of F_R, F_Z and F_λ to within some tolerance actually constitutes an MHD equilibrium state, it is convenient to calculate the flux surface average of the radial force balance component

$$F_s = -\frac{\partial p_{\|}}{\partial s}\bigg|_B + \sqrt{g} K^u B^v - \sqrt{g} K^v B^u \tag{36}$$

given by

$$\left\langle \frac{F_s}{\Phi'(s)} \right\rangle = -\left\langle \frac{1}{\Phi'(s)} \frac{\partial p_{\|}}{\partial s}\bigg|_B \right\rangle - \frac{\partial}{\partial s}\left\langle \frac{\sigma B_v}{\sqrt{g}} \right\rangle - \iota(s) \frac{\partial}{\partial s}\left\langle \frac{\sigma B_u}{\sqrt{g}} \right\rangle. \tag{37}$$

The near vanishing of this quantity constitutes a very useful indicator of the validity of a $3D$ MHD equilibrium that is computed. Furthermore, the firehose stability criterion [22]

$$\sigma > 0 \tag{38}$$

and the mirror stability criterion [22]

$$\tau = \frac{\partial(\sigma B)}{\partial B} = \frac{1}{\mu_0} + \frac{1}{B}\frac{\partial p_\perp}{\partial B} > 0 \tag{39}$$

must be satisfied everywhere to obtain numerical convergence.

4. Application to ATF

We construct $3D$ MHD equilibria with anisotropic pressure in a configuration that models the ATF torsatron [3] with a fixed boundary. The dominant Fourier amplitudes of R and Z that describe the plasma-vacuum interface are presented in Table 1.

The pressure anisotropy factor $p(s, B)$ that appears in eq. (23) is computed from a Maxwellian distribution function that is skewed in the perpendicular velocity direction. This model, invoked previously in helically symmetric geometry [7], is represented as,

$$p(s,B) = p_h(s)\Big[\frac{B_{min}(s)}{B}\Big]^8. \tag{40}$$

This form is particularly appropriate for modelling hot particle layers induced by perpendicular neutral beam injection or radio frequency heating techniques. The energetic particles concentrate very locally along a magnetic field line about its minimum value in this case. A more uniform distribution along a magnetic field line results when the ratio of B_{min} to B is raised to a smaller power than the one chosen. Implicit in the variation of p with respect to t described in eq. (24) is the invariance of the factor $p_h(s)B^8_{min}(s)$. In order to radially localise the hot particle contribution that would characterise a layer, we choose the amplitude function $p_h(s)$ as

$$p_h(s) = p_c s^3(1-s)^2, \tag{41}$$

which has the effect of concentrating the energetic species approximately halfway between the magnetic axis and the plasma-vacuum interface in terms of the radial variable s. The hot particle pressure vanishes at the magnetic axis and at the plasma edge with this functional form chosen for p_h. Note that s is roughly proportional to the volume enclosed because the toroidal flux function Φ is chosen proportional to s. For the mass function, which describes the thermal plasma, we choose the standard profile

$$M(s) = M(0)(1-s)^2, \tag{42}$$

which is peaked at the magnetic axis. The effective toroidal plasma current is prescribed as

$$2\pi J(s) = 0, \tag{43}$$

which implies that the rotational transform profile must be evaluated according to eq. (21).

The different plasma β values are defined as follows. The total β is

$$\beta = \frac{\int\int\int d^3x(2p_\perp + p_\parallel)/3}{\int\int\int d^3x B^2/(2\mu_0)}. \tag{44}$$

The thermal component of the total β is

$$\beta_{th} = \frac{\int\int\int d^3x p_{th}}{\int\int\int d^3x B^2/(2\mu_0)}, \tag{45}$$

where the thermal pressure p_{th} corresponds to

$$p_{th}(s) = \frac{M(s)[\Phi'(s)]^{\Gamma}}{\left\langle 1 + p(s,B) \right\rangle^{\Gamma}}. \tag{46}$$

The energetic particle contribution corresponds to the difference between β and β_{th}.

An example of an ATF case in which the thermal pressure component is nearly vanishing but the hot particle pressure component is finite is realized for $M(0) = 0.4$ and $p_c = 120$. This corresponds to a case where the total $\beta = 0.55\%$, $\beta_{th} = 0.16\%$, $\beta_{th}(0) = 0.48\%$ (the value of β_{th} at the magnetic axis), the peak value of β due to the energetic species is 4.85% and the minimum value of τ is 0.724. The inner third of the plasma volume displays a weak magnetic hill, the middle third of the plasma volume has a weak magnetic well driven by the energetic particle pressure and the outer third of the volume displays a typically large magnetic hill that is characteristic of torsatron devices. The total pressure distribution, given by $(2p_{\perp} + p_{\|})/3$, on 4 toroidal planes that span half of one field period of the device is shown in fig. 1. Partially concealed by the plane $v = 0$ are the planes $v = \pi/36$, $v = \pi/18$ and $v = \pi/12$, in that order. The maximum pressure illustrated in the colour red occurs on the plane $v = \pi/36$ and is localised about the surface corresponding to $s = 0.42$. The periodic repetition of equilibrium structures according to the number of field periods of the device is manifest in fig. 2. The hot particle pressure layers on the magnetic flux surface with $s = 0.4375$ consists of strips of finite poloidal and toroidal extent that are localised around the minimum value of B on this surface. This spatial localisation becomes more evident by comparing the pressure distribution with the corresponding mod-B distribution on this surface which we present in fig. 3. In fig. 3, the maxima in the magnitude of B are depicted in red and the minima in dark blue. The hot particle pressure strips are aligned with the mod-B minima, but are narrower consistent with the functional dependence chosen for p on B as expressed in eq. (40). To plot the mod-B distribution is instructive because it plays an important role in plasma transport processes and has implications for the radio frequency heating problem. For completeness, we present a case in which both the thermal and energetic particle contributions to the pressure are finite in fig. 4. For that purpose, we have chosen $M(0) = 3.67$ and $p_c = 12$ which results in a peak hot particle β almost the same as in the previous example at 4.83%, a total $\beta = 1.79\%$, $\beta_{th} = 1.52\%$, $\beta_{th}(0) = 4.80\%$ and a minimum value of τ of 0.725. The pressure distribution is shown on the same toroidal planes as in fig. 1, namely $v = 0$, $v = \pi/36$, $v = \pi/18$ and $v = \pi/12$. The thermal pressure is strongly peaked on axis. The energetic particle contribution to the pressure remains concentrated about the surface with $s = 0.42$ and the toroidal plane $v = \pi/36$.

5. Summary and conclusions

We have devised the energy functional $W = \int\int\int d^3x[B^2/(2\mu_0)+p_{||}/(\Gamma-1)]$ as a positive-definite variational principle (for $\Gamma > 1$) that can be used to obtain $3D$ MHD equilibria with anisotropic plasma pressure. The minimisation of W using an inverse coordinate method reproduces the horizontal (F_R), the vertical (F_Z) and the perpendicular ($F_\perp$) components of the MHD equilibrium force balance relations that govern plasmas with anisotropic pressure, $p_{||} \neq p_\perp$. The formulation is limited to plasma confinement configurations with nested magnetic flux surfaces and a single magnetic axis. In addition, the pressures are assumed to depend only on two variables, the radial coordinate s and B. The parallel pressure is prescribed and the perpendicular pressure is determined consistent with conditions of parallel force balance. The vanishing of the flux surface average of the radial force balance relation constitutes a useful measure for the accuracy of the equilibrium state that is computed. The anisotropic pressure formulation for the $3D$ MHD equilibrium problem has been implemented in the preconditioned version of the scalar pressure $VMEC$ code [17]. This takes advantage of the existing preconditioning algorithm that is designed to improve the convergence properties of the steepest descent energy minimisation procedure that is utilised in this code to generate $3D$ MHD equilibria with $p_{||} \neq p_\perp$.

This modification of the $VMEC$ code has been applied to an ATF torsatron configuration. We have computed $3D$ MHD equilibria with highly localised anisotropic pressure distributions that could model an energetic particle layer induced by auxiliary heating techniques such as electron cyclotron resonance heating, ion cyclotron resonance heating or perpendicular neutral beam injection. The high degree of localisation of the anisotropic pressure distribution that has been achieved attests to the robustness of the method of solution that has been applied to numerically determine $3D$ MHD equilibria.

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m	n/L	R_{mn}	Z_{mn}
0	-2	0.0001	-0.0002
0	-1	-0.0008	0.0055
0	0	2.0655	0.0000
1	-2	0.0002	0.0002
1	-1	0.0069	0.0069
1	0	0.2638	0.3316
1	1	-0.0724	0.0862
1	2	-0.0013	0.0017
2	0	0.0044	-0.0075
2	1	-0.0129	0.0104
2	2	0.0021	-0.0012

Table 1: Fourier decomposition of the boundary of an ATF configuration.