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Abstract: Stability limits for the ideal internal kink mode are calculated analytically for the Shafranov current profile using the large aspect ratio expansion. For equilibria with $q(a) > 2$ and circular cross section, the maximum stable poloidal beta is below 0.1. In the absence of a conducting wall, an equilibrium with $q(a) < 2$ is unstable at arbitrarily small positive poloidal beta or shear inside the $q = 1$ surface. The effects of non-circularity are discussed and quantitative results are given for elliptic cross sections.

1. INTRODUCTION

Recent experimental observations have revealed many unexpected features of the sawtooth oscillations, in particular in large tokamaks, and have lead to a renewed interest in the stability properties of the internal kink mode. Even within the simplest theoretical framework, ideal magnetohydrodynamics (MHD), the stability of the internal kink mode is sensitive to a large number of parameters: pressure gradients inside the q = 1 surface [1], the current profile [1-6], shaping of the cross section [6-8] and even wall position [6,7]. Although numerical stability results are available, analytical results are valuable because they show parametric dependencies and apply to certain well specified (although, unfortunately, rather specialized) cases. Furthermore, they eliminate uncertainties concerning numerical accuracy. In the present letter, we calculate analytically the stability boundaries of the ideal internal kink mode with toroidal mode number n = 1 for a Shafranov (step-function) current profile, using the large aspect ratio expansion [1, 7, 8]. When q(a) > 2, the maximum stable poloidal beta for the Shafranov profile is less than 0.1, which is considerably lower than the often quoted value of $(13/144)^{1/2} \approx 0.3$, valid for a parabolic current profile in the limit of small q = 1 radius [1]. Furthermore, if q(a) < 2 and there is no conducting wall, the Shafranov equilibria are unstable at arbitrarily small positive poloidal beta or shear inside the q = 1 surface.

2. FORMULATION

2.1 Large aspect ratio expansion

The potential energy for the n = 1 ideal internal kink mode was calculated in Ref. 1 by means of the large aspect ratio expansion. It is a quadratic polynomial in β_p, the poloidal beta at the q = 1 surface:

$$\delta W = \frac{\pi^2}{2} |\xi|^2 R\, B_T^2\, \varepsilon^4\, P_T(\beta_p) \quad . \tag{1}$$

Here $\varepsilon = r_1/R$, r_1 is the minor radius of the q = 1 surface, R the major radius, ξ the amplitude of the m = 1 radial displacement (m = poloidal mode number), and

$$P_T(\beta_p) = -3\beta_p - s - \frac{[3(\beta_p+s-1/4) + (\beta_p+s+3/4)\, A_i]\,[3(\beta_p+s-1/4) + (\beta_p+s+3/4)\, A_e]}{A_i - A_e} \quad . \tag{2}$$

The poloidal beta is defined as

$$\beta_p = -\frac{2}{B_\theta^2(r_1)} \int_0^{r_1} \frac{dp}{dr}\, (r/r_1)^2\, dr \quad , \tag{3}$$

and the global shear inside the q = 1 surface is represented by

$$s = \int_0^{r_1} \left(\frac{r}{r_1}\right)^3 \left(\frac{1}{q^2} - 1\right) \frac{dr}{r_1} \quad . \tag{4}$$

The quantities A_e and A_i denote the logarithmic derivatives of the (m = 2, n = 1) component of the radial displacement just outside and inside the q = 1 surface, respectively. These are obtained from the solutions of the cylindrical Euler equation for (m=2, n=1) in the two separate regions q > 1 and q < 1, with appropriate boundary conditions.

2.2 Shafranov equilibrium

The Shafranov profile gives a simple, yet physically meaningful, class of equilibria for which all quantities appearing in δW can be evaluated analytically. The current profile is a step function,

$$j(r) = \begin{cases} \dfrac{2B_T}{Rq_0} \ , & r < r_0 \ , \\ 0 \ , & r > r_0 \ , \end{cases} \tag{5}$$

and the safety factor profile has $q(r) = q_0$ = constant for $r < r_0$ and $q(r) = q_0\,(r/r_0)^2$ for $r > r_0$, see Fig. 1. The global shear for the Shafranov profile is

$$s = \frac{1}{2} \log(1/q_0) \quad . \tag{6}$$

To obtain the solutions of the Euler equation it is convenient to use the flux perturbation ψ related to the displacement by

$$\psi = irF\xi \quad , \qquad F = \mathbf{k} \cdot \mathbf{B} = (B_T/R)(m/q - n) \tag{7}$$

for arbitrary m and n. The cylindrical Euler equation for ψ reads

$$\frac{1}{r}\frac{d}{dr}\, r\, \frac{d\psi}{dr} - \frac{m^2}{r^2}\psi - \frac{m}{rF}\frac{dj}{dr}\psi = 0 \quad . \tag{8}$$

If no resonant surface q = m/n exists between r = 0 and $r = r_0$, the logarithmic derivatives for x are obtained from (7) and (8) as

$$A_i^{(m,n)} = \frac{m+n}{m-n} + |m| - \frac{2m\, q_0^{|m|}}{m - nq_0 - \mathrm{sgn}(m)\,(1 - q_0^{|m|})} \quad , \tag{9a}$$

$$A_e^{(m,n)} = \frac{m+n}{m-n} - m \frac{1 + (r_1/b)^{2m}}{1 - (r_1/b)^{2m}} \quad . \tag{9b}$$

Here, b denotes the radius at which the external solution for $\psi^{(m,n)}$ vanishes, that is, the resonant surface where $q = m/n$ if such a surface exists in the plasma (i.e., if $1 < m/n < (a/r_1)^2$), otherwise the conducting wall.

3. WALL STABILIZATION, CIRCULAR PLASMA

The potential energy is now obtained from (2) and (9) as

$$P_T(\beta_p) = - \frac{1}{q_0^2 + (1-q_0)\,x} \Big\{ 4(\beta_p + s)^2 (x-2)\,[q_0^2 + 2(1-q_0)] + 6\beta_p\,(1-q_0)\,(x-2) - 2s\,[q_0^2 - 2(1-q_0)\,(x-3)] - \frac{9}{4}\,(1-q_0)] \Big\} \quad , \tag{10a}$$

where

$$x = (b/r_1)^4 \quad . \tag{10b}$$

represents the effect of the wall. We note that the sign of the coefficients for both β_p and β_p^2 in δW is determined entirely by the wall separation. When $(b/r_1) > 2^{1/4}$, both these coefficients are negative and consequently pressure is destabilizing. In this case, there can be at most one positive marginal value of β_p, depending on the sign of δW at zero pressure. It is easy to verify for more general profiles, the same conclusion follows if

$$A_e + 3 > 0 \tag{11}$$

and

$$\Delta = A_e - A_i < 0 \quad ,$$
$$G = \frac{3}{4}(A_i - 1) + s(A_i + 3) > 0 \tag{12}$$

(where the latter two conditions (12) are normally satisfied). It is clear that (11) must be violated when the wall is sufficiently close to the $q = 1$ surface, because $A_e \rightarrow -\infty$ as $b \rightarrow r_1$. However, to violate condition (11) the wall must be very near $q = 1$, and this is possible only when $q(a) < 2$. Specifically, for the Shafranov equilibrium, a tight-fitting wall makes the effect of pressure *stabilizing* on the internal kink if, and only if

$$\frac{b}{r_1} < 2^{1/4} \approx 1.19 \quad . \tag{13}$$

Under this condition, there is no stability limit in β_p within the validity of the expansion. However, whenever $b/r_1 > 2^{1/4}$, we have $\Delta < 0$, $G > 0$ and $A_e + 3 > 0$, so that pressure is destabilizing and there is at most one positive marginal value for β_p. This, of course, applies to the normal case of $q(a) > 2$ where $b/r_1 = \sqrt{2}$ for the Shafranov profile.

Equation (9) shows that for equilibria with $q_0 < 1$, $q(a) < 2$ and no wall stabilization (i.e., $b = \infty$), arbitrarily small positive values of *either* β_p (pressure) or s (shear) inside the $q = 1$ surface lead to instability. Therefore, an equilibrium with $q(a) < 2$ needs stabilization by a close-fitting wall. The stability condition $\delta W < 0$ can be rewritten as a condition on maximum wall distance:

$$\left(\frac{b}{r_1}\right)^4 - 2 < \frac{s\, q_0^2 + (1-q_0)\,(2s + 9/8)}{2\,(\beta_p + s)^2\, q_0^2 + (1-q_0)\,[4(\beta_p + s)^2 + 3\beta_p + 2s]} \quad . \qquad (14)$$

In the limit $\beta_p >> s$ and $s << 1$, this condition simplifies to

$$\beta_p^2 < \frac{13}{8}\,\frac{s}{(b/r_1)^4 - 2} \quad , \qquad (15)$$

which reduces to (13) when $\beta_p >> s^{1/2}$. When $q(a) \geq 2$, i.e., $b = \sqrt{2}\, r_1$, the small s limit gives $\beta_p^2 < 13s/16$. When $\beta_p << s << 1$, the pressure effect is negligible, and Eq. (15) yields $s < (13/24)\,[(b/r_1)^4 - 2]^{-1}$, but as this was obtained by an expansion assuming $s << 1$ the condition is strictly

$$s < \frac{13}{24}\left(\frac{r_1}{b}\right)^4 \quad , \quad \left(\frac{r_1}{b}\right)^4 << 1 \quad . \qquad (16)$$

4. SHAPING EFFECTS

Because of the simple solutions of Eq. (8) for the Shafranov equilibria, it is also possible to give analytic results for the effects of cross-section shaping on the stability of the internal kink mode. Here, we apply the results obtained in [7], and treat the non-circularity as an additive effect, separate from the effects of toroidicity and finite pressure.

The equilibrium ($n = 0$) flux function is given by

$$\psi_0(r, \theta) = \sum_{\ell \geq 0} \psi^{(\ell,0)}(r) \cos \ell\theta \quad , \qquad (17)$$

and the deformation of the flux surfaces is, to first order in the non-circularity,

$$\delta r(r, \theta) = \sum_{\ell \geq 0} \delta_\ell \cos \ell\theta \quad , \quad \delta_\ell = - \frac{\psi^{(\ell,0)}}{d\psi^{(0,0)}/dr} \quad . \tag{18}$$

The potential energy can be written as

$$\delta W = \frac{\pi^2}{2} |\xi|^2 R_0 B_T^2 \varepsilon^2 [\varepsilon^2 P_T(\beta_p) + P_S] \quad , \tag{19}$$

where the shaping effects are contained in P_S which is given by

$$P_S = \sum_{\ell > 1} a_\ell^2 \frac{(\ell+2+A_i^{(\ell+1,1)}) (\ell+2+A_e^{(\ell+1,1)})}{A_e^{(\ell+1,1)} - A_i^{(\ell+1,1)}} + \ell \rightarrow -\ell \quad . \tag{20}$$

The coefficients a_ℓ can be expressed as

$$a_\ell = r^{\ell-1} \frac{d}{dr} \left(\frac{\delta_\ell}{r^{\ell-1}} \right) = \frac{\delta_\ell}{r} [A_i^{(\ell,0)} - \ell+1] \quad , \tag{21a}$$

where we used the fact that the $\ell \neq 0$ components of the equilibrium satisfy (8) (to lowest order) with $m = \ell$, $n = 0$. For the Shafranov equilibrium, Eq. (8a) gives, for $\ell > 0$,

$$a_\ell = \frac{\delta_\ell}{r} 2 (\ell - 1) \frac{1 - q_0^\ell}{\ell - 1 + q_0^\ell} \quad . \tag{21b}$$

Together with the expression for P_T in a circular plasma, Eqs. (19 - 21) now give the potential energy for the internal kink mode including shaping effects. As a specific example showing the relative magnitude of the toroidal and shaping effects, we consider the correction due to ellipticity, $\ell = 2$, for the Shafranov equilibrium

$$P_{ellipse} = 6 \left[\frac{\delta_2(r_1)}{r_1} \right]^2 \left(\frac{1 - q_0^2}{1 + q_0^2} \right)^2 \frac{(6-3q_0+q_0^3) [3 - (b/r_1)^6]}{q_0^3 + (2-q_0) (b/r_1)^6} \quad , \tag{22}$$

which is to be added to $\varepsilon^2 P_T$ as given by (10). Equation (22) shows that ellipticity is destabilizing when

$$\frac{b}{r_1} > 3^{1/6} \approx 1.20 \quad , \tag{23}$$

as found previously by Edery et al [7]. Thus, ellipticity destabilizes the internal kink mode for all realistic wall positions and, in particular, when $q(a) > 2$.

Finally, the deformation δ_ℓ / r at the $q = 1$ surface can be expressed in terms of its value at the plasma boundary. For the Shafranov equilibrium, δ_ℓ / r is proportional to $[(\ell-1)(r/r_0)^\ell + (r_0/r)^\ell]$ in the region $r > r_0$, and therefore,

$$\frac{\delta_\ell(r_1)/r_1}{\delta_\ell(a)/a} = q_a^{\ell/2} \frac{\ell - 1 + q_0^\ell}{(\ell-1)q_a^\ell + q_0^\ell} \quad . \tag{24}$$

5. QUANTITATIVE RESULTS

Figure 2 shows the limits in poloidal beta for the Shafranov equilibrium with different ellipticities as functions of q_0. The limits have been calculated as the solutions of the quadratic equation $\delta W(\beta_p) = 0$ with δW given by Eqs. (10), (19) and (22). The figure applies to the case of $q(a) \geq 3$, so that, in evaluating A_e, we have taken $b = r_{q=2} = \sqrt{2}\, r_1$ for the $m = 2$ side-band and $b = r_{q=3} = \sqrt{3}\, r_1$ for $m = 3$. These beta limits decrease only slightly for the elliptic cross section when $2 < q(a) < 3$ and wall stabilization is disregarded (obtained by setting $b = \infty$ for the $m = 3$ perturbation). The curves in Fig. 1 are labelled by the ratio of ellipticity to inverse aspect ratio at the $q = 1$ surface

$$\hat{e} = \frac{\delta_2(r_1)/r_1}{r_1/R} \quad . \tag{25}$$

Figure 2 shows that the limit in poloidal beta is below 0.1 for a circular Shafranov equilibrium. This is significantly smaller than the often quoted result $(13/144)^{1/2} \approx 0.3$ for a parabolic current profile with a small $q = 1$ radius [1], and it is clear that the stability of the internal kink mode is sensitively dependent on the current profile. It may be noted that the beta limits obtained here are in the same range as those found by de Blank and Schep for equilibrium profiles with high shear at the $q = 1$ surface (see Fig. 7 of Ref. 5).

Figure 2 also shows that the destabilizing effect of ellipticity is significant when $\hat{e}$ is in the range of unity. However, the destabilizing effect is reduced when $1 - q_0$ is small, i.e., for low global shear. (Note, however, that the Shafranov profile with $q_0 < 1$ always gives high local shear at $q = 1$.) We emphasize that the ratio of ellipticity to inverse aspect ratio is generally not small in modern tokamaks. For instance, for typical JET parameters, $\delta_2(a)/a = 0.25$ and $q(a) = 3$, Eq. (24) gives $\delta_2(r_1)/r_1 = 0.15$. Thus, with $a/R = 1/3$, we have $r_1/R \approx 0.18$ and $\hat{e} \approx 0.78$. According to Fig. 2, therefore, ellipticity gives an appreciable reduction of the β_p limit for JET geometry. Naturally for JET, triangularity has a stabilizing affect, which can be estimated for the Shafranov current profile by formulas (19 - 21).

We have verified that the results shown in Fig. 2 are in good agreement with numerical solutions from the MARS stability code [6] of the full-MHD eigenvalue problem at an aspect ratio of $R/a = 4$. Finally, we remark that as the Shafranov profile is rather special, and moreover gives significant differences with the parabolic profile, it is important to consider more general, and more realistic, current profiles. For more *smooth profiles* and circular cross section, numerical computations [9] show limits in β_p that are typically between 0.1 and 0.3. With respect to the effect of ellipticity, we note that for smooth profiles, the local shear at the q = 1 surface tends to be low when q_0 is close to unity. Computations retaining the full geometrical effects [9] show that for weak shear in the q = 1 region, ellipticity can be strongly destabilizing. The destabilizing effect of ellipticity at low shear can be seen, e.g., from the Mercier criterion for an elliptic cross section [10] which contains terms proportional to $e\beta'/rq'^2$ (i.e., of order $\varepsilon^2 e\beta_{pol}$). Terms of this order are not accounted for in expression (20).

In summary, we find that the ideal MHD limit in β_p can be considerably lower than the often quoted value of 0.3. This brings the ideal MHD limit closer to the values typically observed in tokamaks at the time of sawtooth crashes and indicates that the sawteeth may, in many cases, be triggered by instabilities close to the ideal MHD stability threshold. For a Shafranov profile with $q_0 < 0.9$, the β_p limit is considerably reduced by ellipticity of the same order as in JET.

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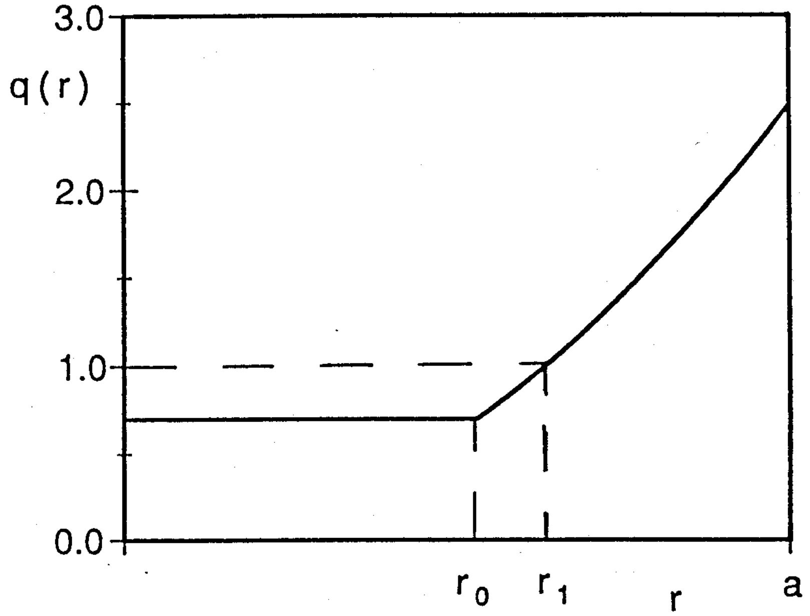

FIGURE 1 Safety factor profile q(r) for Shafranov equilibrium.

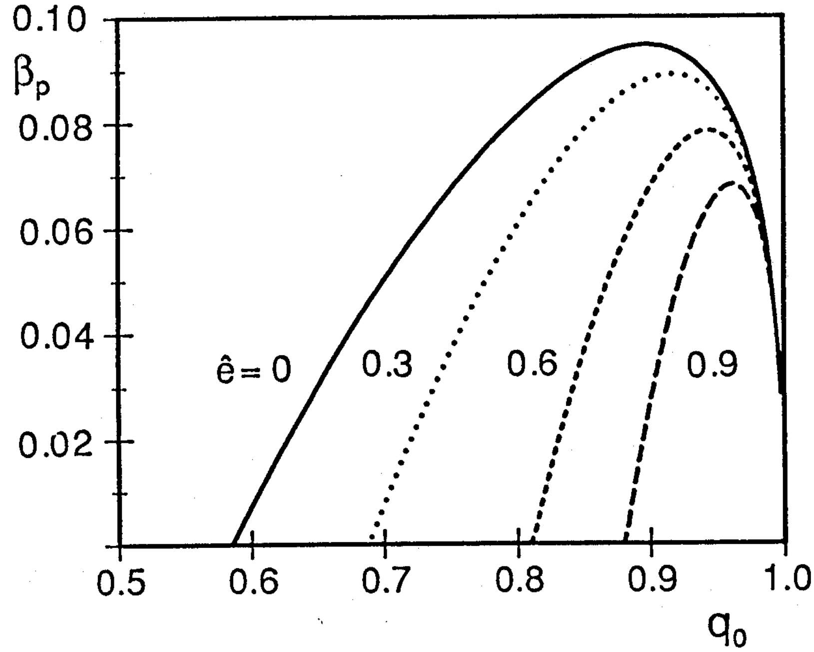

FIGURE 2 Stability limits for the ideal internal kink mode in terms of poloidal beta at the $q = 1$ surface as functions of central q. A Shafranov current profile with $q(a) > 3$, large aspect ratio and elliptical shaping is assumed. The curves refer to different normalized ellipticities at the $q = 1$ surface $\hat{e} = 0, 0.3, 0.6$ and 0.9 (see Eq. (25)).