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MHD EQUILIBRIUM AND STABILITY OF AXISYMMETRIC PLASMAS IN THE PRESENCE OF ROTATION

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1. INTRODUCTION

In this paper, we review some recent developments in the ideal MHD equilibrium and stability theory of axisymmetric plasmas with flow. In tokamaks heated by neutral beams, such as ISX-B,[1] PDX[2] and more recently, TFTR[3] and JET,[4] plasma rotation of the order of the sound speed has been observed. The reported data pertains to purely toroidal rotation. However, when neutral beams are co-injected in TFTR, there are indications that poloidal flows can occur.[5] It has also been recently suggested that strong electric fields can drive poloidal flows in the outer regions of tokamaks such as PDX and ASDEX, and that these flows can play an important role in determining the characteristics of the H-mode observed in these devices.[6,7]

Apart from the experimental observations cited above, motivation for some of the studies reported in this paper has come from the Second Regime Experiment (SRX) jointly proposed by Columbia University and Grumman Corporation.[8] Plasma rotation has been proposed as a possible technique for access to the second regime of stability[9,10] in SRX.

Our understanding of plasmas with flows is far from complete, inspite of the fact that there exists an extensive literature on different aspects of this problem in astrophysics, fusion and space plasma physics. In the present review, we will concentrate on axisymmetric plasmas, with emphasis on tokamaks.

2. AXISYMMETRIC TOROIDAL EQUILIBRIA

The ideal MHD equations are

$$\rho(\partial \underline{v}/\partial t) = \underline{J} \times \underline{B} - \nabla p - \rho\, \underline{v}\cdot\nabla\underline{v} \;, \qquad (1)$$

$$\partial\rho/\partial t \quad = -\nabla \cdot (\rho\underline{v}) \;, \qquad (2)$$

$$\partial s/\partial t \quad = -\underline{v} \cdot \nabla s \;, \qquad (3)$$

$$\partial \underline{B}/\partial t = -\nabla \times \underline{E} \ , \tag{4}$$

$$\underline{E} = -\underline{v} \times \underline{B} \ , \tag{5}$$

$$\nabla \times \underline{B} = \underline{J}, \qquad \nabla \cdot \underline{B} = 0, \tag{6}$$

$$p = S(s)\ \rho^{\gamma}, \tag{7}$$

where ρ is the mass density, $\underline{v}$, the flow velocity, $\underline{B}$, the magnetic field, $\underline{J}$, the current density, p, the plasma pressure and s, the entropy per unit mass. We have assumed that the plasma is an ideal gas, obeying the adiabatic law (7) in which S(s) is a known function of s and γ is the ratio of the specific heats.

In the equilibrium state, the left-hand sides of equations (1)-(4) vanish. Then, equation (4) implies $\underline{E} = \nabla\phi_E$, where ϕ_E is a (single-valued) function of position. From equation (5), we then get $\underline{v} \times \underline{B} = \nabla\phi_E$, which, in turn, implies that

$$\underline{B} \cdot \nabla\phi_E = \underline{v} \cdot \nabla\phi_E = 0 \ , \tag{8}$$

which means that both $\underline{B}$- and $\underline{v}$- lines lie on surfaces of constant ϕ_E. We can then invoke the theorem by Hopf invoked by Kruskal and Kulsrud[11] to assert that surfaces of constant ϕ_E are topologically toroidal and nested. (For the formulation of a Hamilton's principle for toroidal equilibria with flow, the reader is referred to Greene and Karlson.[12])

In the presence of a symmetry direction, the equilibrium equations can be reduced to a partial differential equation (P.D.E.) analogous to the Grad-Shafranov equation for static equilibria.[13-16] We use cylindrical coordinates (r,ϕ,z), where ϕ is the direction of symmetry. The magnetic field $\bar{B}$ is represented as

$$B = \nabla\phi \times \nabla\Psi + B_\phi \hat{\phi} \ , \tag{9}$$

where $\Psi(r,z)$ is the poloidal flux function and B_ϕ is the toroidal field. Since $\underline{B} \cdot \nabla\Psi = 0$, equation (8) implies that $\phi_E = \phi_E(\Psi)$. We define $\Omega(\Psi)$ such that $\nabla\phi_E = \Omega(\Psi)\nabla\Psi$. It follows, from equation (8) and $\nabla\cdot(\rho\underline{v}) = 0$, that the velocity field $\underline{v}$ can be represented as[16]

$$\underline{v} = (\Phi(\Psi)/\rho)\ \underline{B} + r\Omega(\Psi)\hat{\phi} \tag{10}$$

If $\Phi \neq 0$, $v \cdot \nabla s = 0$ implies that $s = s(\Psi)$. If $\Phi = 0$, there need be no constraint on s. However, for specificity, we shall keep the constraint $s = s(\Psi)$ even if $\Phi = 0$.

The ϕ-component of equation (1), at equilibrium, gives $(\underline{J} - \Phi\ \nabla \times \underline{v}) \cdot \nabla\Psi = 0$, which can be manipulated into the form $\underline{B} \cdot \nabla[rB_\phi(1-M^2) - r^2\Phi\Omega] = 0$, where $M^2 \equiv \Phi^2/\rho$ is the square of the Mach number of the poloidal flow with respect to the poloidal magnetic field. We therefore define

$$I(\Psi) \equiv rB_\phi(1 - M^2) - r^2\Phi\Omega\ . \tag{11}$$

If $\Phi = 0$, i.e., the rotation is purely toroidal, $I(\Psi) = rB_\phi$, as in the case of static equilibria.

Taking the scalar product of equation (1) with $\underline{B}$, we get $\underline{B} \cdot \nabla H = 0$, where

$$H(\Psi) \equiv (\Phi^2/2\rho^2)B^2 - R^2\Omega^2/2 + (\gamma/\gamma-1)S\ \rho^{\gamma-1}\ . \tag{12}$$

Equation (18) is the Bernoulli law for plasmas with flows.

Finally, projecting equation (1) along $\nabla\Psi$, we get,

$$\nabla \cdot \left[(1-M^2)\ (\nabla\Psi/r^2)\right] + (B_\phi/R)I' + \rho H' - (\rho^\gamma/\gamma-1)S' + \underline{v}\cdot\underline{B}\ \Phi' + r\rho v_\phi\Omega' = 0, \tag{13}$$

where prime designates derivation. Equation (13) is the analog of the Grad-Shafranov equation, and contains five functions $I(\Psi)$, $H(\Psi)$, $S(\Psi)$, $\Phi(\Psi)$ and $\Omega(\Psi)$, which are subject to the subsidiary relations (7), (10), (11) and (12). It is easy to see why there are five functions, instead of two in the case of static equilibria. The flow $\underline{v}$ introduces two additional functions. The remaining function can be accounted for by noting that in the static problem both the density and the entropy enter the equilibrium only through the pressure. In the presence of flows, this no longer holds, and density enters the equilibrium problem on the same footing as the pressure.

As pointed out by Grad,[17] the P.D.E. (13) is not always elliptic. The boundary conditions necessary to make the problem well-posed[18] (in the sense of Hadamard, i.e., a unique solution exists and depends continuously on the boundary data) depend crucially on the type of the P.D.E. (13), which can be cast in the form[15,16,19,20]

$$a(\partial^2\Psi/\partial r^2) + 2b(\partial^2\Psi/\partial r\partial z) + c(\partial^2\Psi/\partial z^2) + d = 0. \tag{14}$$

The determinant $\Delta \equiv ac-b^2$ determines the type of the P.D.E. If $\Delta > 0(< 0)$, the characteristics are complex (real), and equation (14) is elliptic (hyperbolic). For the P.D.E. (13), Δ is given by

$$\Delta = \frac{(1 - M^2)^2\ (1 - M^2/\beta)}{M^4/\beta_p - M^2/\beta + 1}\ , \tag{15}$$

where $B \equiv |\underline{B}|$, $B_p = |\underline{B} - B_\phi\hat{\phi}|$, $\beta \equiv \gamma p/(\gamma p+B^2)$ and $\beta_p \equiv \gamma p/{B_p}^2$. For $0 < M^2 < \beta$, we obtain the <u>first elliptic region</u>. This region encompasses purely toroidal flow for which equation (13) is always elliptic, like the Grad-Shafranov equation for static equilibria.

For simplicity, we now use the tokamak ordering[21] $B_p/B_\phi \sim p/{B_\phi}^2 = 0(\delta)$, where $\delta \equiv a/R$ is a small parameter, and a and R are the minor and major radii of the tokamak. We obtain the <u>first hyperbolic region</u> for $\beta < M^2 < {M_s}^2$ where ${M_s}^2 \simeq \beta(1+\beta^2/\beta_p)$ and M_s is the Mach number of the slow magneto-acoustic wave with respect to the poloidal Alfvén speed. We obtain the <u>second elliptic region</u> for ${M_s}^2 < M^2 < {M_f}^2$, where ${M_f}^2 \simeq \beta_p/\beta$ and M_f is the Mach number corresponding to the fast magneto-acoustic wave with respect to the poloidal Alfvén speed. Finally, for ${M_f}^2 < M^2$, we obtain the <u>second hyperbolic region</u>. The case of Alfvénic flows, $M^2 = 1$, occurs in the second elliptic region, but is not a transition point of the P.D.E. (13). For fusion plasmas, the regime $M^2 \gg 1$, in which poloidal rotational kinetic energy is much larger than poloidal magnetic field energy, is not interesting. Experimental values of M^2, even when serious attempts are made to measure them,[22] are estimated to be $O(\beta)$.

We note that the hyperbolic region separating the first elliptic and second elliptic regions is very narrow.[19,20] On the M^2-scale, the width of the first hyperbolic region is $M_S{}^2-\beta \simeq \beta^3/\beta_p = O(\delta^4)$. It is perhaps not too surprising that discrete-particle effects, incorporated through a guiding-center model,[23] have been shown to eliminate this narrow region in a low-β limit.[24] This has the important consequence that for all practical purposes, limits on equilibrium flows in tokamak plasmas will be set, not by equilibrium, but by stability considerations.

In recent years, several computer codes have been developed to calculate axisymmetric toroidal equilibria in the first elliptic region.[19,20,25-27] Some of these equilibrium codes are benchmarked with the analytical solutions for purely toroidal flow given in Ref. 28.

One of the distinguishing features of equilibria with flow is that the plasma density generally exhibits poloidal asymmetry on a flux surface.[29] A strong asymmetry can be produced by purely toroidal rotation of the order of the sound speed, but the presence of even a small poloidal rotation can induce the same degree of asymmetry at much lower values of toroidal rotation. Figures 1(a) and 1(b) from the Lausanne code CLIO[19] show the flux and isobaric surfaces, and the density and pressure profiles, respectively, for a purely toroidal rotation profile with a peak value of the order of the sound speed and a peaked temperature profile. We note that the constant-density and constant-pressure surfaces are centrifugally separated from each other, and from flux surfaces. The density profile peaks more towards the outer edge than does the pressure profile. On some intermediate flux surfaces, which lie between the maxima of the pressure and density profiles, the gradients in pressure and density actually oppose each other. This has a subtle consequence for the problem of ballooning stability in the presence of toroidal rotation, to which we shall return later.

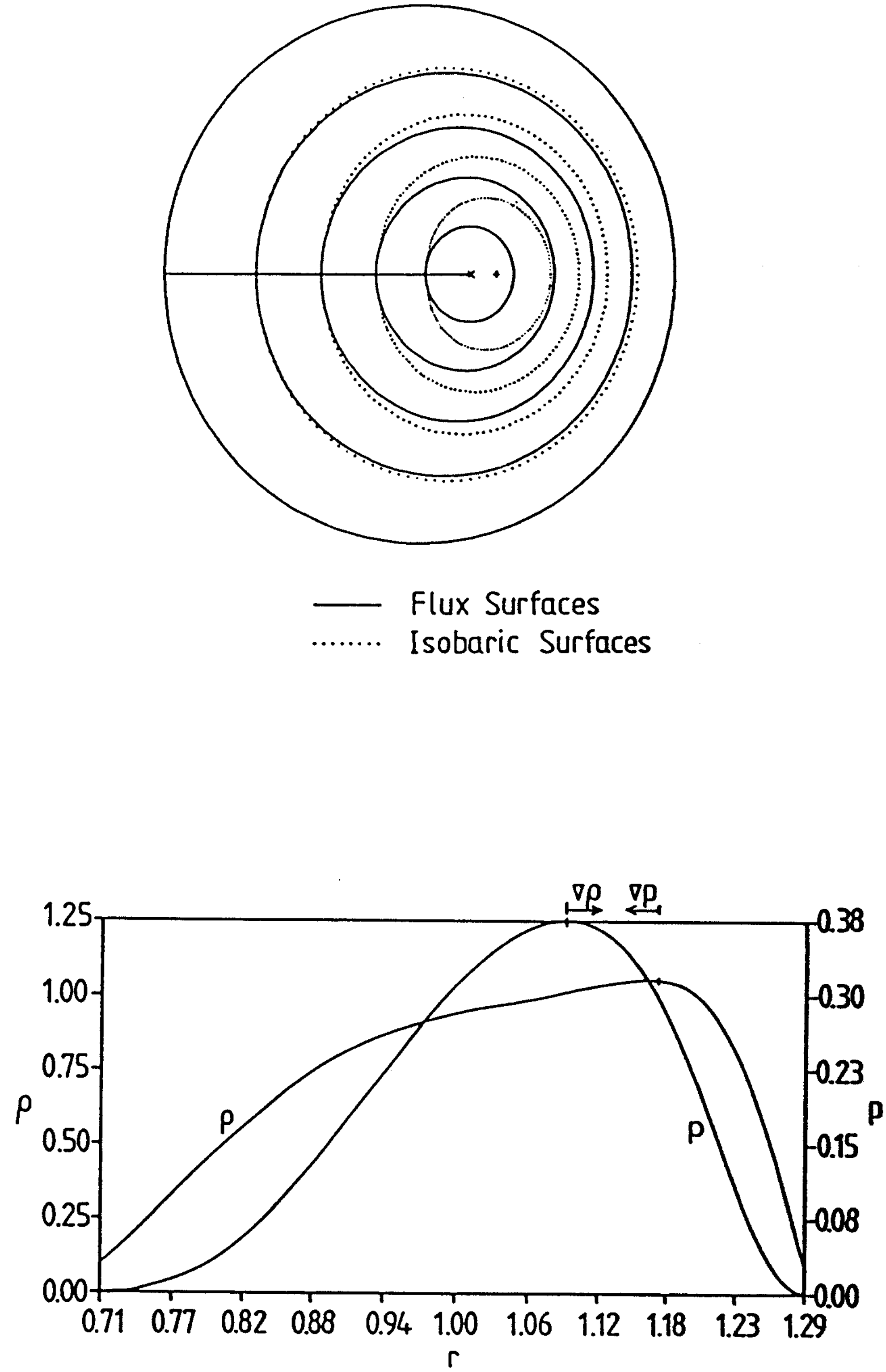
Flux Surfaces
Isobaric Surfaces
∇ρ
∇p
1.25
1.00
0.75
0.50
0.25
0.00
ρ
ρ
p
p
0.38
0.30
0.23
0.15
0.08
0.00
0.71
0.77
0.82
0.88
0.94
1.00
1.06
1.12
1.18
1.23
1.29
r

Fig. 1

3. LINEAR STABILITY OF EQUILIBRIA WITH FLOWS

The linear stability problem of MHD equilibria with flows is formulated by Frieman and Rotenburg.[30] We will restrict ourselves to a plasma bounded by a perfectly conducting wall, on which the boundary conditions are $\hat{n} \cdot \underline{B} = \hat{n} \times \underline{E} = \hat{n} \cdot \underline{v} = 0$, where $\hat{n}$ is the normal to the boundary. The perturbed quantities are most conveniently calculated by introducing a Lagrangian displacement vecter $\underline{\xi}$ about the equilibrium trajectory. Linearising equations (1) - (7), we get, after standard manipulations, the equation[30]

$$\rho\partial^2\underline{\xi}/\partial t^2 + 2\rho\underline{v} \cdot \nabla\partial\underline{\xi}/\partial t - \underline{F}\{\underline{\xi}\} = 0 \ , \tag{16}$$

where the operator $\underline{F}\{\underline{\xi}\}$ is given in the laboratory frame by

$$\underline{F}\{\underline{\xi}\} = \nabla(\gamma p\nabla\cdot\underline{\xi} + \underline{\xi}\cdot\nabla p - \underline{B}\cdot\underline{Q}) + \underline{B}\cdot\nabla\underline{Q} + \underline{Q}\cdot\nabla\underline{B} + \nabla\cdot(\rho\underline{\xi}\underline{v}\cdot\nabla\underline{v} - \rho\underline{v}\underline{v}\cdot\nabla\underline{\xi}), \tag{17}$$

with $\underline{Q} \equiv \nabla \times (\underline{\xi} \times \underline{B})$. We consider normal-mode solutions of the form $\underline{\xi}(\underline{r},t) = \underline{\xi}(\underline{r})\exp(-i\omega t)$, whereupon equation (16) becomes

$$-\omega^2\rho\underline{\xi} + 2i\omega\rho\underline{v}\cdot\nabla\underline{\xi} - \underline{F}\{\underline{\xi}\} = 0 \ . \tag{18}$$

Both the operators $i\rho\underline{v}\cdot\nabla$ and $\underline{F}$ are Hermitian with respect to the inner product $\langle \underline{f},\underline{g}\rangle = \int d\underline{r}\ \underline{f}^*\cdot\underline{g}$, where the integral is taken over the plasma volume. However, the presence of the second term containing ω in equation (18) makes the total operator non-Hermitian in general and does not permit a necessary and sufficient condition for stability analogus to the ideal MHD Energy Principle.[31] Thus, ω^2 is complex in general. However, if

$$\delta W = -\int d\underline{r}\ \underline{\xi}^*\cdot\underline{F}\{\underline{\xi}\} > 0, \tag{19}$$

ω is real and the system is stable. Equation (19), in other words, gives a sufficient condition for stability.

For several reasons, one of which is the absence of a comprehensive stability theory such as the Energy Principle[31], results on the stability problem with flows are neccessarily piecemeal[32-41]. Much can be learned by analysis of the linearized equations in geometries simple enough to be analytically tractable but containing physical effects essential to more complex geometries. In the next section, we give an example of such an analysis[38,39], relevant for tokamaks.

4. LOCAL STABILITY OF A CYLINDRICAL PLASMA

We consider axisymmetric equilibria in a straight cylinder. We use cylindrical polar co-ordinates (r,θ,z) and assume that all equilibrium quantities depend only on the radius r. The magnetic field is given by $\underline{B} = B_\theta\hat{\theta} + B_z\hat{z}$, and the velocity field, by $\underline{v}=v_\theta\hat{\theta}+v_z\hat{z}$. In the stationary state $(p + B^2/2)' + B_\theta{}^2/r - \rho v_\theta{}^2/r = 0$, where prime denotes radial derivative. Though v_z does not enter the equilibrium condition, it enters the stability analysis through the Doppler-shifted eigenfrequency. We assume that all perturbed quantities can be Fourier-analysed as $\exp[i(\omega t+m\theta-kz)]$. Following Ref. 42, equation (18) can be cast as the second-order, radial eigenvalue problem

$$AS\ (r\xi_r)'/r = c_1\xi_r - c_2p_* \ , \tag{20}$$

$$AS\ p_*' = c_3\xi_r - c_1p_* \ , \tag{21}$$

where $p_* = -\underline{\xi}\cdot\nabla p - \gamma p\ \nabla\cdot\underline{\xi} + \underline{B}\cdot\underline{Q}$. The coefficients in equations (20) and (21) are given by $A = \rho\tilde{\omega}^2 - F^2$, $S = (B^2+\gamma p)\rho\tilde{\omega}^2 - \gamma p\ F^2$, where $\tilde{\omega}^2 = \omega + \underline{k}\cdot\underline{v}$ is the local Doppler-shifted frequency, $F = \underline{k}\cdot\underline{B} = k_\parallel B = mB_\theta/r - kB_z$, and c_1, c_2 and c_3 are known functions of r. Equations (20) and (21) have the same form as the analogus equations in the static problem[42], except that equilibrium flows modify the coefficients. As in the static case, the radial eigenvalue problem has singularities when $A = 0$ or $S = 0$, which can occur only when ω is real. When $A = 0$, we obtain the Alfvén continua $\omega = -k\cdot v \pm k_\parallel v_A \equiv \Omega_A(r)$ where $v_A = B/\rho^{1/2}$ is the Alfvén speed. When $S = 0$, we obtain the slow-wave continua $\omega = -k\cdot v \pm k_\parallel v_S \equiv \Omega_S(r)$ where $v_S = v_A\beta^{1/2}$.

In the static problem, ω^2 is always real. This makes the region in the vicinity of the marginal point $\omega = 0$ interesting from the point of view of instabilities. The stability criterion for local modes at the resonant surface $r = r_0$ where $k \cdot B = 0$ is given by Suydam[43]. The generalisation of the Suydam criterion[38,39] can be obtained from a local analysis of equations (20) and (21) near the resonant surface, where A and S vanish quadratically with $X = r-r_0$ when $\omega(r_0) = 0$. Instability is indicated when the solution for ξ_r is oscillatory. The general form of the criterion is given in Ref. 39; we quote here the instablity condition for purely axial flow,

$$D_o \equiv \left(\frac{q}{q' B_z}\right)^2 \frac{2}{1-M^2} \left(-\frac{p'}{r} + \frac{2\beta M^2}{\beta - M^2} \frac{B_\theta^2}{r^2}\right) > \frac{1}{4}\,, \tag{22}$$

where $M = \rho^{1/2}\omega'/F'$ is the Alfvén Mach number, $q = rB_z/RB_\theta$ and $2\pi R$ is the periodicity length of the cylinder. It is clear that as $M^2 \to \beta$ from below, there is instability independent of the pressure gradient. Similarly, if $M^2 \to 1$ from below, there is also instability. In tokamaks, where $v_s{}^2/v_A{}^2 = \beta$ is small, the threshold $M^2 = \beta$ is met more readily the threshold $M^2 = 1$. We note that if we take the incompressible limit $\gamma \to \infty$, these two thresholds coalesce to $M^2 = 1$. The assumption of incompressibility may lead to a more optimistic stability picture in some cases[44]. It should be noted that whereas incompressibility is a strict consequence of the linearized equations for static equilibria (with non-zero shear) at marginal stability, it is not generally so in the presence of equilibrium flows.

There are two infinite sequences of eigenfrequencies which converge geometrically to the marginal point $\tilde{\omega} = 0$ at the resonant surface $\underline{k} \cdot \underline{B} = 0$. This can be demonstrated by an asymptotic analysis[38,39] similar to that given by Greene[45] for the static problem. The advantage of such an analysis, which is given in detail in Ref. 39, is that it prepares one for a surprise. From the generalised Suydam criterion (22), it would appear that just above the critical speed $M^2 = \beta$, the plasma is stable to local modes independent of the pressure gradient. However, we find numerically for a given equilibrium that one of the unstable Suydam sequences is transformed into a sequence of unstable discrete modes when the flow is just supercritical. This phenomenon of "exchange of instability" can be understood as follows.

At the threshold $M^2 = \beta$, the resonant surface is at the edge of a slow-wave continuum, $d\Omega_S/dr = 0$. Then a sequence of global slow modes may appear if the indicial equation for ξ_r has a complex exponent. This is one of the distinguishing features, introduced in the stability problem by equilibrium flows, which invalidates the claim[38] that the modified Suydam criterion (22) is necessary and sufficient for local instability. A local analysis near the edge of slow wave for purely axial flow indicates instability when

$$D_S \equiv \frac{2F^2}{S''} \quad \frac{\beta^2}{1-\beta} \quad \left[2B_\theta\left(\frac{B_\theta}{r}\right)' - \left(\frac{2B_\theta k}{rF}\right)^2 \frac{\gamma p}{\beta} + (1-\beta)F^2\right] > \frac{1}{4}\ , \qquad (23)$$

It can now be seen easily[39] that just above the threshold $M^2 = \beta$, whereas D_0 is large and negative, D_S is large and positive. Thus, the stability of the edge of the continuum is exactly opposite to that of the $k \cdot B = 0$ surface. At the threshold $M^2 = \beta$, where the two surfaces coalesce, the sequence (s) of unstable modes are "exchanged". It should be noted that just above the threshold $M^2 = \beta$, the limit frequency $\omega_0 \equiv -\underline{k}\cdot\underline{v}(r_0)$ will always overlap with a continuum at some other radial location. This has the physical consequence that "global" effects strongly influence the stability of "local" modes.

The analytical results presented above have been substantiated by a detailed numerical study given in Ref. 39. Here we excerpt one illustration from Ref. 39 which shows the presence of instability both below and above the threshold $M^2 = \beta$ for a certain equilibrium. Figure 2a shows the growth-rates of the lowest order modes whose eigenfunctions are localised inside the resonant surface. The instability persists above the threshold $v_{zo} = 0.18$. Figure 2b shows, on the other hand, that the modes localised outside the resonant surface are stabilised for values of v_{zo} around the critical speed. We emphasize that the distinction between "inner" and "outer" sequences may depend on the equilibrium, but the phenomenon of "exchange of stability"is generic.

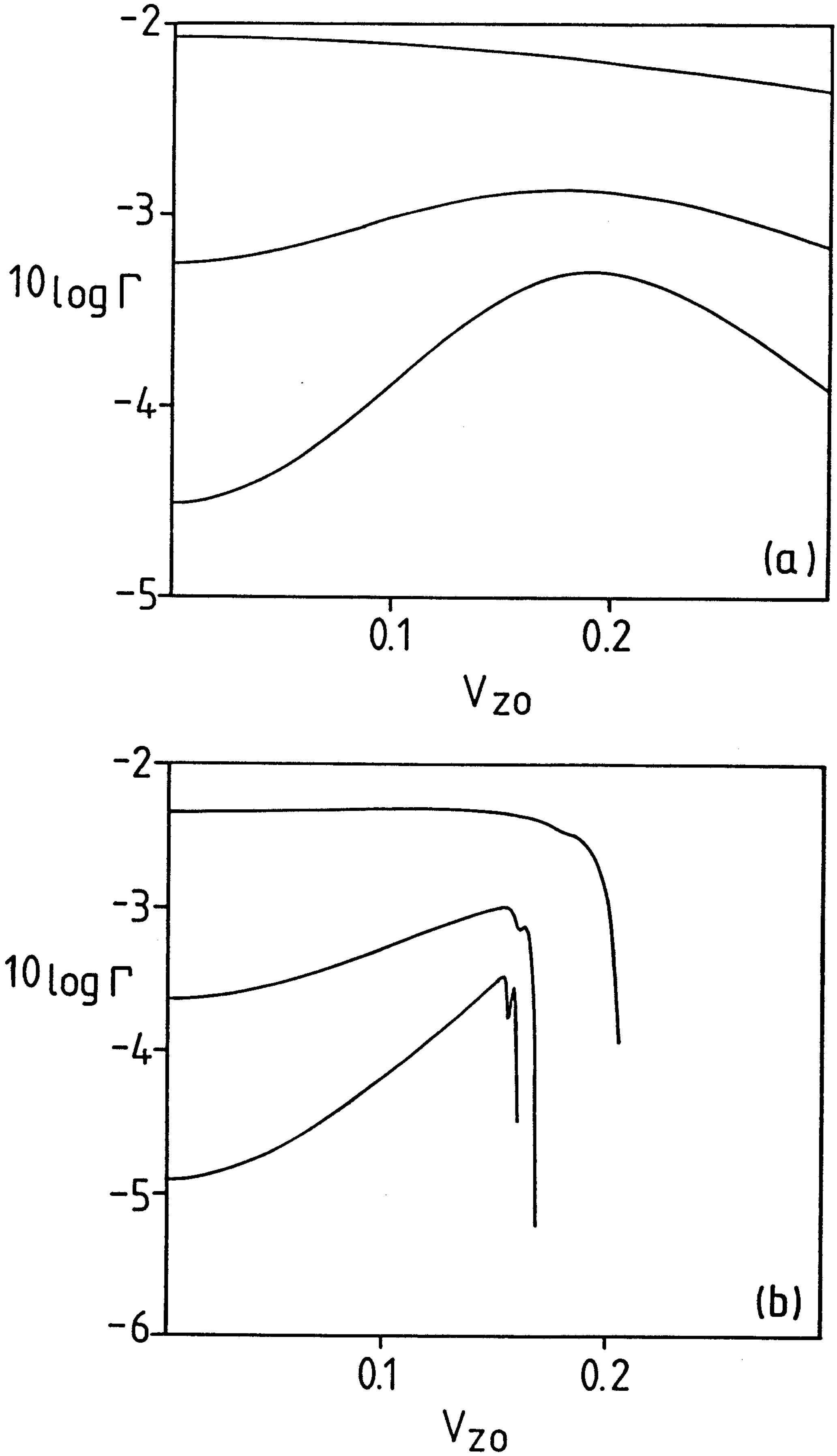

Fig. 2

5. BALLOONING STABILITY OF A TOROIDAL PLASMA

Ballooning modes, for a toroidal plasma in static equilibrium, are pressure-driven instabilities localised in the region of bad curvature. In the presence of equilibrium flows, this description of ballooning modes needs to be broadened to encompass the effects of flow. These effects can be fairly dramatic. In the following, we describe some recent results[35,40,41] obtained by using the standard ballooning-mode theory[46-48], and point to a breakdown of the conventional theory.

In Section 3, we have seen that in the presence of equilibrium flows, the Doppler-shifted eigenfrequency $\tilde{\omega} = \omega + \underline{k}\cdot\underline{v}$ takes the place of the eigenfrequency ω in the static problem. In an axisymmetric torus, in which $\underline{v}$ is given by equation (10), this suggests that the natural coordinate system in which to study ballooning modes, which are dominantly flute-like with $k\cdot B \sim 0$, is a coordinate system rotating toroidally. At a given flux surface, we therefore transform to a coordinate frame rotating in the ϕ-direction with angular velocity Ω_0, i.e., $\Omega_0 = \Omega_0\hat{z}$. Then, equation (1) transforms to

$$\rho\partial\underline{u}/\partial t = -\nabla p + \underline{J}\times\underline{B} - \rho\underline{u}\cdot\nabla\underline{u} - 2\rho\Omega_0\hat{z}\times\underline{u} + \rho\Omega_0{}^2 r\,\hat{r}, \qquad (24)$$

where $\underline{u} = \underline{v} - r\Omega_0\hat{\phi}$. The fourth term on the right in equation (24) is the Coriolis force and the last term, the centrifugal force. In the equations (2) - (7), $\underline{u}$ replaces $\underline{v}$. The analysis of Frieman and Rotenberg[30] can be repeated in the rotating frame to obtain equation (18), where now

$$\begin{aligned}\underline{F}\{\underline{\xi}\} = \;& \nabla(\underline{\xi}\cdot\nabla p + \gamma p\nabla\cdot\underline{\xi} - B\cdot Q) + \underline{B}\cdot\nabla\underline{Q} + \underline{Q}\cdot\nabla B \\ &+ \nabla\cdot(\rho\underline{\xi}\underline{u}\cdot\nabla\underline{u} - \rho\underline{u}\underline{u}\cdot\nabla\underline{\xi}) + 2\Omega_0\hat{z}\times\underline{u}\nabla\cdot(\rho\underline{\xi}) \\ &- 2\rho\Omega_0\hat{z}\times(\underline{u}\cdot\nabla\underline{\xi} - \underline{\xi}\cdot\nabla\underline{u}) - \nabla\cdot(\rho\underline{\xi})\Omega_0{}^2 r\,\hat{r}.\end{aligned} \qquad (25)$$

We consider large-n modes (where n is the mode number in the ϕ-direction) in the ballooning ordering, with long wavelength parallel and short wavelength perpendicular to the field-lines. We use the elegant WKB formalism developed by Dewar and Glasser[49]. The displacement $\underline{\xi}$ is represented in the eikonal form

$$\underline{\xi}(r) = \hat{\xi}(r,\varepsilon)\ \exp[iW(r)/\varepsilon] \tag{26}$$

where $\varepsilon \equiv 1/n$, $k \equiv \nabla W = O(1)$ and $\underline{B}\cdot\nabla W = 0$. The manipulations sketched in Ref. 49 can be executed to give an equation along a field-line only if $\Omega(\Psi) = \Omega_0$, a constant, but $\Phi(\Psi)$ can be arbitrary. In other words, the equilibrium flow in rotating coordinates is constrained to be $\underline{u} = [\Phi(\Psi)/\rho]\underline{B}$ for the standard ballooning theory to apply. This observation is originally due to Hameiri and Lawrence[37] who describe ballooning modes using the theory of singular sequences due to Weyl. The same difficulty appears in the WKB theory.[40,41] We will comment later on this problem.

If the toroidal rotation is rigid, the sufficient condition for stability (19) can be reduced to the one-dimensional form[37,40]

$$\delta W = \int_{-\infty}^{+\infty} \frac{dl}{B}\ [\dot{X}^2|\underline{N}|^2 - X^2\ \{2(\underline{N}\cdot\underline{J}\times\underline{B})(\underline{N}\cdot\underline{\kappa})\ - r\Omega_o^2(\underline{N}\cdot\nabla r)(\underline{N}\cdot\nabla\rho) + \frac{1}{\gamma p}\ (\rho r\Omega_o^2\ \underline{N}\cdot\nabla r)^2\}] \tag{27}$$

where X is the component of ξ in the direction $\underline{N}$, which is parallel to $\underline{B} \times \underline{k}$ and normalised such that $N \cdot \nabla\Psi = 1$, and dot indicates derivation with respect to l, the coordinate along the field-line. It is useful to consider the tokamak ordering used in Section 2, in which the curvature $|\kappa| = O(\delta)$. We consider a toroidal rotation speed of the order of the sound speed $c_s = (\gamma p/\rho)^{1/2}$, i.e. $r\Omega_0/c_s \sim 1$, which implies that $\Omega_0 = O(\delta^{3/2})$. Then, to $O(\delta^2)$, equation (27) reduces to

$$\delta W = \int_{-\infty}^{+\infty} \frac{dl}{B}\ [\dot{X}^2|\underline{N}|^2 - X^2\{2(\underline{N}\cdot\nabla p)(\underline{N}\cdot\underline{\kappa}) - \rho\Omega_0{}^2(\underline{N}\cdot\nabla r)(\underline{N}\cdot\nabla\rho)],\tag{28}$$

which agrees with the corresponding expression in Ref. 37. However, there is an error of interpretation in Ref. 37, where the effect of the relative centrifugal separation of the pressure and the density profiles from each other and from flux surfaces, described in Figure 1, is overlooked. If we consider a flux surface in the intermediate region, lying between the maxima of the pressure and density profiles, the last term in equation (28) is stabilising, and acts against the second term which is always destabilising in the bad-curvature region. Numerical results[40,50] seem to indicate, however, that for rigid toroidal rotation, the stabilising effect of the third term is not strong enough to overcome the destabilising pressure term. As we move towards the outer edge of the plasma, out of the intermediate region between pressure and density maxima, both the second and the third terms are destabilising. We remind the reader that the present theory is limited to the case of rigid toroidal rotation, which is not expected to be applicable near the edge of the plasma.

We now turn to the case when the equilibrium flow in the laboratory frame is purely field-aligned, i.e., $\underline{v} = [\Phi(\Psi)/\rho]\underline{B}$.[41] (The qualitative picture is not altered if an additional, rigid toroidal rotation is included in the calculation). In this case, we obtain

$$\delta W = \int_{-\infty}^{+\infty} \frac{dl}{B} \Big[\dot{X}^2 |\underline{N}|^2 (1 - M^2) - X^2 \Big\{ 2(\underline{N}\cdot\underline{\kappa})\big((1 - M^2)(\underline{N}\cdot\nabla p) + v_A{}^2\Phi(\underline{N}\cdot\nabla\Phi) + (M^2 v_A{}^2/2)(\underline{N}\cdot\nabla\rho)\big) + \frac{\rho M^2 v_A{}^2 (1+\beta)(M^2-1)(M^2-2\beta/(1+\beta))}{\beta(1-M^2/\beta)} \Big\} (\underline{N}\cdot\underline{\kappa})^2\Big] \quad (29)$$

We note that the last term on the right-hand side of equation (29) contains the expression $(1 - M^2/\beta)$ in the denominator which multiplies a negative definite term. In order to make the physical meaning of the various terms in equation (29) more transparent, we again use tokamak ordering. Since the first elliptic region for equilibria lies in the domain $\Phi^2/\rho < \beta$, we take $M = O(\delta^{1/2})$. We now order the various terms in equation (29): the first term is approximately $|\underline{N}|^2\dot{X}^2$, the second term gives $-2X^2 (\underline{N}\cdot\underline{\kappa}) (\underline{N}\cdot\nabla p)$ to $O(\delta^2)$, and the third and fourth terms are each $O(\delta^2)$. In the last term, the expression multiplying the term $(1-M^2/\beta)$, is $O(\delta^3)$. It is clear that as the quantity M^2 approaches the value β from below and is sufficiently close to it, the last term dominates the other terms in δW, and the sufficient condition for stability is violated before the occurence of the transition from the first elliptic region to the first hyperbolic region. We remark that equilibrium quantities, such as $\nabla\rho$, remain well-defined in the first elliptic region, arbitrarily close to the transition $M^2 = \beta$. In other words, the solutions are "strong solutions" which have no discontinuities in Ψ and its derivatives.[18] As we approach the transition $M^2 = \beta$ from below, it is clear that the second and fourth terms in δW continue to be potentially destabilising. (The third term which is equal to $V_A{}^2\Phi\Phi'$ has no dependence on $\nabla\Psi$ and remains well-behaved).

We have stated earlier that $\delta W > 0$ is a sufficient condition for stability. By imposing an ordering on the eigenfrequency ω, it can be shown[37,41] that $\delta W > 0$ is also a necessary condition. Hence, the violation of this stability condition may be considered as evidence of instability.

It is interesting to note that the threshold $M^2 = \beta$ also appears in the generalised Suydam criterion. However, for the straight cylindrical equilibria described in Section 3, there is no equilibrium limit on field-aligned flows, unlike the case of a torus.

It is worthwhile to consider the fluid-dynamical analog of the instabilities investigated in this section. In the presence of purely toroidal, rigid rotation, the instability excited is analogous to the Rayligh-Taylor instability. In this sense, the instability driven by rigid toroidal rotation is similar in character to the standard ballooning mode for static equilibria, with the flow-dependent term destabilising in the bad-curvature region and stabilising in the good-curvature region. On the other hand, the instability due to field-aligned flows, in as much as it is driven by flow shear, is analogous to the Kelvin-Helmholtz instability. Even though the mode is localised to a field-line and hence can be described by the conventional ballooning formalism, it has little in common with standard ballooning modes. Since the destabilizing term in δW is proportional to $|\underline{N}\cdot\underline{\kappa}|^2$, it will occur on both the good and bad-curvature sides of a torus.

6. CONCLUSIONS

The problem of stability of axisymmetric, rotating plasmas has several surprises. Here we discuss some of the many unsolved problems which remain.

In Section 5, we pointed out that the conventional ballooning mode formalism breaks down in the presence of shear in that part of the toroidal flow which is not field-aligned. It seems to us that this is not so much a question of whether modes which are ballooning-like (i.e., have long wavelengths parallel and short wavelengths perpendicular to a field-line) exist as it is to find the appropriate mathematical representation for them. We can, of course, place a bound on how large toroidal flow shear can possibly be before the "quasi-mode" structure of finite-n ballooning modes - to use the appealing notion of Roberts and Taylor[51,52] - is destroyed. Ballooning modes are constructed from the linear superposition of localised Fourier modes on neighbouring rational surfaces. In order for an instability to evolve as a quasi-mode, it is essential that the difference in the Doppler-Shifted frequency between two adjacent rational surfaces in a plasma with flow be much smaller than the growth rate of the quasi-mode. This immediately gives the condition $\gamma \gg |\Omega'(\Psi)/q'(\Psi)|$, which must hold in addition to the condition $n[\Psi\ q'(\Psi)]^2 \gg 1$, required by the standard ballooning theory for static equilibria.[52] (We remark that these inequalities should be regarded as necessary, but not sufficient, conditions for the validity of standard ballooning theory.)

If a suitable mathematical representation can be found to describe ballooning modes in the presence of sheared toroidal flow, we expect that there should be a threshold (akin to $M^2 = \beta$ for field-aligned flow) below which instability should exist independent of the pressure gradient. We are guided to this conclusion due to the appearance of exactly the same threshold in the cylindrical plasma, in which the instability occurs independent of whether the shear is in the axial or the field-aligned component of the flow. Since there is no MHD equilibrium limit for purely toroidal flow, it is tempting to suggest that the toroidal plasma should be stable to ballooning modes just above the threshold, except for the sobering fact that sheared flows in a cylinder have been found to destabilize the edge of continua, and may be expected to do the same in a torus.

An important problem, to be addressed in future work, is the effect of flow on kink modes. This will undoubtedly require a major analytical and computational effort. We also remark that discrete-particle effects, which have been found to eliminate the narrow hyperbolic region separating the first and second elliptic regions of MHD equilibria, may have important consequences for stability.

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