January 1986 LRP 285/86

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Submitted for publication as a Brief Report to Physical Review A

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Abstract

A general method recently developed to establish the conditions for the existence of similarity solutions of the Fokker-Planck-Smoluchowsky equation is applied to the 1-dimensional advection-diffusion equation in the form

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}\left(R(x)P + D(x)\frac{\partial P}{\partial x}\right).$$

The case of power-law behaviour of the coefficient functions R(x), D(x) is investigated in detail and few classes of similarity solutions are presented. In addition, a class of functions R(x), D(x) is identified such that the general method, based on the invariance under continuous group of transformations, is equivalent to a generalized scale transformation of the space and time variables.

The starting point of the present investigation is the work exposed in Ref. [1], in which the authors, starting from the general theory of continuous group of transformations [2], establish the condition that the diffusion coefficient D(x) has to fulfill in order the equation

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x} D(x)\frac{\partial P}{\partial x}$$

to admit similarity solutions.

They suggest that such a condition should be checked case by case providing a quick answer whether or not the specific problem one has to solve admits a solution in similarity form.

In the present investigation we just adopt such a philosophy and extend it to the more general case of a 1-dimensional advection-diffusion equation:

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x} \left(R(x)P + D(x)\frac{\partial P}{\partial x}\right) \qquad (1)$$

which has applications in many fields of physics.

This paper is organized as follows: in Section 1 we derive the general equation that R(x) and D(x) have to fulfill in order the eq. (1) to admit similarity solutions. In Section 2 an application to the case $R(x)=x^p+\lambda x^s$, $D(x)=x^q$ is made and the corresponding solutions briefly discussed. Finally, in Section 3 we give a quantitative insight into the effective capabilities of the similarity method to yield solutions which could not have been obtained with less sophisticated methods, like generalized scale transformation of the original space and time variables.

1. GENERAL THEORY

Following Ref. [1], we look for a couple of functions $R(x)$, $D(x)$ such that the eq. (1), regarded as a form

$$\omega(P,P_t,P_x,P_{xx}) \equiv P_t - aP - bP_x - cP_{xx} = 0 \tag{2}$$

is invariant under the continuous group of transformations

$$x^* = x + \xi(x,t)\varepsilon \tag{3}$$

$$t^* = t + \tau(t)\varepsilon \tag{4}$$

$$P^* = P + \eta(x,t,P)\varepsilon \tag{5}$$

ε being the parameter and ξ,τ,η the infinitesimals of the group.

Once the coefficients $a(x)$,$b(x)$,$c(x)$ are appropriately identified in terms of $R(x)$ and $D(x)$, the invariance condition $\omega=\omega^*$, leads to the following set of differential equations:

$$Df_{xx} + (R+D')f_x + \xi R'' + R'\dot{\tau} - \dot{f} = 0 \tag{6}$$

$$D(2f_x-\xi_{xx}) + (R'+D'')\xi + (R+D')(\dot{\tau}-\xi_x) + \dot{\xi} = 0 \tag{7}$$

$$D'\xi + D(\dot{\tau}-2\xi_x) = 0 \tag{8}$$

where prime denotes the d/dx and dot the d/dt operators respectively and we have chosen $\eta(x,t,P) = f(x,t)P$.

From the solution of eqs (6-8), one can obtain the three unkown functions ξ,τ,f and thus deduce the form of the similarity solution by integrating the characteristic equations

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{dP}{fP} \tag{9}$$

The system of equations (6-8) is better handled by introducing a new variable, z, defined by $z(x) = \int dx'/\sqrt{D(x')}$. After some algebra one obtains:

$$\xi = \frac{\dot{\tau}}{2} z D^{1/2}(z) \tag{10}$$

$$\dddot{\tau} z^2 - 2\ddot{\tau} - b(z)\dot{\tau} - 8\dot{f}_0 = 0 \tag{11}$$

$$f = -\frac{\ddot{\tau}}{8} z^2 - \frac{\dot{\tau}}{8} H(z) + f_0(t) \tag{12}$$

where:

$$H(z) = z\left(\frac{D_z}{D} + \frac{2R}{\sqrt{D}}\right) \tag{13}$$

$$b(z) = H_{zz} + \frac{1}{2z} HH_z - 4z\left(\frac{R_z}{\sqrt{D}}\right)_z - 8\left(\frac{R_z}{\sqrt{D}}\right) \tag{14}$$

and $f_0(t)$ is an arbitrary function.

Any choice of R(x) and D(x) which make it possible to satisfy eq. (11), will lead to a class of similarity solutions.

Eqs (10), (14) represent the direct generalization of those given in Ref. [1] to which they are easily seen to reduce by letting $R \to 0$.

2. APPLICATION TO POWER-LAW DRAG AND DIFFUSION

As an example, we apply the general procedure to the case of power-law drag and diffusion

$$R(x) = x^p + \lambda x^s \tag{15}$$

$$D(x) = x^q \tag{16}$$

where p,q,s,λ are real and λx^s is meant to model the coupling of the system with the exterior. This choice is computationally quite comfortable and at the same time representative from the physical point of view.

As few examples we can cite the case $s=p=1$, $q=0$ which corresponds to the familiar Brownian motion with friction (Ornstein-Uhlenbeck process) [3] or the case $p=-2$, $s=0$, $q=-3$ which can model the behaviour of fast electrons in a hot plasma in presence of a d.c. electric field [4]. In a different context, with the choice $s=p={}^1/_3$, $q={}^4/_3$ one can describe the turbulent two-particle diffusion in configuration space [5].

After some lengthy but staightforward algebra, three classes of similarity solutions can be identified:

I) $\lambda = 0$; $p = 1$, $2-q > 0$

One has:
$$P(x,t) = \text{const.}\ \tau^{-1/2-q} \exp\left(-\frac{x^{2-q}}{(2-q)\tau}\right) \qquad (17)$$

with
$$\tau(t) = 1 - \exp\left[-(2-q)t\right]. \qquad (18)$$

This class is immediately recognized to be a sort of generalized brownian motion with friction ($q=0$ corresponds to the classical case). As the parameter $2-q$ becomes more and more positive the particle population is correspondingly flatter around the origin where diffusion overwhelms the drag effects (see Fig. 1) and depressed for large values of x, where just the opposite tendence occurs. The separation value, x_s, between these two regions is approximately given by $x_s = (2-q)^{1/2-q}\ \tau(t)^{1/2-q}$.

For $2-q < 0$ the diffusion is completely dominated by the drag effects for $x < x_s$, and $P(x,t)$ gets squeezed around the origin where it develops a singularity which corresponds to the absence of a steady state.

II) $p = s = 1$; $(\lambda+1)(2-q) > 0$

We have: $$P(x,t) = \text{const.}\ \tau^{-1/2-q} \exp\left[-\left(\frac{x^{2-q}}{\tau} / 2-q\right)(\lambda+1)\right] \tag{19}$$

$$\tau(t) = 1 - \exp\left(-(\lambda+1)(2-q)t\right) \tag{20}$$

This class is essentially the same as the previous one. The only reason why we keep it distinguished is the occurence of physically acceptable solutions also for $q > 2$.

These solutions correspond to $\lambda+1 < 0$, that is the drag term "pushing" away from the origin, so that the squeezing effect just discussed is no longer effective. On the contrary, the particles tend to accumulate to infinity, $\lim_{x\to\infty} P(x,t) = \text{const.}$, so that these solutions are acceptable only in a finite space.

III) $p = 1$; $s = q-1$; $2-q > 0$

We have: $$P(x,t) = \text{const.}\ \tau^{-1/2-q}(x/\tau^{1/2-q})^{-\lambda} \exp\left(-\frac{x^{2-q}}{(2-q)\tau}\right) \tag{21}$$

$$\tau(t) = 1 - \exp\left(-(2-q)t\right) \tag{22}$$

The new feature is the presence of the factor $(x/\tau^{1/2-q})^{-\lambda}$ which stems from the competition between the "internal" drag, $R_{int} \equiv x^p$, and the external one, $R_{ext} \equiv \lambda x^{q-1}$.

If $\lambda > 0$, we have once more a singular behaviour around $x=0$, which results from the squeezing effect already discussed. For $\lambda < 0$, $P(x,t)$ exhibits a humped shape (Fig. 2) since R_{ext} prevails for small x and is overbalanced by R_{int} for large x (remember that $q-1 < 1$).

The separation value, x_{crit}, is determined by the condition $R_{int} = R_{ext}$, that is $x_{crit} = |\lambda|^{1/2-q}$. Since it is the dragging action which prevails for large x, no runaway phenomenon [6] occurs.

3. DISCUSSION

The three classes of similarity solutions we have just discussed have been presented with the main purpose of illustrating an application of the general procedure outlined in Section 1. Now, independently from their physical relevance, one can notice that they exhibit an important common feature on the mathematical ground. In fact, all of them can be written in the form

$$P(x,t) = \tau(t)^a N(x\tau^b) \qquad a,b \text{ reals} \tag{23}$$

which means that they can be obtained by a generalized scale transformation:

$$t \to \tau(t) \tag{24}$$

$$x \to n = x\tau^b \tag{25}$$

$$P = (\tau)^a N(n) \tag{26}$$

A question arises quite naturally up to which extent the general method is really needed to give more than what one could achieve with a bit of skill and "feeling" in selecting the right ansatz for its particular problem.

To answer this question, at least partially, let us come back to the general equations (9)-(14). Integrating the first of eq. (9) we obtain

$$z = \tau^{1/2} \text{ const.} \tag{27}$$

which yields the form of the natural variable: $n = z\tau^{-1/2}$. To obtain the form of the solution with respect to the ignorable variable τ, we integrate the second of eq. (9) along a characteristic line (n=const.):

$$P = N(n) \exp \int_{n=\text{const.}} (f/\dot{\tau}) \frac{d\tau}{\tau} \tag{28}$$

where N is an arbitrary function.

The eq. (28) tells us that the most general similarity solution of eq. (1) takes the form:

$$P(x,t) = N(n)\ T(\tau,n) \tag{29}$$

where $$T(\tau,n) \equiv \exp \int_{n=\text{const.}} (f/\dot{\tau}) \frac{d\tau}{\tau}$$

which corresponds to a generalized separation of the natural and ignorable variables n and τ. The trivial solutions, eq. (23), are then just a special case of eq. (29) which occurs whenever f is proportional to $\dot{\tau}$ through a constant. To see under which circumstances this can happen, let us go back to eq. (11).

By inspecting this equation, we realize that in order to fulfill it without letting $\tau=0$ (which would correspond to the usual method of separation of variables, [1]), we must require:

$$b(z) \equiv H_{zz} + \frac{1}{2z} HH_z - 4z(R_z/\sqrt{D})_z - 8(R_z/\sqrt{D}) = b_0 + b_2 z^2 \qquad (30)$$

with b_0, $b_2 > 0$ constants. Eq. (11) then splits into

$$\dddot{\tau} - b_2\dot{\tau} = 0 \qquad (31)$$

$$-2\ddot{\tau} - b_0\dot{\tau} - 8\dot{f}_0 = 0 \qquad (32)$$

From eq. (31), we get, after rejecting the growing exponential

$$\tau(t) = (\tau_0-\tau_\infty)\exp(-\sqrt{b_2}t) + \tau_\infty \; ; \; \tau_0, \tau_\infty \text{ arb. constants} \qquad (33)$$

and from eq. (32)

$$f_0(t) = -\frac{1}{4}\dot{\tau} - \frac{b_0}{8}\tau + f_\infty \; ; \; f_\infty \text{ arb. constant} \qquad (34)$$

We are now in condition to calculate f/τ, which reads

$$f/\dot{\tau} = [(-\frac{1}{4} + \frac{b_0}{8\sqrt{b_2}}) - \frac{1}{8}H(z) + \frac{\sqrt{b_2}}{8}z^2] \qquad (35)$$

where we have chosen f_∞ such that $f_\infty + b_0\tau_0/8 = 0$.

The condition for obtaining a "trivial" solution, $f/\dot{\tau}$ = const., therefore reads

$$H(z) = \sqrt{b_2}\, z^2 + h_0 \quad , h_0 \text{ arb. constant} \qquad (36)$$

Inserting this expression in the eq. (30), we have

$$G(z) \equiv 4z(R_z/\sqrt{D}) + 8\, R_z/\sqrt{D} = (2+h_0)\sqrt{b_2}-b_0 \equiv \text{const.} \qquad (37)$$

These two equations have to be fulfilled independently in order for a similarity solution to exist and being of trivial type.

The question concerning the real power of the general method can now be reformulated by asking which ones of the solutions of eq. (29) do also satisfy eq. (36) and (37). This question goes beyond the scope of the present work, here we only want to notice that:

"If the functions R(x), D(x) are such that under the transformation $x \to z = \int dx'/\sqrt{D(x')}$ they take the form $R(z) = z^r$, $D(z) = z^d$, with r and d real, then no similarity solutions of eq. (28) exist other than the "trivial" ones defined by the condition d=2r-2".

This assertion is easily proven by observing that with the choice d=2r-2, which is imposed by eq. (30), eq. (36) and (37) are also fulfilled.

Two relevant classes of functions R(x), D(x) which fall within the range of validity of our assertion are easily identified. The first class is of course that of powers $R(x) = x^p$, $D(x) = x^q$ which has already been investigated in detail. The second class is that of exponentials R(x) = exp(px), D(x) = exp(qx) which also bears a certain physical interest.

IV. CONCLUSION

The formalism developed in Ref. [1] has been extended to the case of the 1-dimensional advection-diffusion equation

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}\left(R(x) + D(x)\,\frac{\partial P}{\partial x}\right).$$

As an application we have studied the case of power-law dependence of R(x) and D(x) and exhibited few classes of similarity solutions.

Conditions have been given under which the results yielded by the transformation group method are not obtainable with more elementary techniques.

Acknowledgements

Dr. J. Vaclavik is kindly acknowledged for his critical reading of the manuscript.

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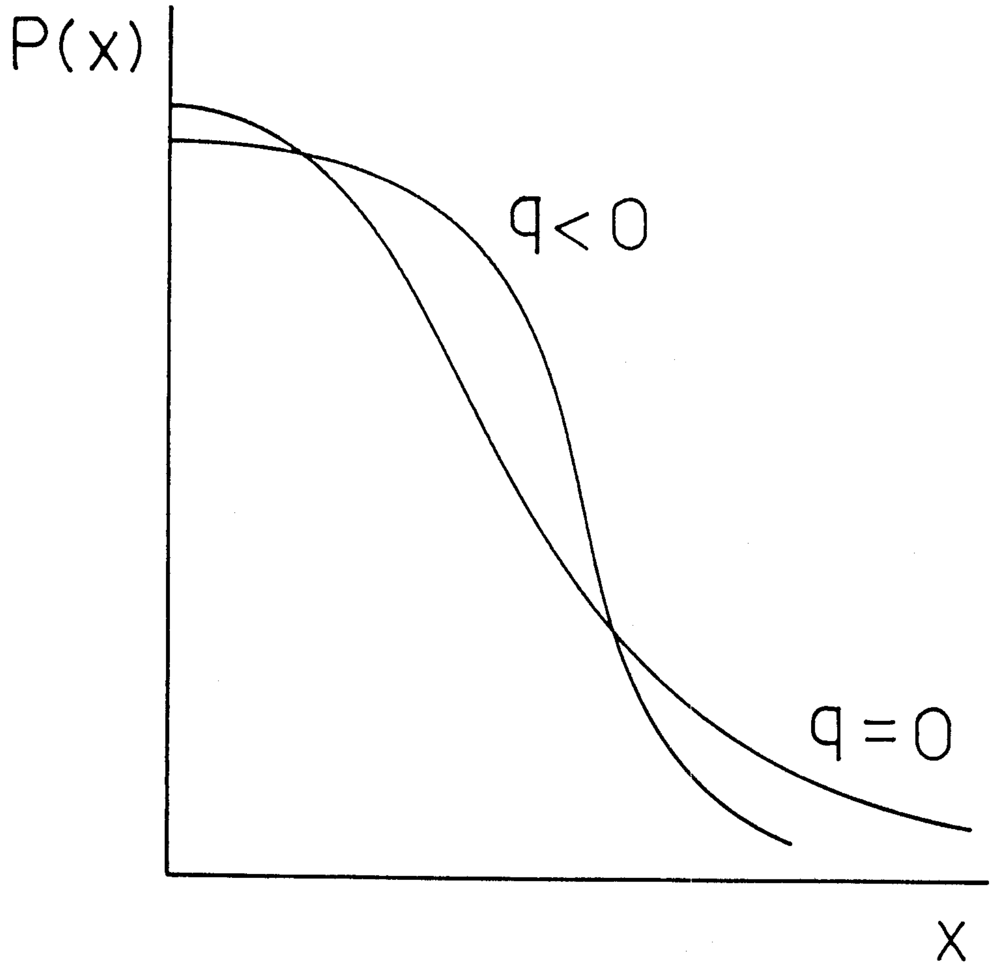

FIGURE 1

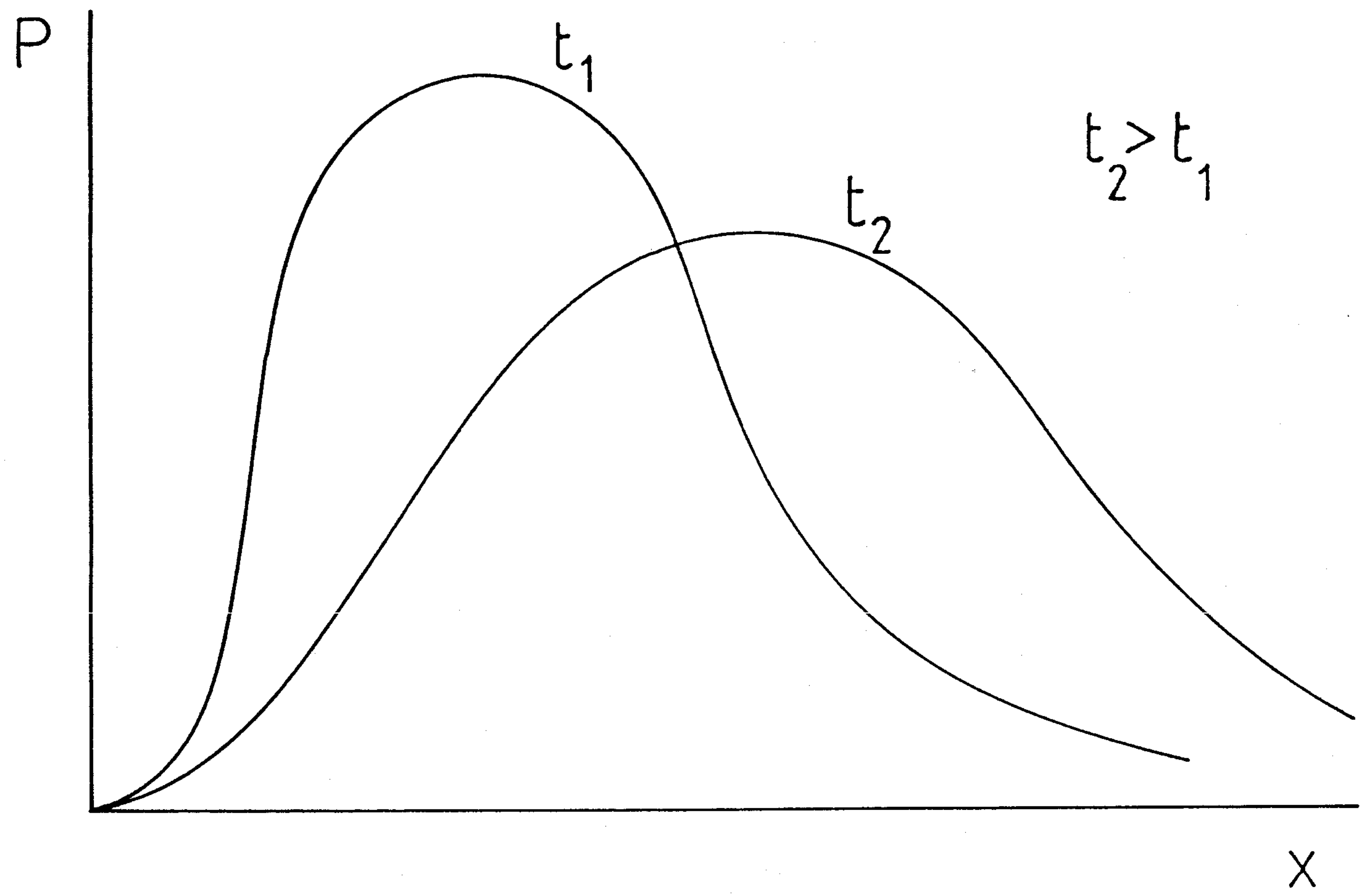

FIGURE 2