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Abstract - The ballooning-mode eikonal representation is applied to the linearised incompressible magnetohydrodynamic (MHD) equations in axisymmetric systems with toroidal mass flows to obtain a set of initial value partial differential equations in which the time t and the poloidal angle θ are the independent variables. To derive these equations, the eikonal function S is assumed to satisfy the usual condition $\mathbf{B} \cdot \nabla S = 0$ to guarantee that the modes vary slowly along the magnetic field. In addition, to resolve the $\mathbf{V} \cdot \nabla$ operator acting on perturbed quantities, the eikonal must also satisfy the condition $dS/dt=0$. This induces a Doppler shift in S. In a flux coordinate system (s, θ, ϕ) with straight magnetic field lines, we have that $S=\phi-q(s)\theta-\Omega(s)t+k(s)$, where q, Ω and k are functions only of the radial coordinate s and represent the safety factor, the plasma rotation frequency and the radial wave number, respectively. This description of the instability, however, is incompatible with normal mode solutions of the MHD equations because the wave vector ∇S becomes time dependent when the velocity shear is finite. Nevertheless, we are able to investigate the effects of sheared toroidal flows on localised ballooning instabilities because the initial value formulation of the problem we have developed does not constrain the solutions to evolve as $\exp(i\omega t)$. Fixed boundary MHD equilibria with isothermal toroidal flows that model the JET device are generated numerically with a variational inverse moments code. As the initial value equations are evolved in time, periodic bursts of ballooning activity are observed which are correlated with the formation of a ballooning structure at the outside edge of the torus that becomes displaced by 2π in the extended poloidal angle domain from one burst to the next. The frequency of these bursts depends only on the velocity shear and the instability growth rates extracted from the peak values of the bursts are virtually linear. The velocity shear has a stabilising influence on plasma ballooning.

1. INTRODUCTION

Auxiliary heating techniques such as neutral beam injection to achieve high temperatures in Tokamak devices can also induce a large toroidal plasma rotation that has approached the sound speed in the TFTR (GREK et al., 1987) and the JET (W.G.F. CORE et al., 1987) devices. The toroidal rotation profiles that have been measured in the TFTR tokamak indicate that the velocity shear is significant. Therefore, the motivation to determine the effect that sheared plasma flows can have on ballooning instabilities is clear and important because this class of mode may limit the beta (β) values that can be obtained.

Previous theoretical studies of ballooning modes in toroidal devices have suggested that ballooning instabilities as described through Weyl sequences do not exist (HAMEIRI and LAURENCE, 1984) or that the WKB ballooning formalism breaks down (BHATTACHARJEE, 1987) when the velocity shear in the toroidal flow is finite. These investigations, however, have been based on expansions of the Frieman and Rotenberg energy principle (FRIEMAN and ROTENBERG, 1960) which constrain any unstable solution to evolve as $\exp(i\omega t)$ and thus be a normal mode of the system.

In this work, we apply the covering space concept (DEWAR and GLASSER, 1983) to the linearised magnetohydrodynamic (MHD) equations to investigate localised ballooning instabilities in general axisymmetric equilibria. In contrast to the energy principle approach that has been previously followed, the initial value scheme we propose does not impose a priori any constraints on the form that the bounded solutions may take. We obtain as a result a set of initial value ballooning equations whose structure will only allow solutions that evolve as $\exp(i\omega t)$ when the toroidal flow is rigid. We thus have resolved the problem of the existence of ballooning type instabilities in plasmas with sheared toroidal rotation; the ballooning formalism is valid, but the solutions obtained are not normal modes of the system. Furthermore, the theory is applied to realistic numerical JET equilibria to determine qualitatively and quantitatively the impact of velocity shear on the ballooning stability properties.

The derivation of the initial value ballooning mode equations in plasmas with toroidal mass flows is presented in Section 2. In Section 3, we discuss the numerical JET equilibrium that is investigated. The ballooning stability results are analysed in Section 4, and the summary and conclusions are discussed in Section 5.

2. BALLOONING EQUATIONS WITH SHEARED TOROIDAL FLOWS

As a starting point, we consider the linearised incompressible MHD equations with toroidal mass flow given by the equation of motion

$$\rho_M \frac{\partial \mathbf{v}}{\partial t} = -\rho_M \nabla(\mathbf{v}\bullet\mathbf{V}) - \rho_M(\nabla\times\mathbf{v})\times\mathbf{V} - \rho_M(\nabla\times\mathbf{V})\times\mathbf{v}$$

$$-\rho_1(\mathbf{V}\bullet\nabla)\mathbf{V} - \nabla p_1 + (\nabla\times\mathbf{b})\times\mathbf{B} + (\nabla\times\mathbf{B})\times\mathbf{b}, \tag{1}$$

the Ohm's Law

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla\times(\mathbf{V}\times\mathbf{b}) + \nabla\times(\mathbf{v}\times\mathbf{B}) - \eta\nabla\times(\nabla\times\mathbf{b}), \tag{2}$$

the convective pressure evolution equation

$$\frac{\partial p_1}{\partial t} = -(\mathbf{v}\bullet\nabla)P - (\mathbf{V}\bullet\nabla)p_1, \tag{3}$$

the convective density evolution equation

$$\frac{\partial \rho_1}{\partial t} = -(\mathbf{v} \bullet \nabla)\rho_M - (\mathbf{V} \bullet \nabla)\rho_1 \tag{4}$$

and the Maxwell equation

$$\nabla \bullet \mathbf{b} = 0 \, . \tag{5}$$

The equilibrium velocity and magnetic fields are **V** and **B**, respectively. The perturbed velocity and magnetic fields are **v** and **b**, respectively. The equilibrium pressure and mass density are P and ρ_M, while the perturbed pressure and mass density are p_1 and ρ_1, respectively. The plasma resistivity is denoted by η.

To represent ballooning type instabilities, the covering space concept (DEWAR and GLASSER, 1983) is invoked so that any perturbation ξ_1 in axisymmetric devices is expressed in a magnetic flux coordinate system (s, θ, ϕ) as

$$\xi_1(s, \theta, \phi, t) = \xi(s, \theta, t) \exp\,[inS(s, \theta, \phi, t)] \tag{6}$$

where s labels the flux surface, θ and ϕ are the poloidal and toroidal angles, respectively, $n \gg 1$ is the toroidal mode number, and the eikonal function S is assumed to satisfy the condition $(\mathbf{B}\cdot\nabla)S = 0$. In order to resolve the $\mathbf{V}\cdot\nabla$operator acting on perturbed quantities, the function S must also satisfy the condition $dS/dt=0$. In flux coordinate systems with straight magnetic field lines, S can be constructed as

$$S = \phi - q(s)\,\theta + k(s) - \Omega(s)t \tag{7}$$

where the flux surface quantities q(s), k(s) and Ω(s) correspond to the safety factor, the radial wavenumber and the toroidal rotation frequency, respectively. That Ω depends only on s is a consequence of the Ohm's Law and axisymmetry. The last term in Eq. (7) represents a Doppler shift of the eikonal which occurs even in the limit of rigid flow. For the incompressible MHD equations under consideration, it is convenient to express the perturbations as

$$\mathbf{v} = \left(v_s \nabla S + v_\perp \frac{\mathbf{B}\times\nabla S}{B^2}\right) \exp(inS), \tag{8}$$

$$\mathbf{b} = \left(b_\parallel \frac{\mathbf{B}}{B^2} + b_s \nabla S + b_\perp \frac{\mathbf{B}\times\nabla S}{|\nabla S|^2}\right) \exp(inS), \tag{9}$$

$$p_1 = p \exp(inS) \tag{10}$$

and

$$\rho_1 = \rho \exp(inS) \tag{11}$$

and then carry out the usual ballooning expansion for large n (CONNOR et al., 1978). This particular decomposition of the vector perturbations that we have chosen simplifies the subsequent algebraic manipulations considerably. A set of initial value ballooning mode equations are obtained from the B×∇S component of the equation of motion

$$\rho_M \;|\nabla S|^2 \;\frac{\partial v_\perp}{\partial t} = \rho_M \;[2(\nabla S \bullet \nabla)\;(\mathbf{V} \bullet \nabla S)]\; v_\perp + B^2(\mathbf{B} \bullet \nabla)\; b_\perp$$

$$- 2\,(\mathbf{B} \times \nabla S \bullet \kappa)p - [\mathbf{B} \times \nabla S \bullet (\mathbf{V} \bullet \nabla)\mathbf{V}]\rho\,, \tag{12}$$

from the $\mathbf{B}\times\nabla S$ component of Ohm's Law

$$\frac{\partial b_\perp}{\partial t} = - [n^2\eta|\nabla S|^2 + \frac{2(\nabla S \bullet \nabla)(\mathbf{V} \bullet \nabla S)}{|\nabla S|^2}]\, b_\perp + \frac{|\nabla S|^2}{B^2}(\mathbf{B} \bullet \nabla)\, v_\perp, \tag{13}$$

from the convective pressure evolution equation

$$\frac{\partial p}{\partial t} = -\frac{\mathbf{B} \times \nabla S \bullet \nabla P}{B^2}\, v_\perp, \tag{14}$$

and from the convective density evolution equation.

$$\frac{\partial \rho}{\partial t} = -\frac{\mathbf{B} \times \nabla S \bullet \nabla \rho_M}{B^2}\, v_\perp, \tag{15}$$

where κ is the magnetic field line curvature. It is evident from these equations that normal mode solutions can only be constructed in the limit that the toroidal flow is rigid (Ω=constant). The initial value approach we have adopted does not constrain the solutions to evolve as $\exp(i\omega t)$. Although $v_\perp$ is expressed in units of

B_0a^2/t and $b_\perp$ is expressed in units of 1/a at the level of equations (12)-(15), a further normalisation is introduced such that distances are normalised to the minor radius a, the equilibrium magnetic field is normalised to its value B_0 at the magnetic axis, the pressure P is normalised to B_0^2, ρ_M is normalised to its value ρ_{M0} on axis, the time is normalised to the poloidal Alfvén time $\sqrt{\rho_{M0}}\, R_0/B_0$ and the rotational frequency is normalised to the inverse Alfvén time. Here R_0 is R on axis.

3. NUMERICAL JET EQUILIBRIUM

The inverse moments equilibrium code ATRIME (COOPER and HIRSHMAN, 1987) is used to generate fixed boundary numerical equilibria that model the JET device. This code generates axisymmetric MHD equilibria with isothermal toroidal mass flows. The profiles that must be specified are the plasma mass function M(s), the inverse of the safety factor, the plasma flow function $U(s)=0.25M_i\Omega^2(s)/T(s)$ where T(s) is the plasma temperature and M_i is the ionic mass, and the toroidal flux function Φ(s). The equilibrium plasma pressure can then be constructed from M(s), U(s) and Φ(s) (COOPER and HIRSHMAN, 1987). In addition, the Fourier amplitudes of R and Z at the plasma boundary must also be specified. For the JET device, these are given in COOPER and HIRSHMAN (1987).

It is worthwhile to note that the equilibrium calculations depend only on the ratio $\Omega^2(s)/T(s)$. However, the stability calculations require the knowledge of both Ω(s) and T(s). Consequently, the effects of plasma rotation and velocity shear on the ballooning stability properties of an invariant equilibrium state can be examined by prescribing a sequence of cases with either different Ω(s) or different T(s) profiles but with fixed U(s). This approach has the advantage of isolating the evaluation of the stability from equilibrium effects that could confuse the results that are obtained.

A numerical JET equilibrium with a thermal component of beta β_p=4.9%, a rotational component of beta β_R=1.1% and a Mach number 0.925 at the magnetic axis is generated by specifying $M(s)=0.04(1-4s^3+3s^4)$, $1/q(s)=1-2s^3/3$ and $U(s)=0.23$. The flux surfaces and the pressure surfaces of this equilibrium state are shown in Fig. 1. This equilibrium is then mapped from the (s, χ, ϕ) magnetic flux coordinates employed in the ATRIME code to the straight field line coordinates using the relation $\theta=\chi+\lambda(s, \chi)$, where λ is a periodic renormalisation parameter calculated internally in the code (HIRSHMAN and WHITSON, 1983).

4. BALLOONING STABILITY RESULTS

The initial value ballooning mode equations (12)-(15) that we have derived are solved explicitly using a technique similar to that successfully applied by HENDER et al. (1984) to the static plasma case for simplicity. The stability results presented in this section concentrate on the flux surface having s=0.85. As the initial value equations are evolved in time, periodic bursts of ballooning activity are observed that resemble the fishbone type of unstability, as is shown in Fig. 2. The mode structures at the peaks of the bursts labelled as a), b) and c) in Fig. 2 are displayed in Fig. 3. The instability structure at each burst has very strong ballooning characteristics localised at the outside edge of the torus, but displaced by 2π in the extended poloidal angle domain when compared with the structure at the previous burst. Although it is clear that the instability has a much more complicated time evolution than can be described by a simple $\exp(i\omega t)$ behaviour, one can nevertheless extract from the peak values of the ballooning bursts a growth rate that is remarkably linear. Using the JET equilibrium that we have calculated, a sequence of stability runs is produced with $\Omega'(s=0.85)=-0.23$ but with variable Ω by choosing the temperature profile as $T(s)=(1-1.026239s^2)^2$, $T(s)=(1-0.804909s^3)^2$ and $T(s)=1-s^3$. The stability results for this sequence are summarised in Fig. 4. As we can see, for fixed $\Omega'=d\Omega/ds$, the instability growth rates extracted from the peaks of the ballooning bursts and the frequency of these bursts which we label as ω are insensitive to variations in Ω. A second sequence of runs with $\Omega(s=0.85)=0.08$ but with variable $\Omega'(s)$ is produced with $T(s)=1-0.7225s$, $T(s)=1-0.85s^2$, $T(s)=1-s^3$ and $T(s)=1-s^4/0.85$. In this case, as is shown in Fig. 5, the frequency of the ballooning bursts varies linearly with Ω' and vanishes in the limit of zero velocity shear. On the other hand, the growth rate of the ballooning bursts decreases with an increase in the absolute value of the velocity shear. This indicates that velocity shear has a stabilising effect on the ballooning modes which can be explained by the fact that two fluid elements on adjacent flux surfaces become physically separated in space as time evolves and this inhibits the formation of an instability structure. In addition, because the system is toroidal in nature, these fluid elements will periodically meet which can account for the observation that the bursts depend only on the magnitude of the differential rotation.

5. SUMMARY AND CONCLUSIONS

The eikonal ballooning representation has been applied to the linearised incompressible MHD equations in axisymmetric systems with sheared toroidal mass flows. In order to construct instabilities with ballooning characteristics, the eikonal function S must satisfy not only the usual condition $\underline{B}\cdot\underline{\nabla}S=0$, but also the condition $dS/dt=0$. This entails a Doppler shift in S. A set of initial value ballooning mode equations is derived that resolves the problem of the existence of solutions in plasmas with sheared toroidal flows. The ballooning formalism remains valid, but the solutions that result are not normal modes of the system. Solutions that evolve as $\exp(i\omega t)$ are only allowed in the limit that the toroidal flow is rigid.

The equations we have derived have been applied to investigate the local stability of a realistic axisymmetric JET equilibrium state with sheared isothermal toroidal rotation. Periodic fishbone-like ballooning bursts are observed that correlate with the formation of a strongly ballooning instability structure at the outside edge of the torus. However, from one burst to the next the structure becomes displaced by 2π in the extended poloidal angle domain. A sequence of studies has demonstrated that the frequency of ballooning burst varies linearly with the magnitude of the velocity shear and is independent of the rotation frequency. Virtually linear instability growth rates are extracted from the peak values of the ballooning bursts. These growth rates depend more sensitively on the velocity shear than on the plasma rotation and the velocity shear has a net stabilising effect.

The calculations that have been carried out so far have been limited to the incompressible model. In future work, we shall address the coupling of the toroidal flows to the sound waves which may introduce important destabilising effects (BONDESON et al., 1987).

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Figure captions

Fig. 1 - The flux surfaces (solid contours) and the pressure surfaces (dashed contours) of a JET equilibrium with β_p=4.9%, β_R=1.1% and Mach number 0.925 at the magnetic axis.

Fig. 2 - The value of $v_\perp^2$ averaged over the extended poloidal angle domain $-41\pi \leq \theta \leq 41\pi$ on the flux surface with s=0.85 as a function of time. The time is normalised to the poloidal Alfvén time.

Fig. 3 - The instability mode structures on the flux surface with s=0.85 at three different times that correspond to the peaks of the ballooning burst labelled with a), b) and c) in Fig. 2, respectively. The parameter θ_k is dk/dq and each tick mark on the horizontal axis corresponds to a multiple of 2π. The instability structures peak at $\theta=14\pi$, $\theta=16\pi$ and $\theta=18\pi$ in a), b) and c), respectively.

Fig. 4 - The frequency ω of the instability bursts and the growth rate γ extracted from the peak values of the bursts as a function of the rotation frequency Ω on the flux surface with s=0.85 at constant $\Omega'(s=0.85)=-0.23$.

Fig. 5 - The frequency ω of the instability bursts and the growth rate γ extracted from the peak values of the bursts as a function of the velocity shear Ω' on the flux surface with s=0.85 at constant $\Omega(s=0.85)=0.08$.

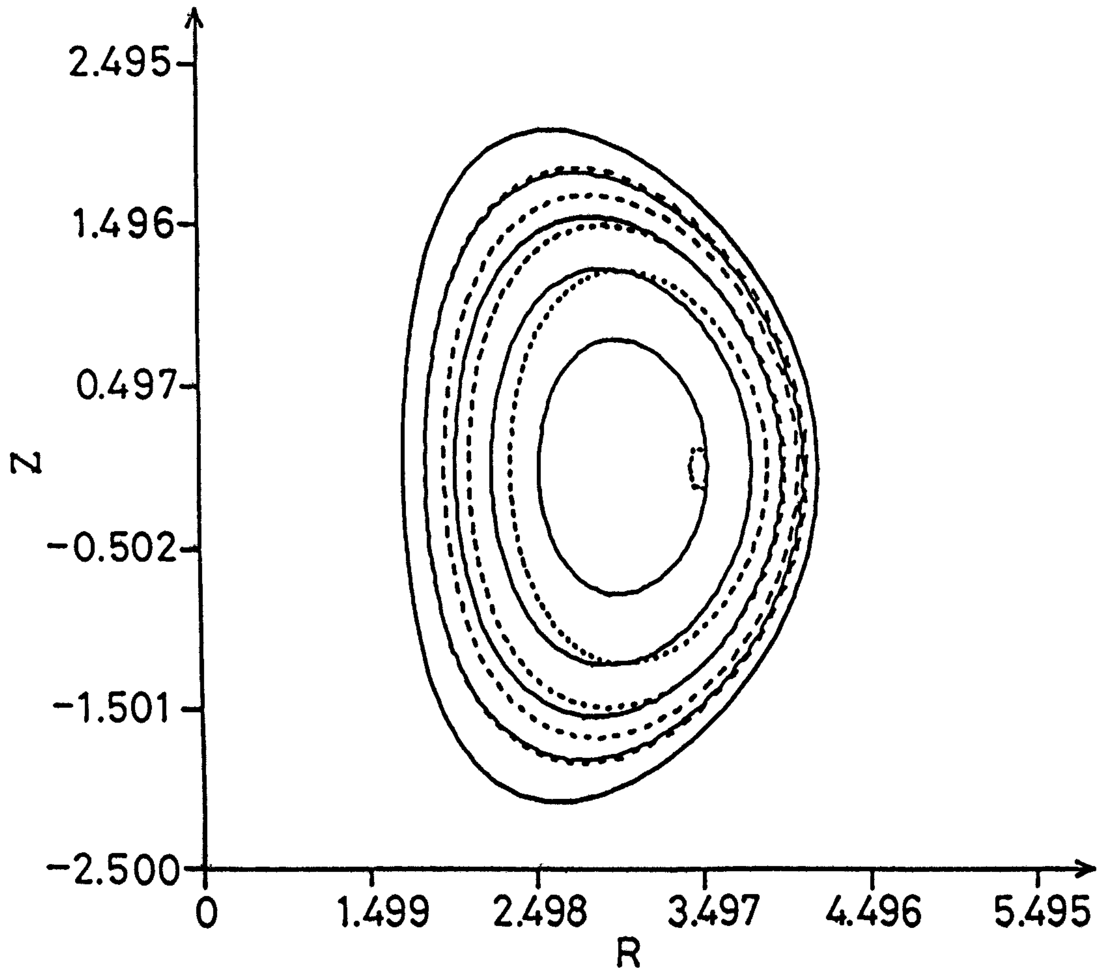


Fig. 1

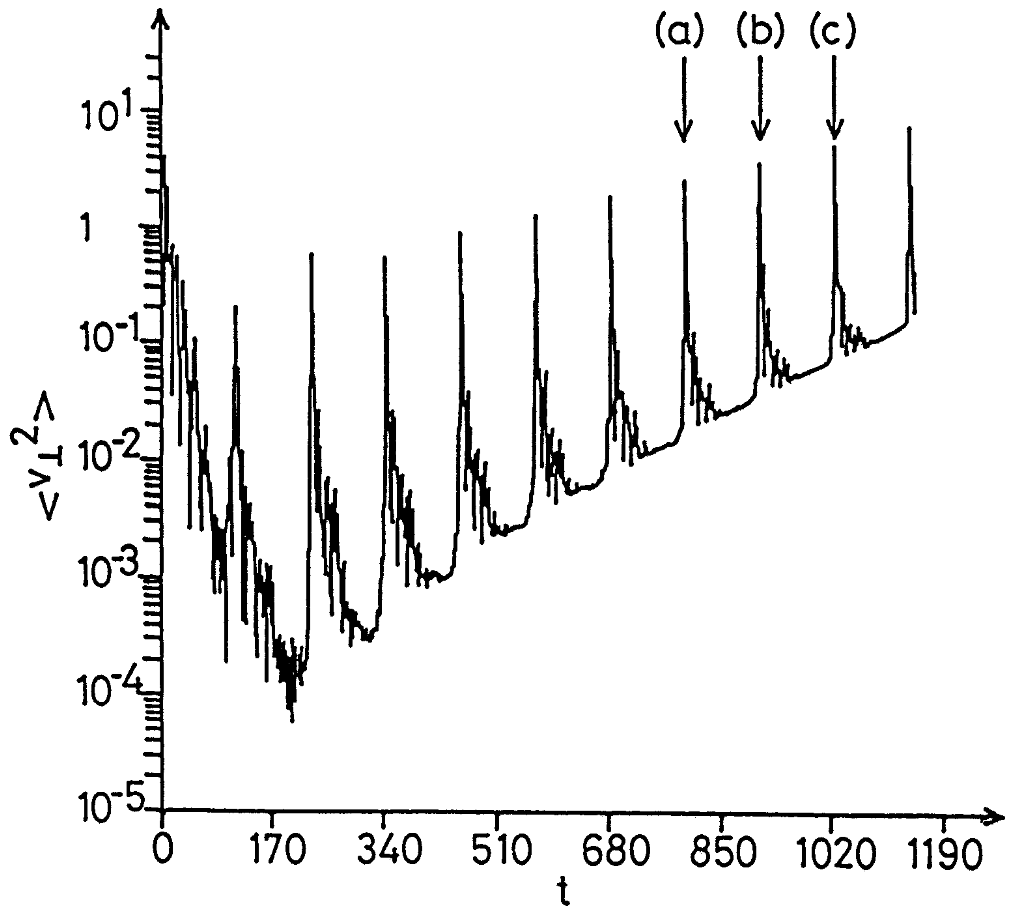
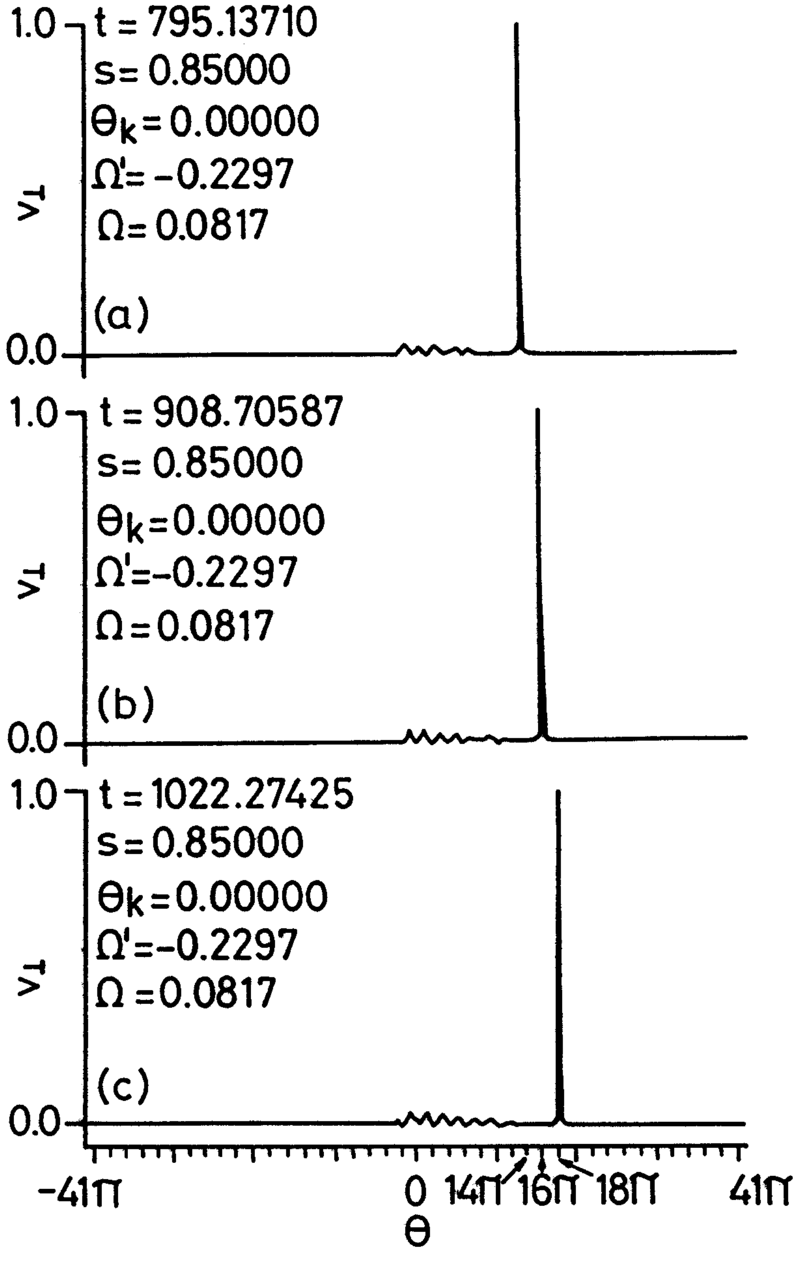

(a) (b) (c)
10^{1}
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
$\langle v_\perp^2 \rangle$
0
170
340
510
680
850
1020
1190
t

Fig. 2



1.0
t = 795.13710
s= 0.85000
Θk = 0.00000
Ω'= -0.2297
Ω= 0.0817
(a)
0.0
1.0
t = 908.70587
s= 0.85000
Θk=0.00000
Ω'=-0.2297
Ω=0.0817
(b)
0.0
1.0
t = 1022.27425
s= 0.85000
Θk=0.00000
Ω'=-0.2297
Ω =0.0817
(c)
0.0
v1
-41π
0
14π
16π
18π
41π
Θ

Fig. 3

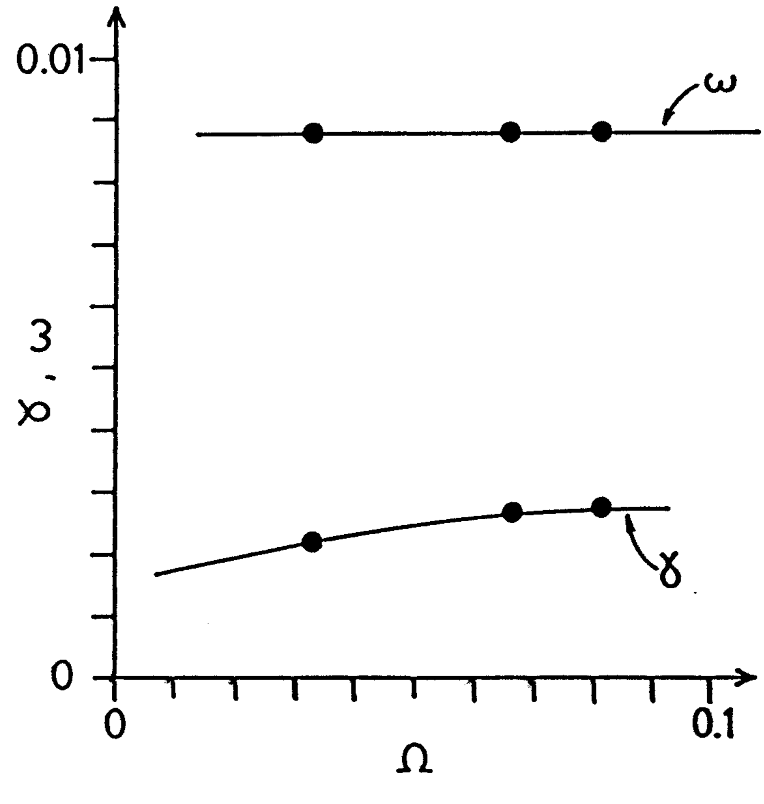


Fig. 4

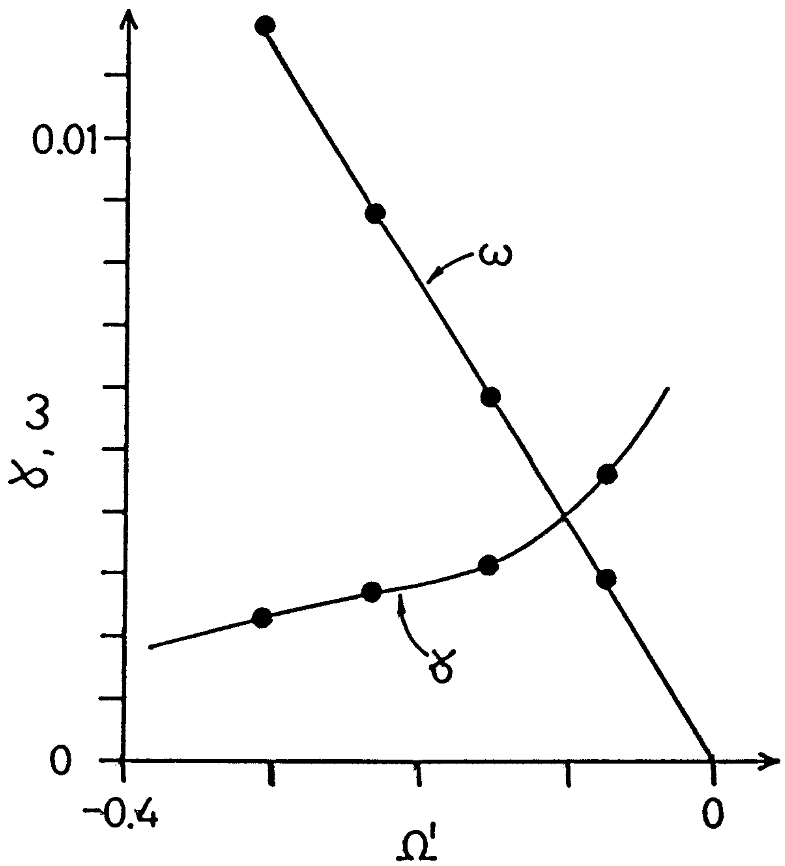


Fig. 5