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Abstract

The ideal MHD stability of cylindrical equilibria with mass flows is investigated analytically and numerically. The flows modify the local (Suydam) criterion for instability at the resonant surfaces where $\vec{k}\cdot\vec{B} = 0$. Sheared flows below the propagation speed for the slow wave are found to be destabilizing for the Suydam modes. At a critical velocity, where the shear of the flow exactly balances the propagation of the slow wave along the sheared magnetic field, and the $\vec{k}\cdot\vec{B} = 0$ surface is at the edge of a slow wave continuum, there is instability regardless of the pressure gradient. Above the critical velocity, the $\vec{k}\cdot\vec{B} = 0$ surface is stable, but an infinite sequence of unstable modes still exists with frequencies accumulating toward the edge of the slow wave continuum at nonzero Doppler shifted frequency. The stability of the infinite sequences becomes a nonlocal problem whenever the accumulation frequency overlaps with a continuum at some other radial location.

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I. Introduction

In tokamak experiments with strong neutral beam injection, equilibrium mass flows can become substantial. For example, flows of the order of the sound speed have been measured in PDX[1] and in ISX-B.[2] So far, with few exceptions, MHD stability theory has been concerned with static equilibria, and the knowledge of MHD stability for equilibria with flows is rather incomplete. As is well known, mass flows render the linearized MHD equations non-selfadjoint and the energy principle no longer gives a simple, sufficient condition for instability. Thus, the stability considerations become considerably more involved than in the selfadjoint static case. Previous stability studies of equilibria with flows have treated the effects of rigid toroidal rotation on ideal ballooning modes[3,4] and of shear flow on resistive tearing modes.[5-7] It has been pointed out[8,9] that in toroidal systems, equilibrium flows can lead, under certain conditions, to an unstable continuous spectrum.

In the present work, we study, analytically and numerically, the ideal MHD stability of circular cylinder equilibria with sheared flows in the azimuthal and axial directions. We are concerned with the so called local instabilities and first derive a modified Suydam criterion for instabilities driven by the local pressure gradient and flow at the resonant surface where $\vec{k}\cdot\vec{B} \equiv k_{\parallel}B = 0$. The modified Suydam criterion shows that sheared flows below the slow wave speed are destabilizing. At a critical velocity, where the shear of the convection precisely balances the propagation of slow waves on the sheared magnetic field, the Suydam modes are unstable regardless of the pressure gradient. At this critical flow, the $k_{\parallel} = 0$ surface is also at the edge of a slow wave continuum. Above the critical speed, the $k_{\parallel} = 0$ surface is stable but we observe numerically that one of the two infinite sequences of Suydam modes is picked up by the edge of the continuum and the unstable modes continuously transform into discrete (or global) slow modes. This transition is discussed analytically and is shown to be possible when the edge of the continuum overlaps in frequency with some continuum at a different radius.

From analytic arguments and a numerical test case it appears that the transfer of instability from the Suydam modes to the discrete slow modes at the critical speed is generic and that, therefore, equilibria with near- or super-sonic flows will generally be unstable. Our numerical computations for the case of axial flow show, however, that the growth rates of the flow driven instabilities can be very small, in particular, in the supersonic regime. We also find that, although the flow decreases the maximum stable pressure gradient, it tends to decrease the growth rate of the most unstable mode for an equilibrium that is already Suydam unstable in the absence of flows. Furthermore, the unstable modes in the supersonic regime have a very complicated structure involving nearly singular behaviour in the vicinity of continuum resonances. Undoubtedly the stability of such modes is a delicate subject; even small changes of the modelling equations may drastically change the picture. Strong candidates for such modifications are of course finite resistivity, ion viscosity and, since the slow wave is essentially an ion sound wave along the field lines, wave-particle resonance. However, as a first step toward a more complete understanding, we believe that the ideal MHD model is useful.

After completion of this work, we were made aware of previous analytic work on the same subject, by Hameiri[10] (also briefly mentioned in Ref. 11). Several of the conclusions of the present paper, in particular those concerning the modification of the Suydam criterion, were reached by Hameiri,[10] who also carried out a boundary layer calculation similar to that in Sec. III.B of the present paper. However, in Ref. 10, the nonlocal aspect of the problem, brought about by overlap with other continua, is overlooked and the modified Suydam criterion is presented as a necessary and sufficient condition for local stability. As we show numerically and analytically, this is not the case, but, in the presence of flows, nonlocal effects can influence the stability of the "local" modes. An important consequence of the nonlocality is the instability of the discrete slow modes for supercritical flows.

The plan of this paper is as follows. Section II derives the second order radial eigenvalue problem. In Sec. III we consider, by an indicial equation plus boundary layer analysis, the different types of "local" instability that can occur: modified Suydam modes and discrete slow and Alfvén modes. We also discuss the consequences of overlap with other continua. Finally, in Sec. IV we present detailed numerical results for a particular equilibrium profile when the flow speed and pressure gradient are varied. The test case shows detailed agreement with the theory and examplifies the importance of overlap with other continua. This effect results in the stabilization of half of the Suydam modes for slightly subcritical flows and instability of the discrete slow modes for supercritical flows.

II. Radial eigenvalue problem in a cylinder

In a pioneering paper on MHD stability in the presence of equilibrium flows, Frieman and Rotenberg[12] showed that the linearized motion of the Lagrangian displacement (i.e., the displacement of a fluid element moving with the equilibrium flow) is described, in Eulerian coordinates, by

$$\rho \frac{\partial^2 \vec{\xi}}{\partial t^2} + 2\rho\, \vec{v}\cdot\nabla \frac{\partial \vec{\xi}}{\partial t} = -\nabla p_* + \vec{B}\cdot\nabla\vec{Q} + \vec{Q}\cdot\nabla\vec{B}$$

$$+ \nabla\cdot[\rho\vec{\xi}(\vec{v}\cdot\nabla)\vec{v} - \rho\vec{v}(\vec{v}\cdot\nabla)\vec{\xi}] \quad , \qquad (1)$$

where

$$\vec{Q} = \nabla\times(\vec{\xi}\times\vec{B})$$

is the perturbed magnetic field,

$$p_* = -\gamma p\, \nabla\cdot\vec{\xi} - \vec{\xi}\cdot\nabla p + \vec{B}\cdot\vec{Q}$$

is the perturbed kinetic plus magnetic pressure and ρ, $\vec{v}$, $\vec{B}$ and p denote the equilibrium density, flow, magnetic field and pressure respectively. Specializing to cylindrical geometry and assuming an $\exp[i(\omega t+m\theta-kz)]$ dependence for the cylindrical components of the displacement, we obtain the following equation of motion

$$- \rho\tilde{\omega}^2 \vec{\xi} = -\nabla p_* + \vec{B}\cdot\nabla\vec{Q} + \vec{Q}\cdot\nabla\vec{B}$$

$$- [\rho \frac{v_\theta^2}{r} \nabla\cdot\vec{\xi} + r\xi_r \frac{d}{dr}\left(\frac{\rho v_\theta^2}{r^2}\right)] \hat{r}$$

$$+ 2i\tilde{\omega}\rho \frac{v_\theta}{r} \vec{\xi}\times\hat{z} \quad , \qquad (2a)$$

where

$$\tilde{\omega} = \omega + \vec{k}\cdot\vec{v} = \omega + mv_\theta/r - kv_z \qquad (2b)$$

is the local Doppler shifted frequency and the cap signifies a unit vector.

Following Appert, Gruber and Vaclavik,[13] we derive from (2) a pair of first order radial differential equations for ξ_r and p_*. This may be done, for instance, by projecting (2a) along the field lines to express $\nabla\cdot\vec{\xi}$ in terms of p_* and ξ_r and eliminating ξ_θ and ξ_z in favour of $\nabla\cdot\vec{\xi}$, p_* and ξ_r. Substitution into the definition of $\nabla\cdot\vec{\xi}$ and the radial equation of motion then yields

$$AS \frac{1}{r} (r\xi_r)' = C_1\xi_r - C_2p_* \quad ,$$

$$AS \; p_*' = C_3\xi_r - C_1p_* \quad , \qquad (3)$$

where prime denotes radial derivative. The coefficients in Eq. (3) are given by

$$A = \rho\tilde{\omega}^2 - F^2 \quad ,$$

$$S = (B^2+\gamma p)\rho\tilde{\omega}^2 - \gamma p F^2 \quad ,$$

$$C_1 = \rho\tilde{\omega}^2 Q/r - 2mST/r^2 \quad ,$$

$$C_2 = \rho^2\tilde{\omega}^4 - (k^2+m^2/r^2)S \quad , \tag{4a}$$

$$C_3 = AS\ C_4 - 4\ ST^2/r^2 + Q^2/r^2 \quad ,$$

$$C_4 = A + r\frac{d}{dr}\left[(B_\theta^2-\rho v_\theta^2)/r^2\right] \ ,$$

where we introduced the notations

$$F = \vec{k}\cdot\vec{B} = k_\parallel B = mB_\theta/r - kB_z \quad , \qquad T = FB_\theta - \rho\tilde{\omega}v_\theta \quad ,$$

(4b)

$$Q = \rho\tilde{\omega}^2(B_\theta^2-\rho v_\theta^2) + \rho(B_\theta\tilde{\omega}-Fv_\theta)^2 = (2B_\theta^2-\rho v_\theta^2)A + 2B_\theta FT \quad .$$

Equations (3-4) state the eigenvalue problem for the ideal MHD spectrum in a circular cylinder with equilibrium flows. The differential equation (3) is identical in form to the static system,[10,11,13] and the flow only modifies the coefficients. In particular, axial flows only enter through the Doppler shift and do not appear in the equilibrium relation $(p + B^2/2)' = (\rho v_\theta^2 - B_\theta^2)/r$. In a toroidal system, toroidal flows also give rise to centrifugal forces which affect both equilibrium and stability.

As in the static case, the radial eigenvalue problem has singularities whenever $A = 0$ or $S = 0$, and this can happen only when ω is real. These singularities give rise to four distinct continua, viz. two Alfvén continua $A = 0$, or

$$\omega = \Omega_A(r) = -\vec{k}\cdot\vec{v} \pm k_\parallel v_A \quad , \tag{5a}$$

where $v_A = B/\sqrt{\rho}$ is the Alfvén speed, and two slow wave continua S = 0, or

$$\omega = \Omega_S(r) = -\vec{k}\cdot\vec{v} \pm k_\parallel v_S \quad , \tag{5b}$$

where $v_S = v_A[\gamma p/(B^2+\gamma p)]^{1/2}$ is the propagation speed of slow waves for $k_r \to \infty$.[14]

Before analyzing Eq. (3) in detail we draw attention to a certain cancellation occuring when AS = 0. This follows from the factorization

$$C_1^2 - C_2C_3 = AS\ (C_5 - C_2C_4), \tag{6a}$$

where C_5 can be written

$$C_5 = r^{-2}(k^2+m^2/r^2)[\rho\tilde{\omega}^2(2B_\theta^2-\rho v_\theta^2)^2 - \rho^2 v_\theta^2(2B_\theta\tilde{\omega}-Fv_\theta)^2]$$
$$+ (4T^2/r^2)(\rho\tilde{\omega}^2-k^2\gamma p) - (4Tm/r^3)\rho\tilde{\omega}^2(2B_\theta^2-\rho v_\theta^2) \quad . \tag{6b}$$

There is considerable simplification when $v_\Theta = 0$

$$C_5(v_\theta \equiv 0) = (4B_\theta^2 k^2/r^2)(\rho\tilde{\omega}^2 B^2 - F^2\gamma p) \quad . \tag{6c}$$

If the factorization (6a) did not occur, Eq. (3) would have essential singularities when AS has a zero of quadratic or higher order.

III. Analysis of local instabilities and singularities

Local conditions for instability can be found by examining the singularities $AS = 0$ of Eq. (3). In the selfadjoint static case, ω^2 must be real and the only singularity that can give rise to instability is at $\omega = 0$ and $k_\parallel = 0$, where both A and S vanish. The stability criterion was given by Suydam.[15] For the case of finite mass flow, a straightforward indicial equation analysis shows that the Suydam criterion is modified. On a more detailed level, by a boundary layer analysis, we find that flow gives rise to a qualitative difference with the selfadjoint static case, viz., the question of stability for the infinite sequences becomes nonlocal whenever the accumulation frequency overlaps with a continuum at some other radial location. In this section we analyze the different singularities of Eq. (3): the continua, the Suydam surface $k_\parallel = 0$, and the edges of the continua.

At a continuum singularity $r = r_c$, AS has a simple zero, $AS(r_c) = 0$ and $(AS)'(r_c) \neq 0$. Expanding (3) to lowest order in $x = r - r_c$, substituting $\xi \propto x^\nu$ and using (6) we find the indicial equation $\nu^2 = 0$. This implies that the continuum solutions are of the same nature as in the static case[14], $\xi(x) \sim \xi_0 \log|x| + \xi_+\sigma(x) + \xi_-\sigma(-x)$, where ξ_0, ξ_+ and ξ_- are constants and σ denotes the Heaviside step function.

III.A The modified Suydam criterion

At the "resonant surface" $r = r_0$ where $k_\parallel = 0$, A and S both vanish quadratically when $\tilde{\omega}(r_0) = 0$. To lowest order around the resonant surface, the coefficients of Eq. (3) behave as

$$AS \simeq c_0 x^4 \quad , \qquad C_1 \simeq c_1 x^3 \quad ,$$

$$C_2 \simeq c_2 x^2 \quad , \qquad C_3 \simeq c_3 x^4 \quad ,$$

where $x = r - r_0$ and the c's can be expressed in terms of

$$f = F' \ , \tag{7a}$$

measuring the shear of the magnetic field,

$$g = \sqrt{\rho}\ (\vec{k}\cdot\vec{v})' = \sqrt{\rho}\ \tilde{\omega}' \ , \tag{7b}$$

measuring the shear of the convection, and

$$V = \sqrt{\rho}\ v_\theta \quad , \tag{7c}$$

measuring the azimuthal flow. Setting $\xi = \xi_0 x^\nu$, $p* = p_0 x^{\nu+1}$, we obtain for the characteristic exponent

$$\nu = -\frac{1}{2} \pm \left(\frac{1}{4} - D_0 \right)^{1/2} \quad ,$$

$$D_0 = (c_2 c_3 - c_1^2 - c_0 c_1)/c_0^2 \quad .$$

In section III.B we find that unless the Suydam frequency $\omega_0 = -\ (\vec{k}\cdot\vec{v})(r_0)$ overlaps with a continuum at some other radial location, a complex characteristic exponent ν implies the existence of two infinite sequences of instabilities. Asymptotically, the eigenfrequencies of these modes converge geometrically toward the marginal point, $\tilde{\omega} = 0$ at $k_\parallel = 0$ and the limiting solution for ξ_r

has an infinite number of oscillations in the vicinity of the resonant surface. The local criterion for instability, corresponding to Suydam's criterion in the static case, can be written

$$D_0 > 1/4 \quad , \tag{8a}$$

where

$$D_0 = \left(\frac{q}{q'B_z}\right)^2 \frac{1}{1-M^2} \quad \{-2[\frac{1}{r}(p'+\frac{V^2}{r}) + \frac{B^2}{B_\theta^2} V \left(\frac{V}{r}\right)' + \frac{q'}{rq}\frac{B_z^2}{B_\theta} MV]$$

$$+ \frac{4}{r^2(1-M^2)} \quad (V-MB_\theta)^2$$

$$+ \frac{1}{r^2B_\theta^2} \frac{1-\beta}{(1-M^2)(\beta-M^2)} \quad [(V-MB_\theta)^2 + M^2(B_\theta^2-V^2)]^2\} \quad . \tag{8b}$$

In Eq. (8b) we introduced

$$\beta \;= \gamma p/(B^2+\gamma p) = v_s^2/v_A^2 \quad , \tag{9a}$$

representing the pressure effects (note $0 < \beta < 1$), and the Alfvén Mach number

$$M = g/f = \sqrt{\rho}\;\tilde{\omega}'/F' \quad . \tag{9b}$$

In order to relate to toroidal devices, we introduced the safety factor $q = rB_z/RB_\theta$, where R denotes the major radius, so that at the resonant surface $F' = f = -\,kB_z\,q'/q$. Furthermore $(B_\theta/r)'$ in C_4 was eliminated by means of the equilibrium relation. In the case of purely axial flow, Eq. (8) simplifies considerably

$$D_0 = \left(\frac{q}{q'B_z}\right)^2 \frac{2}{1-M^2} \left(\frac{-p'}{r} + \frac{2\beta M^2}{\beta-M^2} \frac{B_\theta^2}{r^2}\right) > \frac{1}{4} \quad , \tag{8c}$$

which is readily seen to reduce to Suydam's criterion when M = 0. Equation (8) shows that the flow can play a very significant role for the stability by making either $A'' = -2f^2(1-M^2)$ or $S'' = -(B^2+\gamma p)f^2(\beta-M^2)$ small. In these two cases, shear of the convection exactly balances the propagation of Alfvén or slow waves along the sheared field

$$(\vec{k}\cdot\vec{v})' = \pm\ (k_\parallel v_A)' \quad , \quad A_o'' = 0 \quad , \quad M = \pm\ 1 \ , \tag{10}$$

or

$$(\vec{k}\cdot\vec{v})' = \pm\ (k_\parallel v_S)' \quad , \quad S_o'' = 0 \quad , \quad M = \pm\ \beta^{1/2} \ . \tag{11}$$

In contrast with the static case, the inertial terms now have an effect on the stability as expressed by the appearence of $M = \sqrt{\rho}\tilde{\omega}'/F'$ in the stability criterion (8). The same effect is manifest in (10) and (11) as the balance between convection and wave propagation.

In tokamaks, where $v_S/v_A = \beta^{1/2}$ is small, the resonance with the slow wave (11) is met with much weaker flows than that with the Alfvén wave (10), and the region around the critical speed $M = \pm\ \beta^{1/2}$ is the subject of main interest for the present paper. We emphasize that in the incompressible limit $\gamma \to \infty$, $v_S \to v_A$ and the critical speed (11) is artificially raised from the sound speed to the

Alfvén speed. Thus the assumption of incompressibility, which is usually justified at marginal stability in the static case, is not appropriate here.

The criterion (8c) shows that subcritical ($M^2<\beta$) axial flows decrease the maximum pressure gradient that can be stably confined, while the general case with azimuthal flows (8b) is less clear-cut. However, for flows slightly below the critical speed $M = \pm \beta^{1/2}$, Eq. (8b) indicates instability independent of the pressure gradient. Similar to the original Suydam criterion, the modified form (8) does not depend on the absolute magnitude of k and m, only on the ratio kr_0/m. Thus, the local stability criterion applies to modes of arbitrarily short wavelength so that kr_0/m can be treated as a continuous variable and the stability criterion must be considered everywhere, not only at certain low order rational surfaces.

III.B Location of unstable roots in the complex plane

Interesting information concerning the existence and location of point eigenvalues in the complex ω-plane can be obtained from a boundary layer analysis, slightly modified from that given by Greene[16,17] in the static case. For this purpose, it is more convenient to work with the second order, so called Hain-Lüst[18] equation for ξ_r readily derived from (3) and (6)

$$\frac{d}{dr}\frac{AS}{rC_2}\frac{d}{dr}r\xi_r - \left[\frac{d}{dr}\left(\frac{C_1}{rC_2}\right) + \frac{C_5}{rC_2} - \frac{C_4}{r}\right]r\xi_r = 0 . \qquad (12)$$

As pointed out by Appert et al.,[13] the singularity $C_2 = 0$ (but $AS \neq 0$) is only apparent as it is not a singularity of the original Eq. (3).

To find the scaling relations for the boundary layer, we first note that when the Doppler shifted frequency is zero at the resonant surface $r = r_0$, the leading term in AS/rC_2 is quadratic in $x = r-r_0$, while that of $(C_1/rC_2)' + C_5/rC_2 - C_4/r$ is constant. This implies that the lowest order approximation of (12) is invariant under rescaling of x. Furthermore, around $x = 0$ and $\tilde{\omega}(r_0) = 0$, A and S are both quadratically zero in x as well as in ω. Therefore, if we introduce a finite Doppler shifted frequency $\tilde{\omega} = -i\Gamma$ at $x = 0$, rescaling of x leaves the lowest order equation invariant if Γ is scaled the same way as x. (For the scaling argument to be stricly valid, Γ must be small enough to be negligible outside the region where the lowest order expansion in x is accurate.) Thus, the appropriate scaling is

$$x = (\sqrt{\rho}\Gamma/f)\ X \equiv \hat{\Gamma}X \quad , \qquad (13)$$

$$\Gamma = i\tilde{\omega}(x=0) \quad ,$$

where we shall assume $\mathrm{Re}\hat{\Gamma} > 0$ and where X is a normalized (dimensionless) distance to the resonant surface. In the following, we restrict the analysis to the analytically tractable case of purely axial flows and obtain the scaled, Γ-invariant inner layer equation

$$\frac{d}{dX}\left[(1+iMX)^2+X^2\right]\frac{d\xi_r}{dX}$$

$$+\left[D_0(1-M^2)-\frac{4}{R^2q'^2}\,\frac{\beta^2}{\beta-M^2}\,\frac{1+2iMX}{(1+iMX)^2+\beta X^2}\right]\xi_r=0\ . \qquad (14)$$

Equation (14) shows that within the "inner layer" $|X| < \beta^{-1/2}$, the finite frequency shift is essential. Furthermore, for large $|X|$, there exists an "intermediate" region where the frequency shift becomes negligible, and where the inner layer equation (14) approaches the small $|x|$ limit of the external (marginal stability, $\Gamma = 0$) equation

$$\frac{d}{dx}\,x^2\,\frac{d\xi_r}{dx}+D_0\xi_r=0 \qquad . \qquad (15)$$

The boundary layer analysis consists of matching the solutions of the external Eq. (15) for $|x|$ small with those of the complete inner layer Eq. (14) for $|X|$ large, using (13) to relate x and X, to obtain an equation for Γ.

It follows from Eq. (14) that ξ_r has logarithmic singularities on the imaginary X-axis

$$X_{A\pm} = i(M \pm 1)^{-1} \quad ,$$
$$X_{S\pm} = i(M \pm \sqrt{\beta})^{-1} \quad , \qquad (16)$$

and that one singularity moves to infinity at the critical velocities $M = \pm\,\beta^{1/2}$ and $M = \pm\,1$. Figure 1 shows the complex x and X planes and illustrates the fact that if the assumption $\mathrm{Re}\hat{\Gamma} > 0$ is violated, the singularities (16) move across the real x-axis and the results obtained for $\text{Re}\hat{\Gamma} < 0$ are not the analytical continuation of those for $\text{Re}\hat{\Gamma} > 0$. Therefore, $\text{Re}\hat{\Gamma} > 0$ is required for consistency. Needless to say, the case $\text{Re}\hat{\Gamma} < 0$ can be treated separately. The result is unchanged except for a complex conjugation of the eigenfunctions.

As pointed out by Greene[16], Eq. (14) with M = 0 already has too many singularities to be solved by the well-known hypergeometric functions and we are led to a less specific analysis based on rather general arguments. From now on, we assume that the local criterion for instability (8) is satisfied so that the asymptotic solution of (14) is oscillatory with a characteristic exponent

$$\nu = -\frac{1}{2} \pm is \quad , \; s = (D_0 - \frac{1}{4})^{1/2} \quad , \; D_0 > \frac{1}{4} \quad . \tag{17}$$

Under the scaling (13) the inner layer equation is independent of Γ. However, the asymptotic dependence for the inner layer solution is given by the external solution which, when expressed in terms of X, depends on Γ via (13). For ReX large and negative,

$$\xi_r \sim a_+(-x)^{-1/2+is} + a_-(-x)^{-1/2-is}$$

$$= a_+(-\hat{\Gamma}X)^{-1/2+is} + a_-(-\hat{\Gamma}X)^{-1/2-is} \quad , \tag{18a}$$

where a_+/a_- is fixed by the external solution. Similarly, for ReX large and positive,

$$\xi_r \sim b_+ x^{-1/2+is} + b_- x^{-1/2-is}$$

$$= b_+(\hat{\Gamma}X)^{-1/2+is} + b_-(\hat{\Gamma}X)^{-1/2-is} \quad , \tag{18b}$$

with b_+/b_- fixed by the external solution. Since the scaled inner layer equation is independent of Γ, so is the connection matrix between the expansion coefficients for ξ_r in the two asymptotic regions $\mathrm{Re}X \ll -\beta^{-1/2}$ and $\mathrm{Re}X \gg \beta^{-1/2}$. Thus

$$\begin{pmatrix} b_+\hat{\Gamma}^{is} \\ b_-\hat{\Gamma}^{-is} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} a_+\hat{\Gamma}^{is} \\ a_-\hat{\Gamma}^{-is} \end{pmatrix} \quad , \tag{19}$$

where u_{11}, u_{12}, u_{21} and u_{22} only depend on the coefficients of Eq. (14). The external solution provides the boundary conditions by determining

$$R_+ = b_+/b_-, \quad R_- = a_+/a_- \quad , \tag{20}$$

which allows elimination of $a_\pm$ and $b_\pm$ from (19) to give a quadratic equation for Γ^{2is}

$$\begin{aligned} u_{21}R_+R_-\zeta^2 + (R_+u_{22}-R_-u_{11})\zeta = u_{12} \quad , \\ \zeta \equiv \hat{\Gamma}^{2is} \quad . \end{aligned} \tag{21}$$

To be more explicit, we have to pay attention to the fact that the coefficients of Eq. (21) are not arbitrary. First, by considering the Wronskian of (14) we have

$$u_{12}u_{21} - u_{11}u_{22} = 1.$$

Furthermore, it is readily seen that Eq. (14) has two independent solutions with the symmetry property $\xi(-X) = \xi^*(X)$ for X real. This leads to the four conditions $u_{11}^*u_{22} + |u_{21}|^2 = 1$, $u_{21}u_{22}^* + u_{22}u_{12}^* = 0$ and $1 \leftrightarrow 2$, which, together with the Wronskian condition, imply that

$$\text{Re } u_{11} = \text{Re } u_{22} = 0 \quad ,$$
$$u_{12} = u_{21}^{*} \quad , \tag{22}$$
$$|u_{21}|^2 - u_{11}u_{22} = 1 \quad .$$

If we define

$$\mu = (R_+/R_-)^{1/2} \quad , \tag{23}$$

the solution of (21) can be written

$$\hat{\Gamma}_{\pm}^{2is} = \zeta_{\pm} = (2u_{21})^{-1}(R_+R_-)^{-1/2}$$

$$\times \{ u_{11}\mu^{-1} - u_{22}\mu \pm [4 + (u_{11}\mu^{-1} + u_{22}\mu)^2]^{1/2}\} \quad . \tag{24}$$

Because of the multivaluedness of the complex logarithm, the two solutions for ζ give rise to two distinct geometrical sequences for Γ

$$\Gamma_{\pm}^{(n)} = \Gamma_{\pm}^{(o)} \exp(-n\pi/s) \quad , \tag{25a}$$

$$\Gamma_{\pm}^{(o)} = \frac{f}{\sqrt{\rho}} \exp\left(\frac{\log\zeta_{\pm}}{2is}\right) \quad , \tag{25b}$$

where n is an integer. For each sequence, increasing n by one gives an eigenfunction which has one extra period of oscillation around $r = r_o$.

We emphasize that the analysis of this Section is only asymptotic for $|\Gamma|$ small, i.e., Eq. (25) is accurate only when n is large and positive. The first few eigenfunctions do not usually satisfy the scaling (25) very well, and, in particular, the maximum growth rate always occurs for an eigenfunction for which the local approximation breaks down. Thus, the maximum growth rate cannot be computed from the local analysis.

III.B.1 Selfadjoint case

As is well known, the growth rate in the selfadjoint case M = V = 0 is real. To see that this follows from the general formula (24), we note that when M = 0, the coefficients of the inner layer equation (14) are real and consequently $u_{11} = u_{22}^*$. Together with the general condition (22) this implies that

$$u_{11} = u_{22}^* = i \sinh p \quad ,$$
$$u_{21} = u_{12}^* = e^{i\phi} \cosh p \quad , \tag{26a}$$

where p and ϕ are real. With regard to the R-coefficients, it is seen that in the limit $\Gamma \to 0$, the external equation has real coefficients (and real, homogeneous boundary conditions). Therefore, if the external equation is free of singularities, it must have real valued solutions, thus,

$$|R_+| = |R_-| = |\mu| = 1 \quad . \tag{26b}$$

Substituting (26) into (24) we find that in the static case

$$\arg \hat{\Gamma}_\pm = -\frac{1}{2s} \log |\zeta_\pm| = 0 \quad , \tag{27}$$

hence Γ is real and positive as it should. (Note that $\mathrm{Re}\hat{\Gamma} > 0$ was required for consistency and that for each Γ there is a corresponding eigenvalue $-\Gamma$ with an identical eigenfunction.)

III.B.2 Non-selfadjoint case

When $M \neq 0$, the inner layer equation is complex and we have $u_{22} \neq u_{11}^*$ so that (26) is no longer valid. Instead, Eq. (27) is replaced by the weaker relation

$$2s \arg(\Gamma_+\Gamma_-) = - \log |\zeta_+\zeta_-| = \log |R_+R_-| \quad . \tag{28a}$$

Let us first assume that the limit frequency $\omega_0 = -(\vec{k}\cdot\vec{v})(r_0)$ does not belong to a continuum at some other radius. In this case, the external equation is nonsingular and its solution will be real, hence $|R_+| = |R_-| = 1$. Equation (28a) shows that in this case, the two sequences (25) lie on two lines symmetric about the real Γ-axis,

$$\arg \Gamma_- = - \arg \Gamma_+ \quad . \tag{28b}$$

Note that if ω is an eigenvalue, so is ω^* with the complex conjugate eigenfunction, and therefore there is symmetry about the real ω-axis as well, see Fig. 2. [It may be added that for the opposite signs of m and k, $-\omega$ and $-\omega^*$ are also eigenvalues, and we can say that the eigenvalues of the Suydam modes lie on four, asymptotically straight lines in the complex plane. For vanishing flows, the four lines all collapse onto the imaginary ω-axis.] Finally, if $|\mathrm{Re} \log \zeta_\pm| > \pi/s$, the growth rates do not satisfy the consistency condition $\mathrm{Re}\,\hat{\Gamma} > 0$ and must be rejected, since they do not correspond to integration along the real r-axis.

The results discussed so far are in agreement with those of Hameiri.[10] Assuming that the reality condition (26b) is satisfied, Hameiri shows that $|\mathrm{Re}\ \log \zeta_{\pm}| < \pi/s$ in the subcritical case, and concludes that $D_0 > 1/4$ is sufficient for instability. However, in general, when the limit frequency ω_0 belongs to a continuum at some other radial location, the external solution will be complex and $|R_+R_-| \neq 1$. As a consequence, the phases of the two sequences are no longer related as in (28b). More importantly, it is also possible for one or both of the sequences (25) to be lost by moving off the physical Riemann sheet.

We conclude that the "local" modes can be stabilized by effects having to do with the global solution and hence the global equilibrium profiles. To understand better the influence of the external solution, we first note that R_+ and R_- affect the eigenfunction in the inner layer directly by providing the boundary condition at infinity. Since the two asymtotic solutions of the inner layer equation $\sim X^{1/2 \pm is}$ have the same fall off for large X, a change in the asymptotic boundary condition will be felt throughout the inner layer. Thus, in a sense, the term "local" mode is a misnomer. In the static case, although the criterion for the appearence of an infinite sequence of unstable modes is local, the nonlocal nature of these modes shows up, for instance, in the slow $\sim 1/X$ fall off of the two contributions to the energy density $w = (f^2/2)\ (x^2 |\xi'|^2 - D_0 |\xi|^2)$. Within the inner layer expansion, the contributions to δW from field line bending and interchange (each considered separately) are both divergent at large X. Furthermore, as is well known in the static case, the external parameters R_+ and R_- determine the magnitude, but not the phase of $\Gamma_\pm$. In the presence of equilibrium flows, also the phase of $\Gamma_\pm$ is influenced by the external solution, and stabilization may occur. We emphasize that at marginal stability, where a mode can be created or annihilated by a small change in the equilibrium, the mode belongs to a continuum and is therefore singular. In Sec. IV we show a numerical example where one of the two Suydam sequences is lost somewhat below the critical speed $M = \beta^{1/2}$, even though the local criterion for instability, $D_0 > 1/4$, is satisfied.

III.C Global slow modes

When the flow approaches the critical speed $M = \beta^{1/2}$ from below, the imaginary part of the characteristic exponent ν at the Suydam surface goes to infinity (8). Thus, the marginal solution for ξ_r becomes increasingly oscillatory and the asymptotic ratio of growth rates between successive members of the infinite sequences approaches unity at the critical speed. The numerical computations presented in Sec. IV show that, around the critical speed, the growth rate of the most unstable mode remains finite and well behaved while the growth rates of the higher order, and less unstable, modes increase.

Although the modified Suydam criterion indicates that the $\vec{k}\cdot\vec{B} = 0$ surface is stable for flows just above the critical speed ($D_0 \to -\infty$), we find numerically that there is still instability for supercritical flows. However, the infinite sequence of unstable modes is now connected with the edge of the slow wave continuum, which passes through the resonant surface $k_{\parallel} = 0$ precisely at the critical speed, as expressed by Eq. (11). Simply put, the infinite sequence whose frequencies accumulate at the Suydam frequency, $\tilde{\omega} = 0$ at $k_{\parallel} = 0$, for subcritical flows, is captured by the edge of the slow wave continuum, $d\Omega_S/dr = 0$, at the critical speed when the two points coincide radially. In this subsection we discuss analytically the stability of the discrete slow modes.

We now consider the edge of a slow wave continuum, where S(r) is assumed to have a quadratic zero at nonzero Doppler shifted frequency

$$S = 0, \quad S' = 0, \quad S'' \neq 0, \quad \tilde{\omega}_S \neq 0 \quad . \tag{29}$$

If we write S as

$$\begin{aligned} S &= \rho(B^2+\gamma p)(\tilde{\omega}^2-\Sigma^2) \quad , \\ \Sigma^2 &= \beta(\vec{k}\cdot\vec{B})^2/\rho \quad , \end{aligned} \tag{30}$$

the following conditions hold at the edge of a slow wave continuum

$$\tilde{\omega}_S = \pm\,\Sigma_S \;, \quad \tilde{\omega}'_S = (\vec{k}\cdot\vec{v})'_S = \pm\,\Sigma'_S \quad , \quad \tilde{\omega}''_S \neq \pm\,\Sigma''_S \quad . \tag{29'}$$

III.C.1 Indical equation

At an edge of the slow wave continuum $r = r_S$, the eigenvalue problem (12) has a singularity such that to lowest order in $x = r-r_S$

$$S \simeq S''x^2/2 \quad ,$$
$$A,\ C_1,\ C_2,\ C_4,\ C_5 \simeq \text{constant} \quad ,$$
$$(C_1/rC_2)' \simeq (2B_\theta^2/r^2)' \simeq \text{constant} \quad ,$$

where, once gain, $v_\Theta \equiv 0$ has been assumed. Thus, to lowest order in x, Eq. (12) becomes

$$\frac{d}{dx}\, x^2 \,\frac{d}{dx}\, \xi_r + D_s \xi_r = 0 \quad ,$$

where

$$D_s = \frac{2F^2}{S''} \frac{\beta^2}{1-\beta} \left[2B_\theta (B_\theta/r)' - (2B_\theta k/rF)^2 (B^2+\gamma p) + (1-\beta)F^2\right] . \quad (31)$$

The condition for oscillatory solutions is

$$D_S > 1/4 \quad . \quad (32)$$

If (32) is satisfied, there may exist one or two infinite sequences of regular eigenfunctions of (12), (discrete slow waves) whose eigenfrequencies converge geometrically to the edge of the continuum (29). In the selfadjoint case, the sequences must have purely real eigenfrequencies. In the non-selfadjoint case, this restriction does not apply and, as we show in Sec. III.C.2, the global slow modes can become unstable.

In contrast with the Suydam parameter D_0, D_S in (31) depends on the magnitude of m and k. In the limit of short wavelength, the third term in square brackets in (31) is dominant and the criterion for discrete slow waves becomes $S'' > 0$. On the other hand, when $|F|$ is small (as it is near the critical velocity), the second term in square brackets in (31) is dominant and the condition for the existence of global slow modes becomes $S'' < 0$.

The case when the edge of the slow wave continuum occurs at $r = 0$ (which always gives an at least quadratic zero of S) can be analyzed much in the same way. Here the lowest order expansion of the marginal equation ($\omega = \omega_S$) reads

$$(r(r\xi_r)')' + (D_S - m^2)\xi_r = 0 \quad , \tag{33}$$

where D_S is given by (31). The condition for discrete slow modes at the centre of the plasma is obtained from the indicial equation

$$D_S > m^2 \quad . \tag{34}$$

III.C.2 Inner layer equation

In the Suydam problem, A and S have quadratic zeros in x as well as ω and the radial extent of the inner layer scales linearly with the frequency shift $\tilde{\omega}(r_0)$. At the edge of a slow wave continuum, S has a quadratic zero in x but is linear in ω. Hence, the frequency shift must be scaled with the square of the inner layer dimensions. Setting

$$\omega = \omega_S + \Delta \quad , \tag{35}$$

the appropriate scaling becomes

$$x = X\left[4\rho(B^2+\gamma p)\tilde{\omega}_s\Delta/S''\right]^{1/2} \equiv X\,\hat{\Delta}^{1/2} \quad . \tag{36}$$

The scaled inner layer equation,

$$\frac{d}{dX}(1+X^2)\frac{d\xi_r}{dX} + D_s\xi_r = 0 \quad , \tag{37}$$

is essentially Legendre's equation with the general solution

$$\xi_r = a_1\ {}_2F_1\left(\frac{1}{4}+\frac{is}{2}, \frac{1}{4}-\frac{is}{2}; \frac{1}{2}; -X^2\right)$$

$$+\ a_2X\ {}_2F_1\left(\frac{3}{4}+\frac{is}{2}, \frac{3}{4}-\frac{is}{2}; \frac{3}{2}; -X^2\right) \quad .$$

Here ${}_2F_1$ is the hypergeometric function, $s = (D_S-1/4)^{1/2}$ and $D_S > 1/4$ was assumed. Using the hypergeometric transformation[19] to express ξ_r in terms of ${}_2F_1$ with argument $-X^{-2}$ we obtain explicitly the U-matrix connecting the asymptotic form of ξ_r for $X \to -\infty$ and $X \to +\infty$ as

$$\begin{aligned} u_{11}^* &= u_{22} = i \operatorname{csch} \pi s \quad , \\ u_{21}^* &= u_{12} = i \coth \pi s \quad e^{i\chi} \quad , \\ \chi &= 2s \log 2 + 2 \arg\left(\frac{\Gamma(is)}{\Gamma(1/2+is)}\right) \quad , \end{aligned} \tag{38}$$

where Γ now denotes the gamma function. [Note that (38) satisfies (22) and (23) as it should.] Repeating the same analysis as in the Suydam case we again obtain the quadratic equation (21) but now ζ stands for $\hat{\Delta}^{is}$. The dispersion relation can be written

$$\hat{\Delta}_\pm^{is} = e^{i\chi} \quad (R_+R_-)^{-1/2} \left[c \pm i(1-c^2)^{1/2}\right] \quad , \tag{39a}$$

$$c = \frac{1}{2}\left[(R_+/R_-)^{1/2} + (R_-/R_+)^{1/2}\right] \operatorname{sech} \pi s \quad . \tag{39b}$$

We see that if

$$|R_+| = |R_-| = 1, \tag{40}$$

$|\hat{\Delta}^{is}| = 1$ and, therefore, $\hat{\Delta}$ is real and positive. As in the Suydam case, the asymptotic external equation has real coefficients, therefore Eq. (40) will be satisfied unless ω_s falls within some other continuum.

In most cases, of course, ω_s will belong also to other continua. Under such circumstances, to determine R_+ and R_-, a small but finite imaginary part has to be assigned to Δ so that the external equation can be integrated on the real r-axis past the points where ω_S equals any other continuum frequency. We emphasize that overlap with a continuum destroys the reality of the external solution, since near a continuum singularity at some complex radius r_C, $\xi \simeq \xi_0 \log (r-r_C) + \xi_1$ and unless $\xi_0 = 0$, the solution will be complex on the real r-axis. The coefficients R_+ and R_- characterizing the external solution are well behaved as $\Delta \to 0$, but in general they do not satisfy the reality condition (40) and have cuts at Im $\Delta = 0$. We conclude that there is once again a consistency requirement, namely, the imaginary part of Δ as computed from (39) must be consistent with the value assumed in determining the external solution. This consistency requirement is non-local and difficult to deal with analytically since, depending on the equilibrium, there can be an arbitrary number of such resonances. However, we know, a priori, that in the selfadjoint case, Δ must be real. Then, if ω_S belongs to some other continuum, the potential discrete modes will typically be absorbed into that continuum. In Sec. IV, we show numerical examples demonstrating that unstable discrete slow modes do occur in the non-selfadjoint case when the flow exceeds the critical speed, $M^2 > \beta$.

Under the conditions just dissussed, i.e., $D_S > 1/4$ and overlap with another continuum, Eq. (39) can yield one or two sequences of discrete modes with frequencies accumulating at the continuum edge

$$\Delta_{\pm}^{(n)} = \Delta_{\pm}^{(0)} \exp(-2n\pi/s) \quad . \tag{41}$$

It follows from (39) that $\arg(\Delta_+\Delta_-) = s^{-1} \log |R_+R_-|$. The case of no overlap is special in that $\arg \Delta_+ = \arg \Delta_- = 0$, and both sequences in (41) exist and lie on the real ω-axis. If overlap with other continua occurs, then, in general, $|R_+R_-| \neq 1$, and either none, one or both of the sequences (41) may exist.

When the continuum edge is located at $r_s = 0$, the inner layer equation is more difficult and appears not to be explicitly solvable. However, we can still apply the scaling analysis. This case differs from the previous one where the continuum edge is located at nonzero radius in that there is only one matching condition. [At the origin the boundary condition is regularity rather than matching to an external solution, and the left boundary condition becomes invariant in the scaled variables.] Consequently, the matching gives a linear equation for $\zeta = \hat{\Delta}^{is}$ and there can be at most one sequence of discrete modes.

We believe that the points discussed in this paragraph are significant and of general interest. First, in the presence of background flows, instability can occur for the discrete slow modes whose frequencies accumulate at the edge of a continuum with a finite Doppler shifted frequency. Secondly, the stability considerations for these modes involves, in a nontrivial way, the properties of the external solution and hence the global equilibrium profiles. Therefore, it seems adequate to describe these modes as global rather than local. Examples of such modes will be given in the numerical section IV.

[11] E. Hameiri, J. Math. Phys. 22, 2080 (1981)

[12] E.A. Frieman and M. Rotenberg, Rev. Mod. Phys. 32, 898 (1960)

[13] K. Appert, R. Gruber and J. Vaclavik, Phys. Fluids 17, 1471 (1974)

[14] J.P. Goedbloed, "Lecture notes on ideal Magnetohydrodynamics", Rijnhuizen Report 83-145, FOM-Rijnhuizen, Netherlands (1983)

[15] B.R. Suydam, in Proceedings of Second International Conference on the Peaceful Uses of Atomic Energy (Geneva) p. 157 (1958)

[16] J.M. Greene, "Introduction to resistive instabilities", LRP 114/76, Lausanne, Switzerland (1976)

[17] M.S. Chance, J.M. Greene, R.C. Grimm and J.L. Johnson, Nucl. Fusion 17, 65 (1977)

[18] K. Hain and R. Lüst, Z. Naturforsch. 13a, 936 (1958)

[19] See, e.g., W. Magnus, F. Oberhettinger, and R.P. Soni, "Formulas and Theorems for the Special Functions of Mathematical Physics", 3rd ed. (Springer, 1966), § 2.4.1.

[20] D. Voslamber and D.K. Callebaut, Phys. Rev. 128, 2016 (1962)

and consequently $D_S \to +\infty$ $(-\infty)$ as the critical speed is approached from above (below). This implies that the stability of the continuum edge is exactly opposite to that of the $k_\parallel = 0$ surface (8) and we may expect that the sequence(s) of unstable modes will be transferred between the two at the critical speed.

It should be noted that the range of validity for the boundary layer analyses of Secs III.B and III.C goes to zero at the critical speed. This happens because S" goes to zero at the critical speed and higher order terms in x tend to become important unless the frequency shift is very small. Since $\tilde{\omega}_S$ and S" are both linear in $M^2-\beta$ near criticality, the scaling (36) shows that in order for the boundary layer calculation to be valid, Δ can be at most of order $S''^2 \sim (M^2-\beta)^2$. A similar difficulty occurs in the Suydam case where one pole (16) moves to infinity in the scaled variable X and the last term in (14) blows up at the critical speed. Therefore, the behaviour of the few most unstable modes around the critical speed cannot be inferred from the boundary layer analysis. Numerical results are given in Sec. IV.

III.D Global Alfvén modes

An analysis along the same lines as in Sec. III.C can be carried out for the edge of an Alfvén continuum, where A has a quadratic zero, at radius $r = r_A$. We find that the marginal equation has oscillatory solutions when

$$D_A > 1/4 \quad , \quad r_A \neq 0 \quad ,$$
$$D_A > 1 + m^2 \quad , \quad r_A = 0 \quad , \tag{43}$$

where

$$D_A = \frac{2k^2}{A''} \left\{ \frac{(B_\theta + B_z m/kr)^2}{B^2} \; r \frac{d}{dr} \left[\frac{2FB_\theta B_z}{r(krB_\theta + mB_z)} - \frac{B_\theta^2}{r^2} \right] \right.$$
$$\left. - \frac{4B_\theta^2}{r^2} \; \frac{1-2\beta}{1-\beta} \right\} \quad . \tag{44}$$

Except in the centre, the criterion for oscillatory solutions is independent of the absolute magnitude of m and k. For $r_A \neq 0$, the inner layer analysis is identical to that for the slow waves. Typically, the edge of an Alfvén continuum does not coincide with the other continua and under such circumstances Eq. (43) implies the existence of two infinite sequences of discrete Alfvén eigenmodes with real frequencies.

IV. Numerical study

In this section, we present numerical results for a particular test case where we have followed the unstable frequencies in the complex plane as the flow speed and pressure gradients are varied. Only the case of purely axial flow has been investigated. The equilibrium was a slightly modified form of the force-free Bessel function model analyzed by Voslamber and Callebaut[20]

$$B_\theta = J_1(\lambda r) \quad , \quad B_z = (1-P_1)^{1/2} J_0(\lambda r) \quad ,$$

$$p = P_0 + (P_1/2)\, J_0^2(\lambda r) \quad , \tag{45}$$

$$v_\theta = 0 \quad , \quad v_z = v_{zo}(1-r^2) \quad ,$$

with a conducting wall at $r = 1$, $\xi_r(1) = 0$. The value of λ was fixed at 2, which is well below the threshold for global current driven instabilities, $\lambda = 3.176$ at $P_1 = 0$. [20] We also fixed $m = 1$ and $k_z = 1.2$, so that there was always a resonant surface $k_\parallel = 0$ inside the plasma, and used $\gamma = 5/3$, $P_0 = 0.05$ and $\rho = 1$.

Based on the local parameters D_0 and D_S, we have the stability diagram of Fig. 3 where the stable region is marked A, region B is potentially unstable to the modified Suydam modes $D_0 > 1/4$, region C to the discrete slow waves at $r_S \neq 0$, $D_S > 1/4$ and region D to discrete slow waves at the center $D_S > m^2 = 1$. The boundary between regions B and C occurs where the flow speed is critical $M^2 = \beta$. Along the boundary between C and D, where the edge of the slow wave continuum arrives at the centre, S''_s vanishes. Therefore, this curve is also critical in the sense that $D_S \to \infty$ and the marginal solution for ξ_r becomes infinitely wiggly. We remark that because of the rather flat equilibrium profiles (45), region C, where the edge of the continuum is located at nonzero radius, is rather small.

The numerical results confirm the relevance of the local criteria. They also give concrete examples where the coupling with continua at different radial location gives rise to qualitatively new effects as compared with the static case, notably, the disappearance of one of the two Suydam sequences at slightly subcritical flow and the transfer of the other sequences to the edge of the slow wave continuum at the critical speed $M = \beta^{1/2}$. From the practical point of view, Fig. 3 gives rather a pessimistic view of the stability for equilibria with strong sheared flows, since the entire region where the flow is close to or above the critical speed is unstable.

IV.A Tracing eigenfrequencies

Using the shooting method with an implicit integration scheme for Eq. (3), we have traced eigenfrequencies in the complex plane, varying the flow speed v_{zo}, with the pressure gradient P_1 held at some fixed value. We have chosen two values of P_1, namely $P_1 = 0.10$, which is Suydam unstable already for $v_{zo} = 0$ and also $P_1 = 0$, which is stable in the absence of flows. These two values correspond to the upper and lower edges of Fig. 3.

We discuss the case $P_1 = 0.10$ in detail. We have traced the first few Suydam unstable modes starting at $v_{zo} = 0$. It is useful to note that the two sequences have eigenfunctions which extend primarily to one side of the Suydam surface, i.e., $r < r_0$ or $r > r_0$. These will be referred to as the "inner" and "outer" sequences. For vanishing or small flows, with the equilibrium profile used here, the inner sequence contains the fastest growing mode. The mode with the second largest growth rate belongs to the outer sequence.

Figure 4 shows the dependence of the growth rate on v_{ZO} for the three first modes of the inner sequence $\Gamma_+^{(n)}$. A striking result of Fig. 4 is that the growth rate of the most unstable mode is only weakly dependent on the flow speed, and, in particular, it is well behaved around the critical velocity $M^2 = \beta$. With a bit of hindsight, one can understand that this is what should happen, since the fastest growing mode is always one that violates the local assumption, and its growth rate is dependent more on global conditions than on the local properties in a small region around the resonant surface. As discussed in III.C.3, this will be particularly true near the critical speed.

Secondly, the growth rates of the higher modes in the inner sequence have maxima around the critical speed. This is readily understood since the asymptotic ratio of growth rates between successive members approaches unity at the critical speed as the imaginary part s of the characteristic exponent goes to infinity. If $\Gamma^{(0)}$ is only weakly dependent on v_{ZO}, $\Gamma^{(n)}$ for $n > 1$ will have a maximum around the critical speed, more pronounced the larger n is. As is shown by Fig. 4, the ratio never becomes unity for the first few modes and we reiterate that Eqs. (25) and (41) only apply in the limit of large positive n.

Thirdly, and most importantly, all the modes of the inner sequence remain unstable and become attached to the continuum edge as it crosses the Suydam surface at the critical speed ($v_{ZO*} \simeq 0.18$). The discrete slow modes remain unstable after the edge of the continuum reaches the center (at $v_{ZO} \simeq 0.21$). For even stronger flows, the growth rates decrease with increasing flow speeds. The decrease is more pronounced for the higher modes because s (the imaginary part of the characteristic exponent) is decreasing away from the critical speed.

Certain conclusions can be drawn about the asymptotic eigenvalues for slightly supercritical flows by examining the explicit formula for Δ^{is} obtained from the boundary layer analysis. Equation (39) shows that when $s \to +\infty$, the two sequences of discrete slow modes combine to form <u>one</u> geometric sequence

$$\hat{\Delta}_{\pm}^{2is} = e^{2i\chi} \, (R_+R_-)^{-1} \, [1 \pm O(e^{-\pi s})] \quad .$$

The phases are given by

$$2s \arg \hat{\Delta}_{\pm} = \log|R_+R_-| \pm O(e^{-\pi s}) \quad .$$

Consequently, either both or none of the two sequences should exist depending on the external solution, and if R_+ and R_- do not diverge, both sequences approach the real ω-axis.

Numerically, we find that both sequences of unstable slow modes exist at slightly supercritical flows, but the convergence to the asymptotic behaviour is of course slow. To illustrate this, and also support numerically the conclusion that the unstable slow modes belong to infinite sequences, we show in Table I: the growth rates $\mathrm{Im}\ \Delta^{(n)} = \mathrm{Im}(\omega^{(n)}-\omega^{(\infty)})$, the phases $\arg\ \Delta^{(n)}$, and the ratios $|\Delta^{(n)}/\Delta^{(n-1)}|$ for the infinite sequences at v_{zo} = 0.15, 0.19, and 0.25. The table shows that the ratios of successive frequency shifts converge to the theoretically expected values, $\exp(\pi/s)$ or $\exp(2\pi/s)$, respectively. We have also traced two high order modes (the ninth and tenth) and found that they keep their ordinal number in the sequences, switching from the inner Suydam sequence (region B), to the two combined sequences of slow modes at $r_s > 0$ (region C), to the single sequence of slow modes at the centre (region D).

Figure 5 shows the radial dependence of the first and third inner modified Suydam modes at v_{zo} = 0.15 (below the critical speed) and Fig. 6 shows the corresponding modes at v_{zo} = 0.19, when their eigenfrequencies accumulate toward the edge of the slow wave continuum at $r_s \simeq 0.39$. Figure 7 shows the same modes at v_{zo} = 0.25 when the continuum edge is located at the centre of the plasma. It is evident from these figures that the transition between the different regions B, C and D in Fig. 3 is a very gradual process for the first few modes of the inner sequence, involving slight shifts in where the eigenfunction oscillates faster and a gradual onset of the continuum singularities. Of course, for the higher modes, the changes will be more abrupt.

We now turn to the other Suydam sequence, the "outer" modes. Figure 8 shows the dependence of the growth rate on the flow speed. For subcritical flows, the growth rate of the first mode is essentially independent of v_{zo} and, similar to the inner sequence, the higher modes are destabilized by the flows. However, around the critical speed, $v_{zo*} \simeq 0.18$, the behaviour changes drastically and at certain flow speeds the growth rates go very quickly to zero. The modes disappear when the frequency crosses the real axis and the mode gets absorbed into a continuum. We see from Fig. 8 that the second and third roots disappear below the critical speed. Figure 9 shows the eigenfunctions at $v_{zo} = 0.15$ and Fig. 10 shows the third mode in the outer sequence at $v_{zo} = 0.1592$ where it is almost marginal. The sharp drop in growth rate and stabilization of this mode takes place when its frequency overlaps with the slow wave continuum near the outer edge of the plasma.

To understand why such an overlap must occur, it is helpful to study Fig. 11 where the radial dependence of the slow wave continuum frequency $\Omega_s(r)$ is shown together with Re ω for the first four modes of the inner (Re $\omega > \omega_0$) and outer (Re $\omega < \omega_0$) sequences when $v_{zo} = 0.14$, 0.16, 0.18 (critical), 0.20 and 0.22. At the critical speed, $\Omega_s(r)$ has its maximum at the $k_\parallel = 0$ surface $r_o \simeq 0.48$. For somewhat weaker flows, $\Omega_s(r)$ has a maximum point, $r = r_s$, between $r = r_o$ and the plasma boundary, $r = 1$. As the flow is increased, r_s moves inward and approaches the Suydam surface. It is clearly seen in Fig. 11 that as $r_s \to r_0$, all the outer Suydam modes, which have Re$\omega < \omega_0 = -(\vec{k}\cdot\vec{v})(r_0)$, eventually overlap with the continuum for $r > r_s$. The heavily drawn segment of the ω-axis in Fig. 11 shows the extension of the slow wave continuum outside the maximum point $r = r_s$ and the perpendicular bar marks $\Omega_s(r=1)$. Just above this frequency the eigenvalues of the outer sequence disappear one by one by approaching and being absorbed by the continuum on the real ω-axis. It is also clear why the higher order eigenvalues are absorbed first. For the amusement of the reader we show in Fig. 12 the first outer mode near marginal stability at $v_{ZO} = 0.205$, where the near singular behaviour at the continuum resonances is in clear evidence. Figure 13 shows the location of the first few point eigenvalues in the complex plane [relative to the Suydam frequency $\omega_0(v_{ZO})$] for the same flow speeds as in Fig. 11. The complex plot shows clearly (a) how the inner Suydam modes remain unstable but are captured by the edge of the continuum at the critical speed and (b) how the outer sequence disappears as the mode frequencies go through the cut along the real axis, soon after the overlap with the continuum near $r = 1$ occurs.

To summarize the numerical results for $P_1 = 0.10$; the inner, and most unstable, sequence of Suydam modes transforms into a sequence of unstable discrete slow modes around the critical speed, whereas the outer sequence becomes stabilized as a result of continuum overlap. We emphasize that the stability of the "local" modes really depends on the global profiles.

The scenario described above is modified in a more or less evident way when the pressure gradient is reduced so that the static equilibrium is stable. The growth rates of the first inner and outer modes for $P_1 = 0$ are shown in Fig. 14. Here, we only note that the growth rates of the purely flow driven instabilities are quite small, with a maximum of slightly less than $2\cdot 10^{-4}$ near the critical speed.

In particular, the growth rates of the unstable discrete slow modes become very small when the flow is well above critical.

V. Conclusion

We have studied flow and pressure driven instabilities of a cylindrical plasma with equilibrium flows. One important effect of the flow is a modification of the Suydam criterion, see Eq. (8). Sheared axial flows always decrease the maximum pressure gradient that can be stably confined. The modified Suydam criterion also indicates instability independent of the pressure gradient when the flow speed is slightly below a critical value, $M = \beta^{1/2}$, where the shear of the flow balances the propagation of slow waves on the sheared magnetic field. An interesting effect, qualitatively different from the static case, is that as the flow is increased past the critical speed, the unstable modes originally connected with the Suydam surface $k_\parallel = 0$ become attached to the edge of a slow wave continuum and the unstable modes continuously transform into discrete slow modes accumulating toward a finite Doppler shifted frequency. For the test case we have studied numerically there appears to be no stable region above the critical speed $M^2 = \beta$. However, the growth rates of the purely flow driven instabilities are rather small, and when the flow is well above critical, the growth rates are in the range typical of resistive instabilities.

Instability of the discrete slow modes can occur when two conditions are simultaneously satisfied, namely, (a) the edge frequency of a slow wave continuum gives an oscillatory solution for ξ_r, as expressed by Eqs. (31), (32) and (34), and (b) the edge frequency equals some continuum frequency at another radial location. On the basis of our numerical test case, this destabilization appears typical in the supercritical case, $M^2 > \beta$. We stress that, in contrast with the static case, overlap with continua makes the stability of the infinite sequences depend on the global profiles, and the local criteria are not sufficient for instability.

Finally, we are well aware that the predictions of the ideal MHD model will be significantly modified when, for instance, finite resistivity is taken into account. Another effect not included in MHD is wave particle resonance. Wave particle interaction is likely to be important when the flow is near or above critical, since the unstable modes are then strongly dependent on the propagation of slow waves and, for low β, these are essentially ion sound waves along the field lines. Consequently, kinetic effects must be considered in a more complete description. Furthermore, effects of toroidal geometry have not been addressed here. We believe, however, that an understanding of the intricacies exhibited by the simplest model is useful and may provide guidance for the application of more advanced models.

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References

[1] K. Brau, M. Bitter, R.J. Goldston, D. Manos, K. McGuire and S. Suckewer, Nucl. Fusion 23, 1643 (1983)

[2] R.C. Isler, L.E. Murray, E.C. Crume, C.E. Bush, J.L. Dunlap, P.H. Edmonds, S. Kasai, E.A. Lazarus, M. Murakami, G.H. Neilson, V.K. Papé, S.D. Scott, C.E. Thomas and A.J. Wootton, Nucl. Fusion 23, 1017 (1983)

[3] E. Hameiri, Phys. Fluids 26, 230 (1983)

[4] E. Hameiri and P. Laurence, J. Math. Phys. 25, 396 (1984)

[5] I. Hofmann, Plasma Phys. 17, 143 (1975)

[6] R.B. Paris and W.N.-C. Sy, Phys. Fluids 26, 2966 (1983)

[7] A. Bondeson and M. Persson, Phys. Fluids 29, 2997 (1986)

[8] T.A.K. Hellsten and G.O. Spies, Phys. Fluids 22, 743 (1979)

[9] E. Hameiri and J.H. Hammer, Phys. Fluids 22, 1700 (1979)

[10] E. Hameiri, "The stability of a particular MHD equilibrium with Flow", Courant Institute Report MF-85 (COO-3077-123) (1976)

III.C.3 Exchange of instabilities at the critical speed

It remains to show that just above the critical speed $M = \beta^{1/2}$, D_S is large and positive, so that the Suydam modes can be picked up by the edge of the continuum. We once again consider $v_\theta = 0$, write S as in Eq. (30) and note that at the critical speed the following relations hold at the resonant surface $k_\parallel = 0$ (indexed by sub-*)

$$\tilde{\omega}_* = \Sigma_* = 0 \quad , \quad \tilde{\omega}'_* = (\vec{k}\cdot\vec{v})'_* = \Sigma'_* \neq 0 \quad .$$

If the flow changes slightly $\vec{v} = \vec{v}_* + \delta\vec{v}$, the edge of the continuum (indexed sub-s) will move to a slightly different radius $r_S = r_* + \delta r$ determined by $\tilde{\omega}'_S = \Sigma'_S$. Taylor expansion gives

$$\delta r = (\vec{k}\cdot\delta\vec{v})'/(\Sigma'' - \tilde{\omega}'') \quad .$$

Thus, at the continuum edge

$$S''_S = 2\rho(B^2+\gamma p)\tilde{\omega}_S(\tilde{\omega}'' - \Sigma'') = -2\rho(B^2+\gamma p)(\vec{k}\cdot\vec{v})'(\vec{k}\cdot\delta\vec{v})'$$

$$= f^2(B^2+\gamma p)\ (\beta - M^2) \quad , \tag{42a}$$

hence S" is small and negative for slightly supercritical flows. In this case $F \simeq f\ \delta r$ is also small and D_S in Eq. (31) can be approximated as

$$D_S \approx \left(\frac{q}{rq'}\right)^2 \frac{B_\theta^2}{B_z^2} \frac{8\beta^2}{(1-\beta)(M^2-\beta)} \quad , \tag{42b}$$

Figure Captions

Fig. 1: Complex x and X plane, shown for the case arg Γ = $\pi/3$, β = 0.3 and M = -0.1.

Fig. 2: Schematic picture showing the location of the eigenfrequencies of the Suydam modes, with an accumulation point at $\omega_0 = -(\vec{k}\cdot\vec{v})_0$. The diagram assumes that the external solution is real valued (26b) and does not apply in the case of overlap with another continuum.

Fig. 3: Map of the P_1 - v_{zo} plane characterizing the equilibrium (45). Region A is stable according to the local criteria, region B is potentially unstable to the modified Suydam modes, region C to discrete slow modes at $r_s \neq 0$ (32) and region D to discrete slow modes at the centre (34).

Fig. 4: Logarithmic plot of the growth rates of the three first inner modes as functions of the flow speed for P_1 = 0.10.

Fig. 5: Radial dependence of the first (a) and third (b) inner Suydam modes for v_{zo} = 0.15 and P_1 = 0.10. The solid curve shows $Re(\xi_r)$ and the dashed line $Im(\xi_r)$. The solution was started so that ξ_r is real as $r \to 0$.

Fig. 6: Same as Fig. 5 but with v_{zo} = 0.19. The accumulation frequency is now the edge of the slow wave continuum at $r_s \simeq 0.39$.

Fig. 7: Same as Fig. 5 but with $v_{zo} = 0.25$. The accumulation frequency is now the slow wave continuum frequency at the centre of the plasma.

Fig. 8: Logarithmic plot of the growth rates of the three first outer modes as functions of the flow speed for $P_1 = 0.10$.

Fig. 9: Radial dependence of the first (a) and third (b) outer Suydam modes at $v_{zo} = 0.15$ and $P_1 = 0.10$.

Fig. 10: Radial dependence of the third outer mode at $v_{zo} = 0.1592$ and $P_1 = 0.10$. At this point the mode is near marginal stability. Note the modification of the mode near r=1 from the case $v_{zo} = 0.15$ in Fig. 9b by overlap with a continuum.

Fig. 11: The curve shows the radial dependence of the slow wave continuum frequency and the heavy segment of the ω-axis shows the extension of the slow wave continuum for $r > r_s$. The real part of the frequencies for the four first modes in the inner and outer sequences are indicated by crosses. In all figures, $P_1 = 0.10$ and in (a) $v_{zo} = 0.14$,(b) $v_{zo} = 0.16$, (c) $v_{zo} = 0.18$ (critical), (d) $v_{zo} = 0.20$, and in (e) $v_{zo} = 0.22$. The origin of the diagrams is at $r = r_0 \simeq 0.48$, $\omega = \omega_0(v_{zo})$. The abscissa runs between $\omega_0 \pm 3.11 \cdot 10^{-2}$.

Fig. 12: Radial structure of the first outer mode for $P_1 = 0.10$ and $v_{zo} = 0.205$ where it is close to marginal stability. Note the near singular behaviour at four separate continuum resonances.

Fig. 13: Location of eigenfrequencies in the complex plane for the same case as in Fig. 11. All diagrams are drawn to the same scale $|\omega_r - \omega_0| < 3.11 \cdot 10^{-2}$ and $|\omega_i| < 7.47 \cdot 10^{-3}$, thus the imaginary direction is magnified about 4 times.

Fig. 14: Logarithmic plot of the growth rate as a function of v_{zo} for the purely flow driven case $P_1 = 0$. Shown are the growth rates of the first inner and outer modes.

Table caption

Table I. Numerically computed values of $\Delta^{(n)} = \omega^{(n)} - \omega^{(\infty)}$ for the infinite sequences at various flow speeds and $P_1 = 0.10$.

(a) Modified inner (+) and outer (-) Suydam modes for $v_{zo} = 0.15$, $r_0 \simeq 0.48$, and $\omega^{(\infty)} \simeq \omega_0 = 0.1379656352$. Asymptotic ratio from theory $\exp(\pi/s) \simeq 3.0903$.

(b) The two combined sequences of slow modes at $v_{zo} = 0.19$, $r_s \simeq 0.39$, and $\omega^{(\infty)} = \omega_s \simeq 0.1752635981$. Asymptotic ratio from theory $\exp(\pi/s) \simeq 1.4215$.

(c) Single sequence of slow modes at $v_{zo} \simeq 0.25$, $r_s = 0$, and $\omega^{(\infty)} = \omega_s \simeq 0.2452847071$. Asymptotic ratio from theory $\exp(2\pi/s) \simeq 2.9962$.

n	Im $\Delta(n)$	arg $\Delta(n)$	$\lvert\Delta(n)/\Delta(n-1)\rvert$
1+	$7.04 \cdot 10^{-3}$	0.4140	
2+	$1.28 \cdot 10^{-3}$	0.2728	3.690
3+	$3.63 \cdot 10^{-4}$	0.2482	3.212
4+	$1.14 \cdot 10^{-4}$	0.2428	3.118
5+	$3.65 \cdot 10^{-5}$	0.2413	3.100
6+	$1.18 \cdot 10^{-5}$	0.2407	3.094
7+	$3.80 \cdot 10^{-6}$	0.2405	3.092
8+	$1.23 \cdot 10^{-6}$	0.2405	3.091
9+	$3.98 \cdot 10^{-7}$	0.2405	3.090
1−	$4.47 \cdot 10^{-3}$	2.8502	
2−	$9.71 \cdot 10^{-4}$	2.9008	3.817
3−	$3.02 \cdot 10^{-4}$	2.9095	3.097
4−	$1.01 \cdot 10^{-4}$	2.9068	3.018
5−	$3.37 \cdot 10^{-5}$	2.9038	3.046
6−	$1.10 \cdot 10^{-5}$	2.9023	3.071
7−	$3.59 \cdot 10^{-6}$	2.9017	3.083
8−	$1.16 \cdot 10^{-6}$	2.9015	3.087
9−	$3.77 \cdot 10^{-7}$	2.9015	3.089

Table 1a

n	Im $\Delta^{(n)}$	arg $\Delta^{(n)}$	$\left\|\Delta^{(n)}/\Delta^{(n-1)}\right\|$
1	$6.37\cdot10^{-3}$	0.2885	
2	$1.32\cdot10^{-3}$	0.1494	2.529
3	$4.99\cdot10^{-4}$	0.1088	1.925
4	$2.42\cdot10^{-4}$	0.0894	1.693
5	$1.35\cdot10^{-4}$	0.0782	1.575
6	$8.15\cdot10^{-5}$	0.0713	1.505
7	$5.22\cdot10^{-5}$	0.0667	1.462
8	$3.48\cdot10^{-5}$	0.0637	1.434
9	$2.38\cdot10^{-5}$	0.0619	1.416
10	$1.67\cdot10^{-5}$	0.0608	1.405
11	$1.18\cdot10^{-5}$	0.0602	1.400
12	$8.42\cdot10^{-6}$	0.0601	1.397
13	$6.03\cdot10^{-6}$	0.0602	1.398
14	$4.33\cdot10^{-6}$	0.0605	1.399
15	$3.11\cdot10^{-6}$	0.0608	1.402
16	$2.23\cdot10^{-6}$	0.0612	1.405
17	$1.59\cdot10^{-6}$	0.0616	1.407
18	$1.13\cdot10^{-6}$	0.0619	1.410
19	$8.07\cdot10^{-7}$	0.0622	1.412
20	$5.73\cdot10^{-7}$	0.0625	1.415
21	$4.06\cdot10^{-7}$	0.0626	1.416
22	$2.87\cdot10^{-7}$	0.0628	1.418
23	$2.03\cdot10^{-7}$	0.0630	1.419

Table 1b

n	Im $\Delta^{(n)}$	arg $\Delta^{(n)}$	$\lvert\Delta^{(n)}/\Delta^{(n-1)}\rvert$
1	$5.28 \cdot 10^{-3}$	0.3011	
2	$9.99 \cdot 10^{-4}$	0.2042	3.616
3	$2.92 \cdot 10^{-4}$	0.1874	3.145
4	$9.36 \cdot 10^{-5}$	0.1831	3.049
5	$3.07 \cdot 10^{-5}$	0.1817	3.021
6	$1.02 \cdot 10^{-5}$	0.1813	3.009
7	$3.38 \cdot 10^{-6}$	0.1807	3.003
8	$1.13 \cdot 10^{-6}$	0.1805	2.998
9	$3.77 \cdot 10^{-7}$	0.1808	2.994

Table 1c