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Centre de Recherches en Physique des Plasmas
Association Euratom - Confédération Suisse
Ecole Polytechnique Fédérale de Lausanne
21, Av. des Bains, CH-1007 Lausanne
Switzerland

ABSTRACT: Axisymmetric toroidal plasma equilibria with mass flows and anisotropic pressure are investigated. The equilibrium system is derived for a general functional form of the pressures, which includes both fluid models, such as the magnetohydrodynamic (MHD) and the double-adiabatic models, and Grad's guiding centre model. This allows for detailed comparisons between the models and clarifies how the "first hyperbolic region", occurring in the fluid theory when the poloidal flow is of the order of the poloidal sound speed, can be eliminated in guiding centre theory. In the case of a pure toroidal rotation, macroscopic equations of state are derived from the guiding centre model, characterized by a parallel temperature that is constant on each magnetic surface and a perpendicular temperature that varies with the magnetic field. The outward centrifugal shifts of the magnetic axis and of the mass density profile, due to toroidal rotation, are increased by anisotropy if $p_{\parallel} < p_{\perp}$ or decreased (and can even be inverted) if $p_{\parallel} > p_{\perp}$. The guiding centre model shows that poloidal flow produces an inward shift of the density profile, in contrast with the MHD result.

I. INTRODUCTION

Neutral-beam injection has become a major source of plasma heating on many Tokamaks. New phenomena occurring in these experiments call for important modifications of the usual static description of the plasma equilibrium, based on the MHD model. The plasma can rotate: toroidal velocities up to the ion sound speed have been measured on ISX-B,[1] PDX,[2] and, more recently, on TFTR[3] and JET[4]. Moreover, depending on the angle of injection, the pressure distributions can be strongly non-isotropic.

In this paper we study the influence of toroidal and poloidal flows and of anisotropic pressure on the equilibrium properties of axisymmetric toroidal plasmas. Particular attention is devoted to the analysis of the poloidal variation of the equilibrium quantities, such as the mass density, on the magnetic surfaces. This is motivated by the experimental observation[3,5] of systematic anomalies of the density and electron temperature profiles in plasmas with large mass flows.

Up to now, the effects of flows and pressure anisotropy have usually been investigated separately, using different models. Axisymmetric equilibria with flows have been studied mainly in the framework of MHD, see, e. g., the original work by Zehrfeld and Green[6] and more recent works by Hameiri[7] and Semenzato et al.[8] On the other hand, analyses of static equilibria with tensor pressure have been carried out,[9,10] using more realistic forms of the pressures, obtained by analysis of the distribution functions of the beam particles.

An equilibrium description including both flows and tensor pressure was developed by Dobrott and Greene[11] using Grad's guiding centre plasma (GCP) model.[12] GCP is a semi-macroscopic model in which the particle motion parallel to the magnetic field is described by a one-dimensional collisionless kinetic equation. Dobrott and Greene noted that the guiding centre description gives very different results, compared with MHD. Their analysis suggests, in particular, that the first hyperbolic region, occurring in MHD for poloidal flows of the order of the poloidal sound speed, is removed in GCP.

In this paper we construct an equilibrium formulation which uses fluid equations for the dynamics perpendicular to the magnetic field, but allows for different models of the parallel dynamics. This is achieved by choosing a general functional form of the pressures which includes both MHD and GCP, together with other equilibrium models, such as, for example, the double- adiabatic model of Chew, Goldberger and Low (CGL).[13] We use this formulation to discuss fluid equilibria with flows and tensor pressure and to make a detailed comparison between the fluid and the guiding centre approaches.

In Sec. II we give the basic equations of the model. The pressures are taken as functions of the mass density ρ and the magnetic field B, subject to a relation which ensures energy conservation.

The equilibrium system is derived in Sec. III. The respective models for the parallel dynamics of the plasma are characterized by three coefficients, expressing derivatives of the pressures with respect to ρ and B, which determine the parallel gradients of the mass density and of the pressures. When poloidal flow is present, the fluid models predict that the density gradient becomes singular at the sound speed. This gives rise to the so-called first hyperbolic region of the equilibrium system. We show that, in the guiding centre model, the macroscopic variables (ρ, $v_{||}$, $p_{||}$, $p_{\perp}$) are well behaved when the particle distributions are non-increasing functions of energy. As a consequence, the first hyperbolic region is eliminated in GCP.

The condition for ellipticity of the equilibrium system is studied in detail in Sec. IV.

In Sec.V we analyze the poloidal variation of the mass density. We find both in the fluid models and in the GCP model that toroidal flows lead to an outward shift of the density profile. This shift is increased (if $p_{\|} < p_{\perp}$), or decreased, and can even be inverted (if $p_{\|} > p_{\perp}$), by the pressure anisotropy. For bi-Maxwellian distributions a macroscopic model is derived from GCP, giving a simple analytic expression for the mass density. We solve the generalized Grad-Shafranov equation for this model in the large aspect ratio limit, and give an explicit expression for the Shafranov shift. Related results have been obtained independently by X. Wang and A. Bhattacharjee.[14] Finally, the effect of poloidal flows is investigated. Using model distribution functions, we show that poloidal flows produce an inward shift of the density profile in the GCP model, in contrast to the MHD result.

Part of this work has been presented previously,[15] in a short version. A formalism similar to that employed here, unifying different models of the parallel dynamics, is also applied in a related paper[16] on cylindrical stability with flow.

II. BASIC EQUATIONS

The plasma is assumed to evolve according to

$$\partial_t \rho + \nabla\cdot(\rho \underline{v}) = 0, \tag{1a}$$

$$\rho\, d\underline{v}/dt = \underline{J} \times \underline{B} - \nabla\cdot \underline{\underline{P}}, \tag{1b}$$

$$\partial_t \underline{B} = -\nabla \times \underline{E}, \quad \nabla\cdot\underline{B} = 0, \tag{1c}$$

$$\underline{E} + \underline{v} \times \underline{B} = 0, \tag{1d}$$

$$\underline{\underline{P}} = p_{\perp} \underline{\underline{I}} + \Delta\, \underline{BB}, \qquad \Delta \equiv (p_{\|} - p_{\perp})/B^2 . \tag{1e}$$

where ρ is the mass density, $\underline{v}$ the mass flow, $\underline{B}$ and $\underline{E}$ the magnetic and electric fields, $\underline{J} = \nabla \times \underline{B}$ is the current density and $d/dt = \partial_t + \underline{v}\cdot\nabla$ is the convective derivative. The pressure tensor $\underline{P}$ is defined in (1e), where $p_{\parallel}$ and $p_{\perp}$ are the pressures parallel and perpendicular to the magnetic field, respectively, and the quantity Δ measures the pressure anisotropy.

In a fluid description, such as MHD or CGL, the system (1) is closed by equations of state. Instead, in the guiding center model a one-dimensional collisionless kinetic equation governing the parallel dynamics is added [11,12] and the quantities ρ, $v_{\parallel}$, $p_{\parallel}$, $p_{\perp}$ are obtained in terms of the moments of the guiding center distribution functions. It should be noted that the continuity equation (1a), and the parallel component of the equation of motion (1b) are satisfied automatically in GCP, where they are obtained by taking the first two moments of the kinetic equation. Therefore, if the pressures are computed from a kinetic analysis, all other macroscopic variables can be computed[12] from the fluid system (1).

In order to construct an equilibrium formulation which is common to the fluid models and the GCP model, we consider rather general forms of the equations for the pressures. However, these have to satisfy a certain condition, which is required for energy conservation. If we take the scalar product of the equation of motion with the velocity $\underline{v}$, then the pressure term gives the work done on the plasma. This can be written as

$$\underline{v}\cdot\nabla\cdot\underline{P} = \nabla\cdot[\, p_{\perp}\underline{v} + \Delta\,(\underline{v}\cdot\underline{B})\underline{B}\,] + (p_{\parallel}/\rho)\, d\rho/dt - (\Delta B)\, dB/dt\,, \qquad (2)$$

where the definition of $\underline{P}$, the continuity equation and Faraday's law have been used. In order to get local energy conservation, a function U must exist, such that

$$\rho\frac{dU}{dt} = \frac{p_{\parallel}}{\rho}\frac{d\rho}{dt} - \Delta B\frac{dB}{dt}\,. \qquad (3)$$

The quantity U is of course the internal energy per unit mass for an adiabatic plasma or the Helmoltz free energy for an isothermal one. In CGL, for example, $U = p_\perp/\rho + p_\parallel/2\rho$, as is easily verified using the double-adiabatic equations of state $d(p_\perp/\rho B)/dt = 0$, $d(p_\parallel B^2/\rho^3)/dt = 0$.

For equations of state of the form

$$p_{\parallel,\perp} = p_{\parallel,\perp}(\rho,B,\psi), \tag{4}$$

where ψ is the magnetic flux, Eq. (3) gives a constraint on the pressures which must be satisfied by our physical model. In this case $U = U(\rho,B,\psi)$ satisfies

$$\left(\frac{\partial U}{\partial \rho}\right)_{B,\psi} = \frac{p_\parallel}{\rho^2}, \qquad \left(\frac{\partial U}{\partial B}\right)_{\rho,\psi} = -\frac{\Delta B}{\rho}. \tag{5}$$

The constraint is then simply the integrability condition for this system

$$\frac{1}{B}\left(\frac{\partial p_\parallel}{\partial B}\right)_{\rho,\psi} - \Delta = \beta_\perp - \beta_\parallel . \tag{6a}$$

Here we have introduced the dimensionless quantities

$$\beta_{\perp,\parallel} \equiv \frac{1}{v_A^2}\left(\frac{\partial p_{\perp,\parallel}}{\partial \rho}\right)_{B,\psi} \tag{6b}$$

and $v_A \equiv B/\sqrt{\rho}$ is the Alfvén velocity. It is easily seen that CGL and MHD ($p_\parallel = p_\perp = p$, $(\partial_B p)_{\rho,\psi} = 0$) are included in this model, provided that the initial conditions satisfy Eqs. (4) and (6). In the next section, we show that also the GCP equilibrium expressions for the pressures can be put in the form (4) and satisfy (6).

Another important property of the model is related to the parallel dynamics. In fact, using Eq. (5), the parallel component of the equation of motion can be rewritten, after some algebra, as

$$\rho \frac{d}{dt}\left(\frac{\underline{v}\cdot\underline{B}}{\rho}\right) = \underline{B}\cdot\nabla\left(\frac{v^2}{2} - W\right), \tag{7a}$$

with

$$W = W(\rho,B,\psi) \equiv U + \frac{p_{\parallel}}{\rho}, \tag{7b}$$

showing that $\int \underline{v}\cdot\underline{B}\, d^3x$ is conserved on each flux tube. The function W is the thermodynamic enthalpy for an adiabatic plasma. Equation (7) allows to find a Bernoulli law [6,7] in axisymmetric equilibrium configurations.

III. THE EQUILIBRIUM PROBLEM

In this section we discuss axisymmetric equilibrium ($\partial_t = 0$) solutions of the system (1), with the pressures defined by Eqs. (4) and (6). Cylindrical coordinates (r,ϕ,z) centered on the main axis of the torus are used, where r is the distance from the axis and ϕ is the ignorable coordinate ($\partial/\partial\phi = 0$). The magnetic field $\underline{B}$ is written as

$$\underline{B} = \underline{B}_p + \underline{B}_\phi = \nabla\phi \times \nabla\psi(r,z) + I(r,z)\,\nabla\phi, \tag{8}$$

where $\underline{B}_p$ and $\underline{B}_\phi$ are the poloidal and toroidal magnetic fields, respectively, ψ is the poloidal magnetic flux and I is the poloidal current flux.

We obtain the following equilibrium system:

$$\frac{\rho v_p}{B_p} = \psi_M'(\psi), \tag{9a}$$

$$\frac{v_\phi}{r} - \frac{\psi_M'}{\rho}\frac{I}{r^2} = \phi_E'(\psi), \tag{9b}$$

$$\tau I - r^2\psi_M'\phi_E' = I_M(\psi), \tag{9c}$$

$$\frac{1}{2}\frac{(\psi_M'B)^2}{\rho^2} - \frac{1}{2}(r\phi_E')^2 + W(\rho,B,\psi) = H_M(\psi), \tag{9d}$$

$$\nabla\cdot\left(\tau\frac{\nabla\psi}{r^2}\right) = -\frac{\partial p_{||}}{\partial\psi} - I_M'\frac{I}{r^2} - \psi_M''\underline{v}\cdot\underline{B} + \phi_E''\rho r v_\phi - \rho H_M' + \rho\frac{\partial W}{\partial\psi}, \tag{9e}$$

where $v = |\underline{v}|$, v_p and v_ϕ are the magnitudes of the poloidal and toroidal flows, respectively, and the prime means derivative with respect to ψ. The function τ is given by:

$$\tau \equiv 1 - \Delta - M_p^2, \qquad M_p^2 \equiv \frac{\rho v_p^2}{B_p^2} = \frac{(\psi_M')^2}{\rho}, \tag{10}$$

where the poloidal Alfvén Mach number M_p, relative to the poloidal field, has been introduced. The equilibrium system is completed by the equations of state (4) and by the equations (7b) and (5) for W, which can be combined to give

$$\left(\frac{\partial W}{\partial\rho}\right)_{B,\psi} = \frac{1}{\rho}\left(\frac{\partial p_{||}}{\partial\rho}\right)_{B,\psi}, \qquad \left(\frac{\partial W}{\partial B}\right)_{\rho,\psi} = \frac{1}{\rho}\left[\left(\frac{\partial p_{||}}{\partial B}\right)_{\rho,\psi} - \Delta B\right]. \tag{11}$$

The six functions ψ_M, ϕ_E, I_M, H_M, $\partial p_{||}/\partial\psi$, $\partial W/\partial\psi$ (one more than in MHD, due to pressure anisotropy) can be given arbitrarily.

The system (9) can be derived in the same way as the MHD equilibrium system.[6-8] Equations (9a-b) are simple consequences of Faraday's law and of axisymmetry and Eqs. (9c-e) result from the projection of the equation of motion in the direction of $\hat{\phi}$, $\hat{b}$ and $\nabla\psi$, respectively.

The velocity is *within the magnetic surfaces* and is determined by two free functions of ψ: $\psi_M(\psi)$, which is the poloidal flux of the momentum $\rho\underline{v}$, and $\phi_E(\psi)$, which is the electric potential. Equations (9a-b) can be combined to give the well known representation

$$\underline{v} = \frac{\psi_M'}{\rho}\underline{B} + V\hat{\phi}, \qquad V \equiv -r\,\phi_E', \tag{12}$$

where V measures the flow induced by the electric field. $V = 0$ (i.e. $\phi_E' = 0$) gives a *field-aligned flow,* while $\psi_M' = 0$ corresponds to a *pure toroidal rotation,* with frequency $\phi_E'(\psi)$, of each flux surface.

Once the function W is known, Eq. (9d), which is similar to Bernoulli law, can be regarded as an equation for the local mass density ρ. In MHD and CGL this relation is algebraic. In general, it may not be possible to express W analytically, but its existence is ensured by Eq. (6), which is also the integrability condition for the system (11). Finally, Eq. (9e) is a *second order, quasi-linear, partial differential equation for ψ, analogous to the Grad-Shafranov equation of static MHD.*

III.A MHD and CGL equilibria

The MHD and the CGL equilibrium systems are special cases of (9). The MHD system is obtained by putting $p_\parallel = p_\perp = p = S(\psi)\rho^\gamma$, where S is an arbitrary function of ψ. For the double-adiabatic system we have $p_\perp = S_\perp(\psi)\rho B$ and $p_\parallel = S_\parallel(\psi)\rho^3/B^2$, with $S_{\perp,\parallel}$ arbitrary functions of ψ. In both models the equations of state are of the form (4) and satisfy the integrability condition (6).

The corresponding expressions for W are found from (7b)

$$W_{MHD} = \frac{\gamma}{\gamma-1}\frac{p}{\rho}, \qquad W_{CGL} = \frac{p_\perp}{\rho} + \frac{3}{2}\frac{p_\parallel}{\rho}. \tag{13}$$

A general expression for the variation of the mass density along the field lines can be obtained from Eqs. (9d) and (11) :

$$\frac{d\rho}{\rho} = \left(1 - \frac{\beta_\perp}{\beta_\parallel - M_p^2}\right)\frac{dB}{B} + \frac{(V/v_A)^2}{\beta_\parallel - M_p^2}\frac{dV}{V}. \tag{14}$$

[The differential along a field line is denoted here by a "d" and will sometimes referred to as the parallel gradient.] The MHD and CGL expressions are then obtained by substituting in (14) the respective values of $\beta_\parallel$.

III.B The GCP equilibrium.

In this section we discuss the GCP equilibrium and compare with the formulation (9). We recall some of the results given by Dobrott and Greene,[11] to which the reader is referred for more details.

In the guiding centre model the distribution functions for ions and electrons satisfy[11,12]

$$\frac{\partial f_\ell}{\partial t} + (\underline{v}_\perp + q\hat{b})\cdot\nabla f_\ell + \left[\hat{b}\cdot\nabla\left(\frac{v_\perp^2}{2} - \mu B - \frac{e_\ell}{m_\ell}\phi_\parallel\right) + q\,\underline{\kappa}\cdot\underline{v}_\perp\right]\frac{\partial f_\ell}{\partial q} = 0, \tag{15}$$

where ℓ denotes the different species. Here q is the microscopic parallel velocity, μ is the magnetic moment, e_ℓ and m_ℓ are the charges and the masses of the particles, $\underline{v}_\perp = (\underline{E}\times\underline{B})/B^2$ is the electric drift, $\phi_\parallel$ is the potential for the parallel electric field, $\hat{b} \equiv \underline{B}/B$ and $\underline{\kappa} = \hat{b}\cdot\nabla\hat{b}$ is the magnetic field curvature. Macroscopic quantities are obtained in terms of the moments of the guiding centre distribution functions $f_\ell(\mu,q,\underline{x},t)$

$$(\rho, \rho v_\|, p_\perp, p_\|)_\ell = B\int_0^\infty d\mu \int_{-\infty}^\infty dq\, [1, q, \mu B, (q - v_\|)^2]\, f_\ell . \tag{16}$$

The parallel electric field is determined from the constraint of charge neutrality $\Sigma_\ell\, \rho_\ell (e_\ell/m_\ell) = 0$.

At equilibrium, in axisymmetric geometry, the kinetic equations (15) are solved by $f_\ell = f_\ell(\mu,H_\ell,\psi)$, with the pseudo-Hamiltonians H_ℓ defined by

$$H_\ell = \frac{1}{2}\left(q - \frac{B_\phi}{B}V\right)^2 + \mu B + \frac{e_\ell}{m_\ell}\phi_\| - \frac{V^2}{2} . \tag{17}$$

The parallel gradients of the mass density and of the pressures can be expressed, using the definitions (16) and charge neutrality, as

$$\frac{d\rho}{\rho} = \left(1 + \frac{C_\pm}{\rho}\right)\frac{dB}{B} - \frac{C_\ddagger}{\rho}\, VdV , \tag{18}$$

$$dp_\perp = (2p_\perp + C_*)\frac{dB}{B} - C_+ VdV , \tag{19}$$

$$dp_\| = (p_\| - p_\perp + M_p^2 v_A^2 C_+)\frac{dB}{B} - (\rho - M_p^2 v_A^2 C_\ddagger)\, VdV . \tag{20}$$

The kinetic functions C_+ , $C_\ddagger$ and C_* are given by

$$C_+ \equiv \Sigma_\ell\, C_{\ell,1} - \left(\Sigma_\ell \frac{e_\ell}{m_\ell} C_{\ell,1}\right)\left(\Sigma_\ell \frac{e_\ell}{m_\ell} C_{\ell,0}\right) / \Sigma_\ell \left(\frac{e_\ell}{m_\ell}\right)^2 C_{\ell,0} , \tag{21}$$

$$C_\ddagger \equiv \Sigma_\ell\, C_{\ell,0} - \left(\Sigma_\ell \frac{e_\ell}{m_\ell} C_{\ell,0}\right)^2 / \Sigma_\ell \left(\frac{e_\ell}{m_\ell}\right)^2 C_{\ell,0} , \tag{22}$$

$$C_* \equiv \Sigma_\ell C_{\ell,2} - (\Sigma_\ell \frac{e_\ell}{m_\ell} C_{\ell,1})^2 / \Sigma_\ell (\frac{e_\ell}{m_\ell})^2 C_{\ell,0} \ , \tag{23}$$

with

$$C_{\ell,n} = B^{n+1} \int_0^\infty d\mu\, \mu^n \int_{-\infty}^\infty dq\, \frac{\partial f_\ell}{\partial H_\ell} \ . \tag{24}$$

The variation of ρ, $p_\perp$ and $p_\parallel$ across the flux surfaces, which is not relevant to the present analysis, is given explicitly in Ref. 11.

We note that Eq. (18) for $d\rho$ has the same form of Eq. (14), except that the coefficents of dB and dV are different. If $C_\ddagger, V \neq 0$, we can solve for dV and eliminate it in favour of dB and $d\rho$ in the expressions (19)-(21) for the pressures gradients. By virtue of this transformation of variables, the GCP model gives the pressures as functions of ρ,B and ψ. It is easy to verify that the transformed expression for the pressures satisfy the integrability condition (6). As a consequence, a function W exists and the Bernoulli law (9d) is valid also in GCP. Thus, the equilibrium description that we have developed, also includes guiding centre, although (4) is not the natural form for the kinetic calculation.

We will complete the analysis of the equilibrium system using the general formulation. Then the results for different models can be obtained simply by replacing the corresponding values of the three "thermodynamic" functions $(\partial_B p_\perp)_{\rho,\psi}$, $\beta_\perp$ and $\beta_\parallel$, which determine the gradients of the pressures and of the mass density in the magnetic surfaces according to Bernoulli's law. The values of these quantities in MHD, CGL and GCP are given in the Table 1. The pressure gradients in the "GCP variables" B, V and ψ can also be obtained, using Eq. (14) to eliminate $d\rho$. For example,

$$\left(\frac{\partial p_\perp}{\partial B}\right)_{V,\psi} = \left(\frac{\partial p_\perp}{\partial B}\right)_{\rho,\psi} + B\beta_\perp\left(1 - \frac{\beta_\perp}{\beta_{//} - M_p^2}\right). \tag{25}$$

Comparison between the expressions (14) and (18) for $d\rho$ and between Eqs. (19) and (25) indicates what is the main difference between the fluid and the guiding centre approaches. In the fluid models $\beta_\perp$ and $\beta_\parallel$ are always positive quantities, so that the coefficients of dB and dV in (14) diverge for sonic flows, $M_p^2 = \beta_\parallel$, and similarly for the pressure derivative (25). As will be shown in the next section, this singularity is at the origin of the first hyperbolic region in the Grad-Shafranov equation, for the MHD and the double-adiabatic models. In the guiding centre model singular behaviour of the density and the pressure can occur only if the kinetic integrals C_+ and $C_\ddagger$ diverge. Using the definitions (21)-(23) and charge neutrality, these integrals can be rewritten as

$$C_+ = \rho \frac{\sum_\ell \rho_\ell \frac{C_{\ell,1}}{C_{\ell,0}}}{\sum_\ell \frac{(\rho_\ell)^2}{C_{\ell,0}}}, \qquad C_\ddagger = \frac{\rho^2}{\sum_\ell \frac{(\rho_\ell)^2}{C_{\ell,0}}}, \tag{26}$$

with $C_{\ell,0}$, $C_{\ell,1}$ given in (24). Therefore, in order to recover an hydromagnetic type of behaviour, the denominators of the right hand sides of the expressions (26) must vanish. This is possible only if the functions $C_{\ell,0}$ for ions and electrons have opposite signs. However, by inspection of the definition (24), it is immediately seen that $C_{\ell,0}$ are *negative definite for distribution functions with* $\partial f/\partial H \leq 0$ and the integrals C_+ and $C_\ddagger$ remains *finite* (the same result holds also for C_*). In other words,*the MHD resonance can be recovered only if the ion distribution has either large regions with* $\partial f/\partial H > 0$, *or positive jumps*.

IV. ON THE SOLUBILITY OF THE EQUILIBRIUM SYSTEM

In this section we discuss the well-posedness of the equilibrium system, i.e. we look for the conditions on the flow and the pressure anisotropy under which the equations for the mass density, Eq.(9d), and for the magnetic field, Eq. (9c), can be solved along the field lines. We shall also determine the conditions under which the generalized Grad-Shafranov equation for ψ, Eq. (9e), is elliptic.

IV.A The characteristic determinant

The type of the equilibrium system is determined by the differential operator on the left hand side of Eq. (9e). This can change from elliptic to hyperbolic, due to the presence of the function τ, which depends on $|\nabla\psi|$ through ρ, B and the pressures. Following the analysis of Hameiri[7] for the MHD case, we observe that the second order derivatives of ψ are the same as in $\tau\nabla^2\psi + \dot{\tau}\,\nabla\psi\cdot\nabla|\nabla\psi|^2$, where $\dot{\tau} = \partial\tau/\partial|\nabla\psi|^2$. In cylindrical coordinates this becomes $[\tau + 2\,\dot{\tau}(\partial_r\psi)^2]\,\partial^2{}_r\psi + 4\,\dot{\tau}\,\partial_r\psi\,\partial_z\psi\,\partial^2{}_{rz}\psi + [\tau + 2\,\dot{\tau}(\partial_z\psi)^2]\,\partial^2{}_z\psi$. The characteristic determinant D, which is the determinant of the matrix formed of the coefficients of the second order derivatives, is therefore given by

$$D = \tau^2\,(1 + 2|\nabla\psi|^2\ \dot{\tau}/\tau)\,. \tag{27}$$

The equation is *elliptic (hyperbolic) if $D > 0$ ($D < 0$).* The derivative $\dot{\tau}$ can be computed from the Eq. (9c), using Bernoulli law to eliminate the density gradient. In GCP variables the resulting expression for D is very simple:

$$D = \tau^2 B^2 \frac{K}{B_p^2\tau + B_\phi^2 K}\ , \tag{28a}$$

with

$$K \equiv 1 + \frac{1}{B}\left(\frac{\partial p_\perp}{\partial B}\right)_{V,\psi} . \qquad (28b)$$

The ellipticity (hyperbolicity) condition is given by the positivity (negativity) of the fraction in (28a). The GCP result, where $K = 1 + (2p_\perp + C_*)/B^2$ was obtained by Dobrott and Greene.

In the fluid models K can be expressed using Eq. (25). Then, in the absence of flows, the equilibrium is elliptic for arbitrary poloidal and toroidal magnetic fields if

$$\alpha - \frac{\beta_\perp^2}{\beta_{//}} > 0 , \qquad \alpha \equiv 1 + \frac{1}{B}\left(\frac{\partial p_\perp}{\partial B}\right)_{\rho,\psi} + \beta_\perp ,$$

$$1 - \Delta > 0 . \qquad (29)$$

In the double-adiabatic model the first condition becomes $B^2 + 2p_\perp > p_\perp^2 / 3p_{||}$ and (29) coincide with the ellipticity conditions given by Grad[12] for static equilibria with anisotropic pressures.

However, the expression of the characteristic determinant does not contain explicitly the toroidal flow. Therefore the inequalities (29) are also the ellipticity conditions for anisotropic equilibria with arbitrary toroidal rotation. They are certainly satisfied in tokamaks, where β is at most of the order of the inverse aspect ratio. Thus, *the equilibrium problem remains elliptic for Tokamaks with toroidal flow and anistropic pressure.*

Next, we consider the case $M_p \neq 0$. We first note that in the fluid models $\beta_{||} > 0$ (in CGL, for example, $\beta_{||} = 3 p_{||}/B^2$). Therefore α is positive when the first of the conditions (29) is satisfied. Eq. (25) shows that when the poloidal flow approaches the parallel poloidal sound speed (i.e. M_p^2 approaches $\beta_{||}$)

from below, K decreases, goes through zero at $M_p^2 = \beta^* \equiv \beta_{||} - \beta_{\perp}^2/\alpha$, and becomes negative for $\beta^* < M_p^2 < \beta_{||}$. In the fluid models the behaviour of the denominator of (28) around the sound speed is similar to that of K. However, the zero and the change of sign occurs for a value of M_p^2 slightly larger than β^*, giving rise to a small interval where D is negative. This is the first hyperbolic region of the equilibrium. To see that in detail, we rewrite D, using the Eq. (25), as

$$D = \tau^2 B^2 \alpha \frac{\beta^* - M_p^2}{B_p^2 M_p^4 - X M_p^2 + Y}, \tag{30}$$

with

$$Y = B_\phi^2 \alpha \beta^* + B_p^2 \beta_{//}(1 - \Delta), \qquad X = B_\phi^2 \alpha + B_p^2 (1 - \Delta + \beta_{//}),$$

which is obtained by eliminating the term $\beta_{||} - M_p^2$ from the numerator and the denominator of (28). D vanishes for $M_p^2 = \beta^*$ and $M_p^2 = 1 - \Delta$. When (29) is satisfied, the coefficients X and Y in (30) are positive. Since the discriminant $X^2 - 4YB_p^2$ is always positive, the denominator of (28a) has two positive roots, corresponding to the speeds of the slow and fast compressional waves, that we call M_s^2 and M_f^2. If $\beta^* < 1 - \Delta$, which is certainly true for tokamaks, then $\beta^* < M_s^2 < 1 - \Delta < M_f^2$. Therefore we find two elliptic regions for $0 \le M_p^2 < \beta^*$ and $M_s^2 \le M_p^2 < M_f^2$, separated by a *narrow hyperbolic region for* $\beta^* < M_p^2 < M_s^2$, and a second hyperbolic region for $M_p^2 > M_f^2$. It may be noted that in the small region between M_s^2 and $\beta_{||}$ both K and the denominator of (28) are negative, giving an elliptic equilibrium.

The alternance of elliptic and hyperbolic regions that we have obtained is the same previously found[6-8] for MHD. The MHD expression for the characteristic determinant can be obtained by setting $\alpha = X = 1+\beta$, $\beta^* = \beta/(1+\beta)$ and $Y = \beta = \gamma p/B^2$ in (30). Thus pressure anisotropy in the fluid models only introduces the conditions (29) together with quantitative changes in the characteristic speeds delimiting the different regions .

Instead, in the GCP model the kinetic function C_*, appearing in the expression of K, is well behaved at the sound speed, except for pathological distribution functions. Therefore K is positive (in a low-beta plasma $K \cong 1$) and the first hyperbolic region is removed.

The ellipticity or hyperbolicity of the equilibrium problem is also studied in Ref. (16) by means of a local analysis of the linearized equilibrium equations. It is shown there that in MHD in the first hyperbolic region (which occurs when K is small and negative) perturbations having zero frequency in the laboratory frame propagate across the flux surfaces as slow MHD waves.

IV.B The equation for the mass density

To complete the discussion of the equilibrium system, we analyze the solubility conditions for Eqs. (9c-d). If we take the parallel gradient of (9c) and eliminate dI by using the definition of B, the resulting equation will form, together with Eq. (14), a system of two linear equations for dρ and dB. The determinant of the coefficients of dρ and dB in this system coincides with the denominator of the characteristic determinant in (30). Therefore, except at the transitions between the elliptic and hyperbolic regions corresponding to zeroes of the denominator, there is no additional complication in solving for ρ and B. By eliminating dρ , we can solve for dB:

$$\frac{dB}{B} = \frac{\tau}{D_0} \{ - \left[1 - 2 \frac{M_p}{\tau} \frac{B_\phi}{B} \frac{V}{v_A} - \frac{1}{\tau} \frac{B_\phi^2}{B^2} \frac{V^2}{v_A^2} \frac{B}{\rho} \left(\frac{\partial \rho}{\partial B}\right)_{V,\psi} \right] \frac{dr}{r} + \frac{B_p^2}{B^2} \frac{d|\nabla\psi|}{|\nabla\psi|} \}, \qquad (31)$$

where $D_0 = (B_p{}^2 \tau + B_\phi{}^2 K)/B^2$. Then, to obtain the explicit variation of the mass density along the field lines, we must substitute this expression into Eq. (14) for $d\rho$. However, for flow velocities in the range of the sound speed the terms proportional to V/v_A and $(V/v_A)^2$ can be neglected (in the fluid models this is true far enough from the transition speeds), and the main modifications with respect to the static case are introduced by the factor τ/D_0. For sub-Alfvénic flows ($M_p^2 \ll 1$) and usual Tokamak betas ($|\Delta| \ll 1$), we have that $\tau \cong 1$ and $D_0 \cong 1$, so that we can assume, for a first comparison with the experiments, that B is unchanged by the flow and the pressure anisotropy. As a consequence, the relevant information for the density distribution on the magnetic surfaces is contained in Eq.(14), with B considered as a decreasing function of the radius.

V. THE DENSITY ASYMMETRY

In this section we consider the effect of flows and pressure anisotropy on the poloidal variation of the mass density, for different models of the parallel dynamics. The expressions (14) and (18) for $d\rho$ are used and the magnetic field is taken as a decreasing function of radius. Therefore, the following analysis only holds for sub-Alfvénic flows (see Sec. IV.B). This is not a severe restriction for the comparison with experimental results. It should be noted that the point $M_p^2 = 1 - \Delta$ represents a singularity of the equilibrium problem in GCP[11] as well as MHD, so that the restriction to sub-Alfvénic velocities appears, indeed, as a necessary choice.

To quantify in a simple way the poloidal asymmetry of the equilibrium quantities, we introduce for any given quantity f an asymmetry factor A_f, defined as the normalized radial derivative of f at the radius r_a of the magnetic axis, i.e.

$$A_f \equiv \frac{r_a}{f_a} \frac{\partial f}{\partial r}(r_a, \psi_a), \qquad (32)$$

where ψ_a is the value of ψ at the magnetic axis and $f_a = f(r_a, \psi_a)$. The density asymmetry A_ρ vanishes for a static MHD equilibrium. Experimental results where A_ρ is of order of unity have been reported for the density profile on PDX in a strongly beam-heated discharge with large toroidal rotation.[5]

The mass density can be computed explicitly for an MHD equilibrium with purely toroidal flow, both for adiabatic and isothermal equations of state. In the case of an isothermal plasma with temperature $T = T(\psi)$, Eq. (9d) with $\psi_M' = 0$ and $W = T \ln\rho$ gives the well known result

$$\rho = \hat{\rho}(\psi) \exp\left(\frac{r^2}{2} \frac{\phi_E'^{\,2}(\psi)}{T(\psi)}\right), \tag{33}$$

where $\hat{\rho}(\psi) \equiv \exp\,[H_M(\psi)/T(\psi)]$. It follows from (33) that $A_\rho = V_a^2 / T_a$, showing that *in order to have* $A_\rho \sim 1$, *the velocity must be close to the sound speed* .

In the following we will see how this conclusion is modified by the introduction of pressure anisotropy and poloidal flow .

V.A Toroidal flow and pressure anisotropy

We first consider equilibria with purely toroidal flow and anisotropic pressure. In the fluid models $\beta_{\perp,\parallel}$ are positive quantities, so that the coefficient of dV/V in (14) is positive, corresponding to a centrifugal shift (note that dV/V = dr/r). However, we now have an additional term, the sign of which depends on the relative values of $\beta_\perp$ and $\beta_\parallel$. The outward shift of the density is *increased (decreased)* if $\beta_\parallel < \beta_\perp$ ($\beta_\parallel > \beta_\perp$). The effect of the two terms is comparable when $\beta_\parallel - \beta_\perp \cong (V/v_A)^2$.

Similar results can also be obtained for GCP. When the flow is purely toroidal, we can take distribution functions which are even about the average parallel velocity. If, moreover, the $f^\pm$ are monotonically decreasing functions of $H^\pm$, it follows from (26) that the functions C_+, $C_\ddagger$ are negative definite.

Therefore the corresponding $\beta_{\parallel}$ and $\beta_{\perp}$ are positive and GCP gives the same conclusions as the fluid models.

To be more specific, we consider the case of two-temperature Maxwellian distribution functions. Then "macroscopic" equations of state are obtained from the guiding centre description:

$$p_{\parallel} = \rho T_{\parallel}(\psi), \qquad p_{\perp} = \rho T_{\perp}(B,\psi) = \rho T_{\parallel} \frac{B}{B - \theta T_{\parallel}} , \tag{34}$$

where $\theta = \theta(\psi)$ measures the ratio $(T_{\perp} - T_{\parallel})/T_{\parallel}$ and $T_e = T_i$ has been assumed for simplicity. For this model $\beta_{\perp,\parallel} = p_{\perp,\parallel}/B^2 = T_{\perp,\parallel}/v_A{}^2$ and the kinetic functions C_*, C_+ and $C_\ddagger$ are negative definite, as can easily be verified using Table 1. We observe for completeness that the ellipticity conditions (29) become $B^2 + 2p_{\perp} > 2p_{\perp}{}^2 / p_{\parallel}$ and $1 - \Delta > 0$, which are certainly satisfied for the beta values attainable in tokamaks. The model (34) is characterized by a simple closed expression for ρ as a function of B, r and ψ

$$\rho = \hat{\rho}(\psi) \frac{T_{\perp}}{T_{\parallel}} \exp\left(\frac{r^2}{2} \frac{\phi_E'^{\,2}(\psi)}{T_{\parallel}(\psi)}\right), \tag{35}$$

where $\hat{\rho}(\psi) \equiv \exp\,[H_M(\psi)/\,T_{\parallel}(\psi)]$. This comes about as a natural generalization of the MHD result (33), which is recovered by taking $T_{\perp} = T_{\parallel} = T(\psi)$. The asymmetry factor corresponding to (35) is

$$A_{\rho} = \left(1 - \frac{T_{\perp a}}{T_{\parallel a}}\right) A_B + \frac{V_a^2}{T_{\parallel a}} . \tag{36}$$

Therefore, *if $p_{\perp} > p_{\parallel}$, the amount of toroidal flow needed to produce a given asymmetry of the density profile can be strongly reduced, compared to the isotropic case.* This is in agreement with the experimental situation described in Ref. 5, where nearly perpendicular injection was used and the measured toroidal velocities were significantly lower than the sound speed. In fact, when the injection is perpendicular to the magnetic field, most of the particles are trapped on the outside of the torus, contributing to the outward shift of the density profile.

On the other hand, Eq. (36) also indicates that, *in the case of parallel injection, the density asymmetry can be reverted, particularly for moderate toroidal flows.*

The model (34) is well suited for analytic investigations. In the appendix we solve explicitly the generalized Grad-Shafranov equation for this model, in the limit of small beta and small ratio of poloidal to toroidal magnetic field. For small flows and pressure anisotropy, the resulting Shafranov shift δ of the magnetic axis, defined as $\delta = (r_a - r_0)/a$, is found to be

$$\delta \cong \frac{\varepsilon}{2}\left[1 + \frac{2}{3(1+E)}\left(\mathcal{M}_0^2 + \frac{T_{\perp 0} - T_{\parallel 0}}{T_{\perp 0}}\right)\right], \tag{37}$$

where r_0 is distance of the geometric center of the plasma from the main axis, $\varepsilon = a/r_0$ is the usual inverse aspect ratio, $\mathcal{M}^2 = (r\,\phi_E')^2 / T_\parallel$ is the Mach number relative to the parallel thermal velocity, E is a constant related to the ellipticity of the flux surfaces and the quantities with subscript 0 are computed at r_0. Eq. (37) shows that the outward shift of the magnetic axis is *increased (if $p_\parallel < p_\perp$)* or *decreased (if $p_\parallel > p_\perp$)* by the pressure anisotropy, similarly to what happens for the density asymmetry. The contributions to the shift from rotation and pressure anisotropy can be of the same order. They can both become of the order of the conventional Shafranov shift due to finite pressure (although this is taking Eq. (37) somewhat beyond its strict domain of validity).

It is interesting to note that as long as the parallel temperature is constant along the field lines, the effect of pressure anisotropy on the density asymmetry remains *qualitatively unchanged* from that discussed before, for *any* perpendicular equation of state. This is immediately seen from the parallel component of the force balance (for $\underline{v} = 0$)

$$T_{\parallel}(\psi)\, \underline{B}\cdot\nabla\rho \;=\; (p_{\parallel} - p_{\perp})\; \hat{b}\cdot\nabla B, \tag{38}$$

which shows that the sign of the parallel derivative of the density is determined by that of $p_{\parallel} - p_{\perp}$. Moreover, for perpendicular equations of state of the form $T_{\perp} = T_{\perp}(B,\psi)$, the expression (36) for the density asymmetry remains valid and the mass density can be computed explicitly, giving

$$\rho = G(B,\psi)\, \hat{\rho}(\psi) \exp\left(\frac{r^2}{2}\, \frac{\phi_E'^{\,2}(\psi)}{T_{\parallel}(\psi)}\right), \tag{39}$$

where G is the solution of $\partial_B\,(\ln G)\big|_{\psi} = B^{-1}\,(1 - T_{\perp}/\,T_{\parallel})$ and $\hat{\rho}$ is the same as in (35). In the special case $T_{\perp} = T_{\perp}(\psi)$, we have $\ln G = (1 - T_{\perp}/\,T_{\parallel}) \ln B$.

V.B The effect of poloidal flows

Finally, we consider poloidal flow ($\psi_M' \neq 0$). The poloidal variation of the density in the MHD model is given by

$$\frac{d\rho}{\rho} = \frac{1}{\beta - M_p^2}\left[-M_p^2 \frac{dB}{B} + \frac{V^2}{v_A^2}\, \frac{dr}{r} \right], \tag{40}$$

which is obtained from (14) by putting $\beta_{\parallel} = \beta_{\perp} = \beta \equiv \gamma p/B^2$. (The effect of the pressure anisotropy is similar to that discussed in the previous subsection and it will not be further investigated.) Equation (40) shows that in MHD: **a)** the effects on the density asymmetry of the poloidal flow and of the rotation induced by the electric field are additive; **b)** even small poloidal flows can produce a large shift of the density: the two terms in the square bracket are comparable for $v_p^2 = (B_p^2 / B^2) V^2$.

The sign of the asymmetry is determined by the coefficent in front of the square bracket, which is *positive* in the first elliptic region (giving an *outward* shift of the density), diverges at the sound speed and then becomes *negative* (giving rise to an *inward* shift). [These conclusions are somewhat modified if we take into account the true expression for dB, Eq. (31). Then the term $\beta - M_p^2$ in (40) is replaced by $(M_s^2 - M_p^2)(M_f^2 - M_p^2)$, the denominator of the characteristic determinant in (30). Therefore, a first resonance and change of sign of the asymmetry occurs at *the slow wave speed* instead of the sound speed ($M_p^2 = M_s^2 \cong \beta/(1 + \beta B_\phi^2/B^2)$, instead of β). Another resonance appears at the fast wave speed ($M_p^2 = M_f^2$). For velocities in this range the full expression (31) should be used.]

In the guiding centre model the distribution functions must be even about $q = VB_\phi/B$, in the trapping region, but asymmetric outside[11] in order to produce a net poloidal flow. To simplify, we restrict the analysis to the case of field aligned flow, so that only the coefficient of dB in the expression (18) for $d\rho$ needs to be considered and seek to determine the sign of the asymmetry as a function of the flow. Here we consider a single ion species and denote ions and electrons by ± respectively. We take a specific model distribution that is Maxwellian $f^\pm = \exp(-\beta^\pm H^\pm)$ in one direction of the parallel velocity (say $q < 0$). In the opposite direction we choose:

$$f^\pm = \begin{cases} \exp(-\beta^\pm H^\pm) & \mu B + \phi_{//}(e/m)^\pm \le H^\pm \le H_1^\pm \quad \text{(region I)} \\ \exp(-\beta^\pm H_1^\pm) & H_1^\pm \le H^\pm \le H_1^\pm + \delta \quad \text{(region II)} \\ \exp(\beta^\pm \delta)\exp(-\beta^\pm H^\pm) & H^\pm \ge H_1^\pm + \delta \quad \text{(region III)} \end{cases} \qquad (41)$$

Here $H_1^\pm \equiv \mu \tilde{B}^\pm + \tilde{\phi}^\pm$, with the constants $\tilde{B}$, $\tilde{\phi}$ chosen in such a way that $\mu \tilde{B}^\pm + \tilde{\phi}^\pm > \max [\, \mu B + \phi_{//}(B)\,(e/m)^\pm \,]$ for all B and $\phi_{//}$ on the flux surface, to ensure that the trapping region is inside the region I (note that the left extreme of this region corresponds to q = 0). The distributions (41) are sketched in Fig. 1 as functions of $H^\pm$, at a given μ value. The positive constant δ, which determines the width of the region II, measures the magnitude of the poloidal flow (for $\delta = 0$ Maxwellian distributions with $\beta^\pm = 1/T_{//}^\pm(\psi)$ are recovered).

Observing that $\partial f^\pm/\partial H^\pm$ vanishes in the region II and is equal to $-\beta^\pm f^\pm$ outside, $C_0^\pm$ and $C_1^\pm$ become

$$C_0^\pm = -\beta^\pm [\rho - \rho^{II}]^\pm, \qquad C_1^\pm = -\beta^\pm [p_\perp - p_\perp^{II}]^\pm, \tag{42}$$

where ρ and $p_\perp$ are the total density and perpendicular pressure, respectively, and the quantities with superscript II are integrated over the region II. As pointed out in Sec. III.B, both $C_0^\pm$ and $C_1^\pm$ are negative definite, irrespective of the value of the poloidal flow.

To make more explicit the dependence from the flow, we write

$$\rho^\pm = \rho_0^\pm + \rho^{II\pm} - \mathfrak{R}^\pm,$$

$$p_\perp^\pm = p_{\perp 0}^\pm + p_\perp^{II\pm} - \mathfrak{P}^\pm, \tag{43}$$

$$\mathfrak{R}^\pm \equiv [\rho_0^{II} - \rho^{III} + \rho_0^{III}]^\pm, \qquad \mathfrak{P}^\pm \equiv [p_{\perp 0}^{II} - p_\perp^{III} + p_{\perp 0}^{III}]^\pm,$$

where the quantities with index 0 are computed using the Maxwellian distributions (in the static limit $\delta = 0$, we find $\rho^\pm = \rho_0^\pm$ and $p_\perp^\pm = p_{\perp 0}^\pm$).

Using (42) and (43), we find

$$1+\frac{C_\pm}{\rho} = \frac{\sum_\pm \frac{m^\pm}{\beta^\pm}\frac{1}{\rho_0^\pm - \mathcal{R}^\pm}[\rho^{II} - \mathcal{R} + \beta \mathcal{P}]^\pm}{\sum_\pm \frac{m^\pm}{\beta^\pm}\frac{\rho^\pm}{\rho_0^\pm - \mathcal{R}^\pm}}, \tag{44}$$

where we have taken $(\rho_0/\beta)^\pm = p_{\perp 0}^\pm$, which corresponds to an isotropic static equilibrium. Note that, for $\delta = 0$, $\mathcal{R}^\pm = \mathcal{P}^\pm = \rho^{II\pm} = 0$ and the coefficient vanishes. As the quantities $(\rho^{II} - \mathcal{R})^\pm$, $(\rho_0 - \mathcal{R})^\pm$ are positive, the sign of the density asymmetry is determined by the sign of the quantities $\mathcal{P}^\pm$, which, following the definitions (43), are given by:

$$\mathcal{P} = \frac{B^2}{\sqrt{2}}\int_0^\infty d\mu\, \mu\, \{\int_{H_1}^{H_1+\delta} dH \frac{e^{-\beta H}}{\sqrt{H - H^*}} - (e^{\beta\delta} - 1)\int_{H_1+\delta}^\infty dH \frac{e^{-\beta H}}{\sqrt{H - H^*}}\}$$

where $H^{*\pm} \equiv \mu B + \phi_{//}(e/m)^\pm$. By simple manipulations we obtain

$$\mathcal{P} = \frac{B^2}{\sqrt{2}}\int_0^\infty d\mu\, \mu \int_{H_1}^\infty dH\, e^{-\beta H}\{\frac{1}{\sqrt{H - H^*}} - \frac{1}{\sqrt{H - H^* + \delta}}\}, \tag{45}$$

which is a positive definite function of δ. Therefore $d\rho/dB$ is positive, giving rise to an *inward shift* of the density profile. Comparing with (40), we see that, for sub-sonic flows, this is *opposite to the MHD result.*

A similar behaviour is obtained in Ref. (16) for the response of the plasma density to magnetic field perturbations in cylindrical geometry. Due to the elimination of the slow magneto-acoustic waves in the guiding centre model, squeezing a flux tube leads to an increase of the plasma density. In the equilibrium problem this corresponds to saying that, following a fluid element, the density of the plasma increases when the section of the tube decreases. Notably, in the subsonic region MHD predicts that the density decreases when the flux tube is squeezed, due to the presence of sound waves.

Finally, we note that the other kinetic integral $C_{\ddagger}$, is negative definite for any distribution with non-positive $\partial f/\partial H$. Therefore, the second term on the right hand side of (18), representing a toroidal flow, always gives an outward centrifugal shift of the density.

VI. CONCLUSION

The MHD theory of axisymmetric equilibria with flows has been extended to include pressure anisotropy. The equilibrium equations reduce to a second order quasi-linear partial differential equation for the magnetic flux, similar to the Grad-Shafranov equation of static MHD, coupled with six algebraic constraints expressing conservation laws. Our analysis applies for a general functional form of the pressures, which includes the double-adiabatic model, but also allows for different equations of state, such as, for example, isothermal.

The expressions for the pressures in the guiding centre model[11,12] can be cast in the same form as in the fluid models by a simple variable transformation. Then the difference between the fluid and the guiding centre approaches lies in three coefficients, expressing derivatives of the pressures with respect to the mass density and the magnetic field. These coefficients determine the variation of the pressures and the density in the magnetic surfaces. In the fluid models, when poloidal flow is present, the density gradient becomes singular for flows of the order of the poloidal sound speed, giving rise to the "first hyperbolic region" of the Grad-Shafranov equation. Furthermore, for small betas, pressure anisotropy does not change qualitatively the type of the equilibrium system as a function of the flow speed. In sharp contrast with the fluid models the guiding centre model predicts that the density gradient is well behaved at the sound speed and there is no "first hyperbolic region". We prove that this is true for distribution functions which are non-increasing functions of the particle energies. However, resonant hydromagnetic behaviour can occur for distribution functions that are sufficiently non-monotonic.

For purely toroidal flows, a fully macroscopic, anisotropic, "isothermal" model has been derived from the guiding centre equations, by using two-temperature Maxwellian distributions. An approximate solution of the Grad-Shafranov equation is given for this model in the limit of small beta and small B_p/B. The Shafranov shift and the density distribution on the magnetic surfaces are explicitly computed. For both of these quantities the outward centrifugal shift due to the rotation is increased (if $p_{\|} < p_{\perp}$), or decreased, and can even be inverted (if $p_{\|} > p_{\perp}$), by pressure anisotropy.

Finally we have investigated the effect of poloidal flows on the density profile. In this case the guiding centre results, obtained by using model distribution functions, are very different from the fluid results. In the guiding centre model, not only is the resonance at the sound speed removed, but in addition poloidal flows lead to an inward shift of the density profile. For subsonic flows, this is opposite to the MHD result. It appears, therefore, that the equilibrium of high-temperature collisionless plasmas with poloidal flows can be properly described only by models that treat the parallel dynamics kinetically.

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Appendix. Generalization of the solution of Maschke and Perrin

In this appendix we give analytic solutions of the Grad-Shafranov equation for the model (34), in the limit of small beta and small ratio of poloidal to toroidal magnetic field. Maschke and Perrin[17] obtained analytic equilibria with flows in axisymmetric geometry using MHD and isothermal equations of state. Here, we generalize that solution to the case of anisotropic pressure. For the model (34), $W = T_{\|} \ln(\rho T_{\|}/T_{\perp})$ and we can rewrite (9e) as:

$$r^2\nabla\cdot\left[\left(1-\frac{p_{\|}}{B^2}\frac{\zeta}{1-\zeta}\right)\frac{\nabla\psi}{r^2}\right] = -II'_M - \rho r^2\frac{T_\perp}{B}(\theta T_{\|})' - \left[\eta' + \eta\frac{r^2}{2}\left(\frac{\phi_E'^2}{T_{\|}}\right)'\right]\frac{r^2}{1-\zeta}\exp\left(\frac{r^2}{2}\frac{\phi_E'^2}{T_{\|}}\right), \tag{A1}$$

where

$$\eta = \eta(\psi) = T_{\|}\exp\left(\frac{H_M}{T_{\|}}\right) = p_{\|}(1-\zeta)\exp\left(-\frac{r^2}{2}\frac{\phi_E'^2}{T_{\|}}\right),$$

$$\zeta = \frac{\theta T_{\|}}{B} = \frac{T_\perp - T_{\|}}{T_\perp}.$$

For $T_\perp = T_{\|}$ (i.e. $\zeta = 0$), Eq. (A1) reduces to Eq. (4.6) of Ref. 17.

Now we assume small beta, where Eq. (9c) gives $I \cong I_M(\psi)$, and small B_p/B, so that $B \cong I/r \cong I_M/r$. We choose $\theta(\psi)$ such that $\theta T_{\|} = \text{const}$. Then the anisotropy term in the operator on the l.h.s. of (A1) is negligible. Therefore, using the expression (35) for the density, we obtain

$$\nabla^*\psi = -I_M I'_M - \left[\eta' + \eta\frac{r^2}{2}\left(\frac{\phi_E'^2}{T_{\|}}\right)'\right]\frac{r^2}{1-\zeta}\exp\left(\frac{r^2}{2}\frac{\phi_E'^2}{T_{\|}}\right), \tag{A2}$$

with $\nabla^* \equiv r\,\partial_r (r^{-1}\partial_r) + \partial^2_z$. This equation contains four arbitrary functions of ψ : $I_M, \eta, T_{\parallel}, \phi_E'$. We now introduce the Mach number $\mathcal{M}^2 = (r\,\phi_E')^2 / T_{\parallel}$, *relative to the parallel thermal velocity* . The source term in (A2) can be made a function of r only, with the following choices of the free functions:

$$\phi_E'^2 / T_{\parallel} = \text{const.} = \mathcal{M}_0^2 / r_0^2 \;\Rightarrow\; \mathcal{M}^2 = \mathcal{M}_0^2\, r^2 / r_0^2 \;,$$

$$\eta = (P / r_0^4)(\psi - \psi_b), \qquad I_M^2 = I_0^2 + 2\,(\mathcal{J} / r_0^2)(\psi - \psi_b)\,,$$

where the subscript 0 indicates the value at the radius r_0 , which is for the moment undetermined. Using these profiles, Eq. (A2) finally becomes

$$\tilde{\nabla}^* \psi = -\frac{\mathcal{J}}{r_0^2} - \frac{P}{r_0^2}\frac{\tilde{r}^2}{1 - \zeta_0 \tilde{r}} \exp\left(\frac{\mathcal{M}_0^2}{2}\tilde{r}^2\right), \tag{A3}$$

where the normalized coordinates $(\tilde{r},\tilde{z}) = (r,z)/r_0$ have been introduced.

The problem still contains an arbitrary function of ψ (only the ratio $\phi_E'^2/T_{\parallel}$ has been fixed), which can be chosen to be the parallel temperature, or the electric potential, deduced by the measured rotation velocity.

Following Maschke,[17] we look for a solution of (A3) of the form $\psi(\tilde{r},\tilde{z}) = h(\tilde{r},\tilde{z}) + g(\tilde{r})$, where $g(\tilde{r})$ is a particular solution of

$$\tilde{r}\frac{d}{d\tilde{r}}\left(\frac{1}{r}\frac{dg}{d\tilde{r}}\right) = -P\frac{\tilde{r}^2}{1 - \zeta_0 \tilde{r}} \exp\left(\frac{\mathcal{M}_0^2}{2}\tilde{r}^2\right), \tag{A4}$$

and the function h, which is solution of $\nabla^* h + \mathcal{J}/r_0^2 = 0$, is given by

$$h(\tilde{r},\tilde{z}) = CP\tilde{r}^2 - \frac{\mathcal{J}}{2}\tilde{z}^2 + \frac{P(E-1)}{4}\tilde{r}^2\left(\tilde{z}^2 - \frac{\tilde{r}^2}{4}\right) + \psi_s\;,$$

where C, E and ψ_s are constants (here E is a constant related to the ellipticity of the magnetic surfaces and not the electric field). In general, (A4) must be solved numerically (unless $\zeta_0 = 0$, which gives the MHD case treated by Maschke). However, in the limit of small flow and pressure anisotropy ($\mathcal{M}_0^2, \zeta_0 << 1$), the equation can be solved by expansion. The solution for ψ (for $\mathcal{J} = 0$) reads

$$\frac{\psi - \psi_s}{P} = C\tilde{r}^2 + (E-1)\frac{\tilde{r}^2}{4}(\tilde{z}^2 - \frac{\tilde{r}^2}{4}) - \frac{\tilde{r}^4}{8}(1 + \frac{8}{15}\zeta_0\tilde{r} + \frac{1}{6}\mathcal{M}_0^2\tilde{r}^2). \qquad \text{(A5)}$$

From Eq. (A5) we can compute the shift of the magnetic axis due to the flow and the pressure anisotropy. In order to do that, we fix the horizontal position of the plasma by giving the two points ($\tilde{r} = 1 \pm \varepsilon$, $\tilde{z} = 0$) where the plasma boundary crosses the r-axis (here r_0 is taken as the geometric center and $\varepsilon = a/r_0$ is the usual inverse aspect ratio). The constant C and the value ψ_b of the magnetic flux at the plasma boundary are then determined by the system $\psi(1 \pm \varepsilon, 0) = \psi_b$, and the position r_a of the magnetic axis can be computed from $\partial_r \psi(\tilde{r}, 0) = 0$. The resulting Shafranov shift of the magnetic axis is given in Eq. (37) in Sec. V. By comparing with the numerical results obtained by Maschke (Fig. 3 in the Ref. 17), we can see that, for small values of E, in spite of the approximations done the expression (37) for the shift is very accurate for sound Mach numbers up to 0.5-0.6, and even at sound speed ($\mathcal{M}_0 = 1$) the difference with the numerical values is less than 10%.

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TABLE 1

<table>
<tr><th></th><th>GCP</th><th>CGL</th><th>MHD</th></tr>
<tr><td>$\left(\frac{\partial p_\perp}{\partial \rho}\right)_{B,\psi}$</td><td>$\frac{C_+}{C_\ddagger}$</td><td>$\frac{p_\perp}{\rho}$</td><td rowspan="2">$\frac{\gamma p}{\rho}$</td></tr>
<tr><td>$\left(\frac{\partial p_\parallel}{\partial \rho}\right)_{B,\psi}$</td><td>$\frac{B^2}{\rho} M_p^2 - \frac{\rho}{C_\ddagger}$</td><td>$\frac{3p_\parallel}{\rho}$</td></tr>
<tr><td>$\left(\frac{\partial p_\perp}{\partial B}\right)_{\rho,\psi}$</td><td>$\frac{2p_\perp + C_*}{B} - \frac{C_+}{C_\ddagger} \frac{\rho + C_+}{B}$</td><td>$\frac{p_\perp}{B}$</td><td>0</td></tr>
</table>

Table 1. Values of the coefficients characterizing the parallel dynamics for GCP, CGL and MHD.

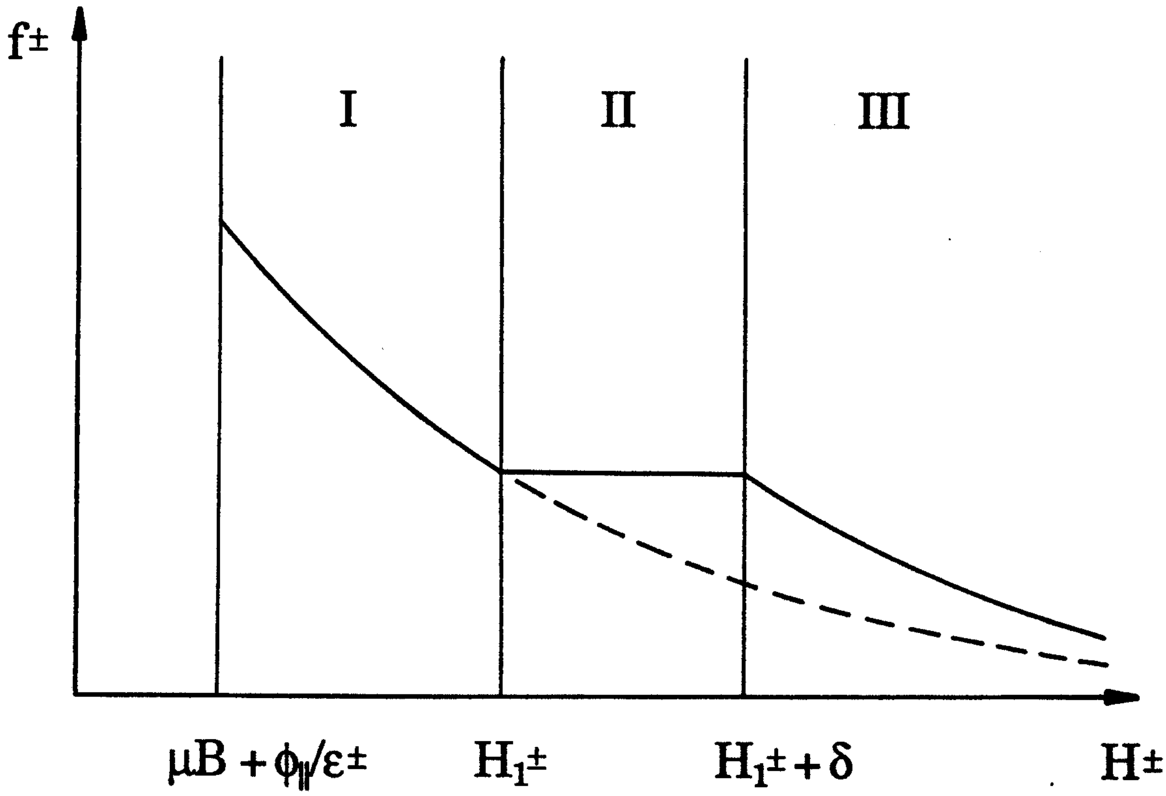

Fig. 1 Distribution functions for ions and electrons as a function of H, at given μ. The solid (dashed) line gives the distributions for $q > 0$ $(q < 0)$. The two coincide in the region I.