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Abstract

A general formulation of the local power absorption density is obtained from the Vlasov equation. An explicit expression is derived for a specific case in a two-dimensional geometry.

Various electromagnetic waves may be used for plasma heating in fusion devices. In the theoretical modelling of the heating schemes an important problem, common to all types of waves, is to determine the amount of wave energy deposited into a species in different spatial regions of the plasma. In other words, one would like to obtain the power absorption density per species as a function of suitable spatial coordinates.

Consider an electromagnetic field oscillating at a frequency ω. If the spatial structure of the field may be described within the WKB approximation, i.e., represented by a travelling weakly-damped wave, then the time averaged local power absorption density can be calculated from the well-known formula (Stix, 1962; Bernstein, 1975)

$$Q = \frac{\omega}{8\pi} \vec{E}^{*} \cdot \overleftrightarrow{\varepsilon}^{a} \cdot \vec{E} \,. \qquad (1)$$

Here $\vec{E}$ is the electric field component of the electromagnetic field ($\vec{E}^{*}$ denoting the complex conjugate of $\vec{E}$) and $\overleftrightarrow{\varepsilon}^{a}$ is the anti-Hermitian part of the local dielectric tensor for the species in question.

In many circumstances, however, the field structure is much more complicated: for example, the propagating waves may be reflected in some regions and consequently, standing waves are partially formed; or in other regions (around particle resonances) the damping may become too strong so that the imaginary part of the wavenumber is comparable with its real part; or the global eigenmodes of the whole system may

be excited. In all these cases expression (1) is clearly not applicable and therefore a more general one is needed.

Recently, a formulation of local power absorption has been proposed by McVey et al. (1985) using an heuristic argument. The purpose of this note is to show that the formulation can be obtained using a more rigorous approach and to provide an explicit expression for the local power absorption in a two-dimensional geometry.

Consider a collisionless plasma immersed in a magnetostatic field $\vec{B}_o$. In the presence of an electromagnetic field $\vec{E}$, $\vec{B}$, the distribution function f of a species with charge q and mass m obeys the Vlasov equation

$$\frac{\partial f}{\partial t} + \vec{v}.\nabla f + \frac{q}{m}\left[\vec{E} + \frac{1}{c}\vec{v}\times\left(\vec{B}_o + \vec{B}\right)\right].\frac{\partial f}{\partial \vec{v}} = 0 \,. \qquad (2)$$

The mean energy density of the particles that are in a volume element $\langle\vec{x},\vec{x}+d\vec{x}\rangle$ at a time t is given by the quantity

$$\int \frac{m v^2}{2} f \, d\vec{v} \,. \qquad (3)$$

In order to ascertain how this quantity varies with time we shall construct an energy balance equation corresponding to (2). For this purpose we multiply (2) by $mv^2/2$ and integrate over the velocities. After simple manipulations this yields

$$\int \frac{mv^2}{2} \frac{\partial f}{\partial t} d\vec{v} = q\vec{E} \cdot \int \vec{v} f \, d\vec{v} - \int \frac{mv^2}{2} \vec{v} \cdot \nabla f \, d\vec{v} . \qquad (4)$$

From this equation one can see that the mean energy of the particles in the volume element considered varies with time owing to two effects: the work done by the electric field on these particles (the first term on the right-hand side), and the flux of energy of those particles that stream into or out of the volume element. Thus, if we want to relate a time derivative of quantity (3) to the local power absorption we must evaluate it in a frame of reference where the particle streaming is absent. This can be achieved if we transform equation (4) into suitable Lagrangian coordinates.

Let $\vec{x}'$ and $\vec{v}'$ represent the position and velocity of a particle at the time t' as it moves along an unperturbed trajectory (in the absence of the electromagnetic field) with the "initial" conditions $\vec{x}$ and $\vec{v}$ at the time t. Choosing $\vec{x}'$ and $\vec{v}'$ as the new variables we transform equation (4) into

$$\int \frac{mv^2}{2} \left(\frac{\partial f}{\partial t} \right)_{\vec{x}', \vec{v}'} d\vec{v} = q \int \vec{E}(\vec{x}', t) \cdot \vec{v}' f(\vec{x}', \vec{v}', t) \, d\vec{v}', \qquad (5)$$

where the relations $v^2 = v'^2$ and $d\vec{v} = d\vec{v}'$ have been used.

In order to show explicitly that the particle streaming is absent

in the new frame of reference we shall construct an equation of continuity corresponding to (2). Integrating equation (2) over the velocities yields

$$\int \left(\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f \right) d\vec{v} = 0 . \tag{6}$$

On transforming (6) into the new variables we finally obtain

$$\int \left(\frac{\partial f}{\partial t} \right)_{\vec{x}', \vec{v}'} d\vec{v} = 0 . \tag{7}$$

Noting that $d\vec{x} = d\vec{x}'$, this equation shows that the number of particles contained in the volume element is conserved. Consequently, equation (5) may be used to calculate the local power absorption.

We now assume the amplitude of the elctromagnetic field to be small. We may then expand the distribution function in powers of $\vec{E}$:

$$f = f_0 + f_1 + f_2 + \dots \quad . \tag{8}$$

Here f_0 describes an equilibrium and f_1 is a solution of the corresponding linear problem. Since a harmonic time-dependence of the field is considered, the lowest order non-vanishing contribution to the time average of equation (5) is given by

$$\tilde{Q} \equiv \int \frac{mv^2}{2} \left(\frac{\partial f_2}{\partial t} \right)_{\vec{x}_1, \vec{v}_1} d\vec{w} = q \int \vec{E}(\vec{x}_1, t) \cdot \vec{v}_1 f_1(\vec{x}_1, \vec{v}_1, t) \, d\vec{w}. \tag{9}$$

To make a practical use of equation (9) we must perform a time average. Before doing so, however, let us note that the lowest frequency involved in the quantity $\tilde{Q}$ is $|\omega - \omega_c|$, where ω_c is the cyclotron frequency of the species. For a class of resonant particles we have $|\omega - \omega_c| \sim v_\parallel / \lambda_\parallel$, where $v_\parallel$ is a typical particle velocity component parallel to the magnetostatic field and $\lambda_\parallel$ is a characteristic length of the variation of the electromagnetic field in the same direction. Thus, in general, if we perform a time average over the scale $|\omega - \omega_c|^{-1}$, for consistency we have to perform also a space average over the scale $\lambda_\parallel$, since a resonant particle of the above-mentioned class traverses this distance during the time $|\omega - \omega_c|^{-1}$. Therefore, without loss of generality we can set

$$\vec{E}(\vec{x}, t) = Re \left\{ \vec{E}(\vec{x}_\perp) \exp \left[i(k_\parallel x_\parallel - \omega t) \right] \right\} \tag{10}$$

and the same for f_1. Substituting these into (9) and averaging over $x_\parallel$ and t we finally obtain a general expression for the local power absorption density as

$$Q(\vec{x}_\perp) = \langle \tilde{Q} \rangle_{x_\parallel, t} = \frac{q}{2} \int d\vec{w} \, Re \left\langle \vec{E}^*(\vec{x}_\perp) \cdot \vec{v} f_1(\vec{x}_\perp, \vec{v}) \right\rangle_t . \tag{11}$$

Let us now derive an explicit expression for this quantity by considering a specific case. We shall assume that the Larmor radius of the species is much smaller than the characteristic inhomogeneity lengths of macroscopic quantities: density, temperature and magnetostatic field. To the lowest order, i.e., neglecting the explicit gradients of these quantities, the equilibrium distribution function f_0 may then be approximated by a local Maxwellian, f_M, with a temperature T and density n. Moreover, to the same order, the particle trajectories may be evaluated assuming a locally uniform magnetostatic field. Thus, choosing the Cartesian coordinate systems with the z-axis along $\vec{B}_0$ we can write

$$\vec{x}_\perp' = \vec{x}_\perp + \frac{v_\perp}{\omega_c}\left[(\sin\alpha - \sin\alpha')\vec{e}_x + (\cos\alpha' - \cos\alpha)\vec{e}_y\right], \tag{12}$$

$$\vec{v}' = v_\perp(\cos\alpha'\vec{e}_x + \sin\alpha'\vec{e}_y) + v_z\vec{e}_z, \tag{13}$$

$$\alpha' = \alpha + \omega_c(t-t'), \quad v_\perp = (v_x^2 + v_y^2)^{1/2}, \quad \alpha = \operatorname{tg}\frac{v_y}{v_x}. \tag{14}$$

To proceed further, we introduce a Fourier transform

$$\left\{\vec{E}(\vec{x}_\perp), f_1(\vec{x}_\perp, \vec{v})\right\} = \int d\vec{k}_\perp e^{i\vec{k}_\perp\cdot\vec{x}_\perp}\left\{\vec{E}(\vec{k}_\perp), f_1(\vec{k}_\perp, \vec{v})\right\}. \tag{15}$$

The solution of the linear problem is then easily obtained in the form

$$f_1(\vec{k}_\perp, \vec{v}) = \exp\left[i\xi \sin(\alpha-\psi)\right] \sum_\ell e^{i\ell(\psi-\alpha)} A_\ell(\vec{k}_\perp, v_\perp, v_z) \,, \tag{16}$$

$$A_\ell(\vec{k}_\perp, v_\perp, v_z) = \frac{iq}{T} \frac{f_M}{\omega - \ell\omega_c - k_z v_z} \left\{ v_\perp \left[E_x(\vec{k}_\perp) \right.\right. \tag{17}$$

$$\times \left(\cos\psi \frac{\ell}{\xi} J_\ell(\xi) - i \sin\psi \, J'_\ell(\xi) \right) + E_y(\vec{k}_\perp) \left(\sin\psi \frac{\ell}{\xi} J_\ell(\xi) \right.$$

$$\left.\left. + i \cos\psi \, J'_\ell(\xi) \right) \right] + v_z E_z(\vec{k}_\perp) J_\ell(\xi) \Big\} \,,$$

where $\xi = |\vec{k}_\perp| v_\perp / \omega_c$, J_ℓ and J'_ℓ are the Bessel function and its derivative, and $\mathrm{tg}\psi = k_y/k_x$. In order to satisfy causality the frequency ω is assumed to have a small, positive, imaginary part.

Using the Fourier transform (15) we now rewrite (11) as

$$Q = \frac{q}{2} \mathrm{Re} \int d\vec{\omega} \, d\vec{k}_\perp \, d\vec{k}'_\perp \, \vec{E}^*(\vec{k}'_\perp) \cdot \left\langle \vec{v}' f_1(\vec{k}_\perp, \vec{v}') e^{i(\vec{k}_\perp - \vec{k}'_\perp)\cdot \vec{x}'_\perp} \right\rangle_t \,. \tag{18}$$

Further, substituting for $\vec{x}'$, $\vec{v}'$ and f_1 expressions (12), (13) and (16), and applying the identity (Gradshteyn and Ryzhik, 1965)

$$\exp(ia\sin b) = \sum_{\ell} J_{\ell}(a)\, e^{i\ell b} \tag{19}$$

we transform equation (18) into

$$Q = \frac{q}{2} \mathrm{Re} \int d\vec{w}\, d\vec{k}_{\perp}\, d\vec{k}'_{\perp}\, e^{i(\vec{k}_{\perp} - \vec{k}'_{\perp})\cdot\vec{x}_{\perp}} \exp\left\{i\left[\xi \sin(\alpha - \psi) - \xi' \sin(\alpha - \psi')\right]\right\}$$

$$\times \sum_{j,\ell,m,p} J_j(\xi) J_m(\xi') J_p(\xi) \exp\left\{i\left[\psi(p+\ell-j) - m\psi'\right]\right\} A_{\ell}(\vec{k}_{\perp}, w_{\perp}, w_z) \tag{20}$$

$$\times \left\langle e^{i(j+m-\ell-p)\alpha'} \left\{ w_{\perp}\left[E_x^*(\vec{k}'_{\perp})\cos\alpha' + E_y^*(\vec{k}'_{\perp})\sin\alpha'\right] + w_z E_z^*(\vec{k}'_{\perp}) \right\} \right\rangle_t ,$$

where $\xi = |\vec{k}'_{\perp}|\, v_{\perp}/\omega_c$ and $\mathrm{tg}\psi' = k'_y/k'_x$.

Upon carrying out the time average and using the summation theorem (Gradshteyn and Ryzhik, 1965)

$$\sum_{\ell} J_{\ell}(a)\, J_{\ell+\ell'}(a) = \delta_{0,\ell'} , \tag{21}$$

where δ is the Kronecker delta, equation (20) reduces to

$$Q = \frac{q}{2} \mathrm{Re} \int d\vec{v}\, d\vec{k}_\perp\, d\vec{k}'_\perp\, e^{i(\vec{k}_\perp - \vec{k}'_\perp)\cdot\vec{x}_\perp} \exp\left\{ i\left[\xi \sin(\alpha-\psi) - \xi' \sin(\alpha-\psi') \right] \right\}$$

$$\times \sum_\ell e^{i\ell(\psi-\psi')} A_\ell(\vec{k}_\perp, v_\perp, v_z) \left\{ \frac{v_\perp}{2} \left[E_x^*(\vec{k}'_\perp) \left(J_{\ell+1}(\xi') e^{-i\psi'} \right.\right.\right.$$

$$\left.\left.\left. + J_{\ell-1}(\xi') e^{i\psi'} \right) + i E_y^*(\vec{k}'_\perp) \left(J_{\ell+1}(\xi') e^{-i\psi'} - J_{\ell-1}(\xi') e^{i\psi'} \right) \right] \right.$$

$$\left. + v_z E_z^*(\vec{k}'_\perp) J_\ell(\xi') \right\} . \qquad (22)$$

Next, applying again identity (19), we perform the integration over α and transform the expression in the last braces using the recursion relations for the Bessel functions to obtain

$$Q = \frac{q}{2} \mathrm{Re} \int d\vec{v}\, d\vec{k}_\perp\, d\vec{k}'_\perp\, e^{i(\vec{k}_\perp - \vec{k}'_\perp)\cdot\vec{x}_\perp} \sum_p J_p(\xi) J_p(\xi') e^{ip(\psi'-\psi)}$$

$$\times \sum_\ell e^{i\ell(\psi-\psi')} A_\ell(\vec{k}_\perp, v_\perp, v_z) \left\{ v_\perp \left[E_x^*(\vec{k}'_\perp) \left(\cos\psi' \frac{\ell}{\xi'} J_\ell(\xi') \right.\right.\right.$$

$$\left.\left.\left. + i \sin\psi' J'_\ell(\xi') \right) + E_y^*(\vec{k}'_\perp) \left(\sin\psi' \frac{\ell}{\xi'} J_\ell(\xi') - i \cos\psi' J'_\ell(\xi') \right) \right] \right.$$

$$\left. + E_z^*(\vec{k}'_\perp) v_z J_\ell(\xi') \right\} .$$

(23)

Finally, substituting for A_ℓ expression (17) and making some re-arrangements we can cast equation (23) into a compact form

$$Q(\vec{x}_\perp) = \frac{\pi q^2}{2T} \int d\vec{v} f_M \sum_{l,p} \delta(\omega - \omega_c l - k_z v_z)$$

$$\times \left| \int d\vec{k}_\perp e^{i\vec{k}_\perp \cdot \vec{x}_\perp} e^{i\psi(l-p)} J_p(\xi) \left\{ v_\perp \left[E_x(\vec{k}_\perp) \left(\cos\psi \frac{l}{\xi} J_l(\xi) \right. \right. \right. \right. \tag{24}$$

$$\left. - i \sin\psi J_l'(\xi) \right) + E_y(\vec{k}_\perp) \left(\sin\psi \frac{l}{\xi} J_l(\xi) + i \cos\psi J_l'(\xi) \right) \Bigg]$$

$$\left. + v_z E_z(\vec{k}_\perp) J_l(\xi) \right\} \Bigg|^2 .$$

Equation (24) is one of the main results of this note. It is valid for an arbitrary field structure and absorption strength. As one can see from the equation the local power absorption density, for a species close to a local thermodynamical equilibrium, is a positive-definite quantity. A result which one should expect. Moreover, we note that for a weakly-damped travelling wave with a wave vector $\vec{k}_{\perp o}$, i.e., for $\vec{E}(\vec{k}_\perp) \sim \delta(\vec{k}_\perp - \vec{k}_{\perp o})$, (24) reduces to expression (1).

In many situations of a practical interest the Larmor radius of the species, ρ, appears to be much smaller than a characteristic length, $\lambda_\perp$, of the variation of the electromagnetic field. In such cases we can expand the Bessel functions in (24), to any desired order, to obtain more explicit expressions. In what follows we confine ourselves to accuracy up to $(\rho/\lambda_\perp)^2$.

Let us define

$$Q \equiv \frac{\omega_p^2}{8\pi} \sum_l Q_l \,, \tag{25}$$

where ω_p is the plasma frequency of the species. Upon expanding the Bessel functions to the required order we invert the resulting expressions from the Fourier space to real space and perform the remaining velocity integrations. This finally yields

$$Q_0 = Y_0 \left[\rho^2 \left(\left| rot_z \vec{E} \right|^2 + \left| rot_z \vec{E} - \frac{2\omega\omega_c}{k_z v_t^2} E_z \right|^2 \right.\right.$$
$$\left.\left. + \left(\frac{\omega}{k_z \omega_c}\right)^2 \left(\left| \frac{\partial E_z}{\partial x} \right|^2 + \left| \frac{\partial E_z}{\partial y} \right|^2 + 2 Re\left(E_z^* \Delta_\perp E_z \right) \right) \right], \tag{26}$$

$$Q_1 = Y_1 \left[\frac{1}{2} \left| E_+ + \frac{\Omega_1}{k_z \omega_c} \left(\frac{\partial E_z}{\partial y} - i \frac{\partial E_z}{\partial x} \right) \right|^2 \right.$$
$$+ \rho^2 \left\{ \frac{1}{2} Re \left(\vec{E}_\perp^* \cdot \left[5 \Delta_\perp \vec{E}_\perp - 2 \nabla_\perp div \vec{E}_\perp \right] \right.\right.$$
$$\left.\left. + i \left[\vec{E}_\perp^* \times \left(5 \Delta_\perp \vec{E}_\perp - 2 \nabla_\perp div \vec{E}_\perp \right) \right]_z \right) \right.$$
$$\left. + \left| \frac{\partial E_+}{\partial x} \right|^2 + \left| \frac{\partial E_+}{\partial y} \right|^2 \right\} \Bigg], \tag{27}$$

$$Q_2 = \gamma_2 \frac{\rho^2}{2} \left| \frac{\partial E_+}{\partial x} + i \frac{\partial E_+}{\partial y} \right|^2 , \tag{28}$$

where

$$E_+ = E_x + i E_y \ , \quad \gamma_\ell = \frac{\pi^{1/2}}{|k_z| v_t} \exp\left[- \left(\frac{\Omega_\ell}{k_z v_t} \right)^2 \right] ,$$

$$\Omega_\ell = \omega - \ell \omega_c \ , \quad v_t^2 = \frac{2T}{m} \ , \quad \rho^2 = \frac{T}{m \omega_c^2} \ . \tag{29}$$

The subscript $\perp$ refers to the two-dimensional space x,y. The corresponding $Q_{-\ell}$ are obtained from Q_ℓ by the replacements $\ell \to -\ell$ and $i \to -i$. If we set $\partial/\partial_y \equiv 0$ in expressions (26) - (28), they reduce to those, in the limit of vanishing explicit gradients, obtained by McVey et al. (1985).

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