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FINITE ELEMENTS APPLIED TO PLASMA WAVES

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Abstract

An introduction is given into two subjects of the theory of plasma waves and their numerical treatment with the finite element method : Linear propagation in an inhomogeneous medium and quasilinear theory. It is attempted to present the two subjects in such a way that the knowledge of plasma physics is not a prerequisite. With the information contained here a non-specialist should be able to appreciate at least the numerical part of the specialized research papers devoted to the subjects.

1. Introduction

As examples of the application of the finite element method we present two physically and mathematically very different problems. The first one is the (linear) propagation of small amplitude waves in an inhomogeneous, anisotropic dissipative medium. Mathematically, we deal with a simple linear boundary value problem. The operator in question can be either elliptic or hyperbolic in different regions of the solution domain. The second physical question we shall approach is the temporal evolution of particle and wave distribution functions under the influence of external fields and mutual interactions. This problem leads to a linear diffusion-advection equation for the particle distribution funtion in velocity space. The coefficients of the equation are partly given by integrals in wavenumber space over the wave distribution function. The latter in turn is given by an ordinary differential equation (ODE) in time whose coefficients depend on the particle distribution function. In plasma physics this set of equations is known as the Quasilinear Equations.

An important aim of the present lecture is to justify our conviction that the finite element method is an ideal numerical tool for a theoretical physicist. Very rarely the method requires ad-hoc decisions. For this reason experience is not as an important factor as when finite differences are used. After all the finite element method can be viewed as a method to derive robust conservative finite difference schemes. Once the basis- and test-functions are defined no further question should arise and perseverance only is needed. Moreover, it is difficult to evade a mathematically sane formulation of boundary conditions. Often such a formulation is not easy to find from the physics but easy to implement.

This lecture heavily leans on two recently published papers [1,2] to which we frequently shall refer. In the present written version of the lecture we try to give the information a non-plasma physicist or a student might need in order not to be terrified by Refs. 1 and 2. Chapter 2 deals with the linear wave propagation and Chap. 3 with the Quasilinear Theory.

2. Linear Wave Propagation

2.1 General wave equation

Assuming a harmonic time-dependence exp(-iωt) of a small-amplitude electromagnetic wave motion the pertinent equation can be written in the form

$$\text{rotrot}\ \vec{E} = \overset{\rightrightarrows}{\varepsilon} \cdot \vec{E} \qquad (1)$$

where $\vec{E}$ is the wave electric field. In a plasma, the dielectric tensor, $\overset{\rightrightarrows}{\varepsilon}$, can have the most different forms [1] depending on the plasma parameters and on the plasma model used. It can be a simple tensor, a complicated differential operator or even an integral operator. Usually it is highly anisotropic due to the fact that the plasma is immersed in a magnetic field $\vec{B}_0$ which impedes the free charged-particle motion across the field lines. Along the field lines, the particles can move freely and at small frequencies (electron inertia negligible) one sometimes even assumes $\vec{E} \cdot \vec{B}_o = 0$ because the electrons can shorten out an eventual parallel electric field component.

2.2 Plane slab geometry

In the past we have treated different geometrical arrangements with different plasma models leading to differential equations of various orders in one and two dimensions [1-4]. Here, we restrict ourselves to a plane slab geometry and to a plasma model where $\vec{E} \cdot \vec{B}_o = 0$ is assumed [1].

Specifically, we adopt the geometry shown in Fig. 1. The extension of the arrangement is infinite in the ignorable directions y and z. The plasma is situated in the domain $x_{p\ell} < x < x_{pr}$, and is inhomogeneous along x. On both sides it is separated from an ideally conducting wall by vacuum regions (III: $x_{s\ell} < x < x_{p\ell}$ and II: $x_{pr} < x < x_a$, I: $x_a < x < x_{sr}$). An antenna is placed into the vacuum on the right side at $x = x_a$ and carries an infinitely thin current sheet varying like $\exp(-i\omega t)$.

As the plasma is dissipative we may ask for the steady state solution of the driven wave motion. As y and z are ignorable coordinates we may search solutions of the form $\exp[i(k_y y + k_z z - \omega t)]$. The ultimate goal is to determine the dependence of the wave field on x.

2.3 Wave equation in the slab plasma [1]

As here our interest is rather numerical than physical the detailed form of the wave equation for $\vec{E} = (E_x, E_y)$ [1] is not of importance and we write the simplified form

$$\mathrm{rot}\,\mathrm{rot}\,\vec{E} = \left[\vec{\vec{\epsilon}}_0 - \frac{\partial}{\partial x}\vec{\vec{\alpha}}_x\frac{\partial}{\partial x}\right]\vec{E}, \tag{2}$$

where

$$\vec{\vec{\epsilon}}_0 = \begin{pmatrix} \epsilon_\perp & \epsilon_{xy} \\ -\epsilon_{xy} & \epsilon_\perp + k_y^2\alpha_1 \end{pmatrix}$$

$$\vec{\vec{\alpha}}_x = \begin{pmatrix} \alpha_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \alpha_1 = \alpha + i\nu . \tag{3}$$

The quantities $\varepsilon_\perp$, $i\varepsilon_{xy}$, α and ν are all real and depend in general on x. The notation adopted is the same as in ref. 1. The essential feature of eqs. (2) and (3) is the fact that, in general, $|\alpha|$ and $|\nu|$ are much smaller than $|\varepsilon_\perp|$ and $|\varepsilon_{xy}|$. In relation to the boundary conditions it is important to note that they even vanish at the plasma edge.

If we eliminated one component of $\vec{E}$ from eq. (2) we would obviously obtain a 4th order ODE. It is, however, less obvious that $\alpha = \nu = 0$ leads to a 2nd order ODE with, in general, singular coefficients. These singularities [1,6] are of the same nature as those occuring in ideal MHD [5], i.e.

$$E_x \sim 1/|x-x_s|, \; E_y \sim \log|x-x_s| \qquad (4)$$

and must be avoided by adding a small imaginary part to ε in order to get unique, physically meaningful results. For the time being we just note that the case $\alpha = \nu = 0$ might cause some numerical troubles (or be at the origin of an interesting physics).

It is important to note that, in general, one does <u>not</u> eliminate any component from eq. (2). There are several reasons for this. First, eq. (2) is transparent because $\vec{\vec{\varepsilon}}_o$ and $\vec{\vec{\alpha}}_x$ are directly given by a pertinent physical theory. A code based on eq. (2) hence can be modular and easy to check. Second, the weak form (see Sect. 2.6) possesses a nice symmetry and does not involve derivatives of $\vec{\vec{\varepsilon}}_o$ or $\vec{\vec{\alpha}}_x$. Such derivatives could only be obtained from lengthy calculations which do not necessarily come out correct in the first attempt. The third reason is that low order finite elements can be used if one wishes so.

2.4 Characteristic scales of solution

We need some preliminary information on the possible solutions of eq. (2) in order to formulate the boundary conditions in a physically meaningful way. Let us assume for the moment that all the coefficients of the equation $\varepsilon_\perp$, ε_{xy} and α_1, exhibit a much weaker space variation than the solution itself. It is then justified to search solutions of eq. (2) in the form $\exp(ik_x x)$. Using this ansatz in eq. (2) we find the dispersion relation [1]

$$a\,k_x^4 + b\,k_x^2 + c = 0, \tag{5}$$

where $a = -\alpha_1$, but b and c are finite when $\alpha_1 = 0$. Hence, in general, $|a| \ll |b|$, $|c|$ and the two solutions of eq. (5) can be approximated by

$$k_x^2 = \begin{cases} k_F^2 = -c/b, \\ \\ k_S^2 = -b/a, \end{cases} \tag{6}$$

with the property $|k_S^2| \gg |k_F^2|$. This latter result shows that eq. (2) is stiff.

The real part of both k_F^2 and k_S^2 can have either sign, positive or negative, representing propagatory and evanescent waves, respectively. If the short-wavelength mode k_S^2 is evanescent in one or several regions of the plasma (which usually is the case) it is almost impossible to solve eq. (2) by a shooting method. It is, on the other hand, straightforward to apply a global method as the one presented here.

2.5 Boundary conditions

In general we have a forth order ODE and hence we need four boundary conditions, two on each side of the plasma domain, Fig. 1. As this ODE reduces to a second order ODE when $|\alpha_1| \to 0$ the boundary conditions, when correctly formulated, also should reduce to two.

There is a problem in the formulation of the boundary conditions caused by the fact that the coefficient α_1 depends essentially linearly on the plasma temperature T which is usually assumed to tend to zero at the plasma edge. As $\alpha_1 \to 0$ implies $|k_s^2| \to \infty$, eq. (6), we have to be very careful in the formulation of the boundary conditions.

Let us first repeat those statements of ref. 1 which can be made without subtle considerations. The wave field in the vacuum is given by Maxwell's equations neglecting the displacement current because the frequency ω is supposed to be small. The field equation simply reads

$$d^2B_z/dx^2 - \kappa^2 B_z = 0, \tag{7}$$

where $\kappa^2 = k_y^2 + k_z^2$. The general solution obviously is

$$B_z = C_1\, e^{\kappa x} + C_2\, e^{-\kappa x}, \tag{8}$$

C_1 and C_2 being constants. Using boundary conditions on the conducting shell x_{sj} $(j = \ell, r)$, and matching conditions at the antenna site x_a one can derive the relations [1]

$$(\beta_j B_z - B_z')_{x = x_{pj}} = s_j, \; j = \ell, r \qquad (9)$$

where β_j depends on κ and the geometry, $s_\ell = 0$ and s_r is a function of k_y and k_z, and depends linearly on the amplitude of the antenna current. The prime ' denotes the spatial derivative d/dx. When expressed in terms of the electric field in the vacuum at $x = x_{pj}$

$$\left.\begin{aligned} i\frac{\omega}{c} B_z &= E_y' - i k_y E_x , \\ i\frac{\omega}{c} B_z' &= k_z^2 E_y , \end{aligned}\right\} \qquad (10)$$

eq. (9) starts to look like a boundary condition for eq. (2). The Maxwell equations in the vacuum, $\mathrm{rot}\vec{B} = 0$ and $(i\omega/c)\vec{B} = \mathrm{rot}\vec{E}$, and $E_z(x_{pj}) = 0$ have been used to obtain eq. (10).

Condition (9) holds on the outside of the plasma-vacuum interfaces which we now denote with $x_{p\ell} - \sigma$ and $x_{pr} + \sigma$, respectively, understanding that $\sigma > 0$ and eventually $\sigma \to 0$. In order to define carefully the boundary conditions we now deal with a transition region of finite width 2σ, Fig. 2, for the temperature or the coefficient α_1, respectively. As we expect troubles from formulating the boundary conditions at $x_{pr} + \sigma$ we try to formulate them for $x_{pr} - \sigma$ where α_1 is finite and integrate analytically over the transition region. Assuming the physically meaningful fields E_x and E_y to be finite in the transition region (finite energy!) one can integrate eq. (2) over it and finds

$$\left[\!\left[i k_y E_y \right]\!\right] = - \left[\!\left[\alpha_1 E_x' \right]\!\right] ,$$

$$\left[\!\left[i k_y E_x - E_y' \right]\!\right] = 0 , \qquad (11)$$

where E = E/outside - E/inside or, for the case in Fig. 2, $[\![E]\!] = E(x_{pr} + \sigma) - E(x_{pr} - \sigma)$. Further, from $(i\omega/c)\vec{B} = \mathrm{rot}\vec{E}$ we conclude that $[\![E_y]\!] = 0$. If we assume that E_x' is finite at $x = x_{pr} + \sigma$ we obtain from eqs. (9), (10) and (11) the two boundary conditions at $x = x_{pr} - \sigma$ we were looking for (the left boundary $x_{pl} + \sigma$ is treated analogously):

$$\left.\begin{aligned} \alpha_1 E_x' &= 0 , \\ \beta_j \left(E_y' + i k_y E_x \right) - k_z^2 E_y &= \frac{i\omega}{c} s_j . \end{aligned}\right\} \qquad (12)$$

These conditions have the required property that they reduce to one when $\alpha_1 = 0$.

2.6 Variational form

The finite element method is usually applied to a variational form of the differential equations to be solved. In the case of equations describing a dissipative system like ours, only a weak (Galerkin) variational form can be constructed. Let $\vec{G}(x) = (G_x(x), G_y(y))\exp[i(k_y y + k_z z - \omega t)]$ be an arbitrary test function in some functional space of sufficient regularity; in particular $d\vec{G}/dx$ must exist. The weak form is obtained by multiplying eq. (1) by $\vec{G}^*$ and integrating over the plasma domain. After the partial integration of the terms containing second derivatives one has

$$\int_{x_{pl}+\sigma}^{x_{pr}-\sigma} \left[\vec{G}^* \cdot \vec{\vec{\epsilon}}_0 \cdot \vec{E} + G_x^{*\prime} \alpha_1 E_x' - \mathrm{rot}\, \vec{G}^* \cdot \mathrm{rot}\, \vec{E} \right] dx = T(x_{pr}-\sigma) - T(x_{pl}+\sigma) \quad , \tag{13}$$

where

$$T = G_x^* \alpha_1 E_x' - G_y^* \left(E_y' - i k_y E_x \right) . \tag{14}$$

The boundary conditions, eq. (12), are obviously both natural and are imposed by using them in eq. (14):

$$T(x_{pj}) = - G_y^* \left(\frac{i\omega}{c} s_j + k_z^2 E_y \right) / \beta_j \quad . \tag{15}$$

2.7 Finite element discretization

The discretization procedure by standard finite elements is described in details in Ref. 1 and does not need to be repeated here. In particular, cubic Hermite (Method 1) and linear elements (Method 2) are discussed. Both methods are appropriate for the solution of eq. (2). Because of the existence of the short-wavelength mode k_S^2, eq. (6), the spatial mesh on which the discretization is undertaken might have to be very fine. For this reason, in our two-dimensional (2D) work [4], we have neglected the physics leading to the short-wavelength mode (cold plasma model), i.e. $\alpha_1 = 0$ but $\varepsilon_\perp \to \varepsilon_\perp + i\nu$. The resulting equation is then solved with finite hybrid elements. For the cold plasma model in ref. 2 we have compared the performance of the polluting [7] methods 1 and 2 with two non-polluting ones.

2.8 Results [1,2]

Using initially only cubic elements it is shown in Ref. 1 that all the well-known basic physical phenomena can be obtained numerically. The discretized system conserves energy. The wavelengths measured on the solution for E_x and E_y agree with those obtained from the dispersion relation, eq. (5). The energy flux agrees, where appropriate, with a WKB-analysis. The eigenfrequencies and eigenfunctions of a homogeneous plasma agree with their analytical values. Energy is exchanged between the long-wavelength mode (k_F^2) and the short-wavelength mode (k_S^2) at (confluence) points where $k_F^2 \approx k_S^2$. The process is known as linear mode conversion. If the confluence is caused by a resonance at $x = x_S$ of the long-wavelength mode, $k_F^2 = -c/b$, $b(x_S) = 0$, the global solution obtained with $\alpha_1 \neq 0$ oscillates around the solution given by the "cold" approximation $\alpha_1 = 0$ as predicted by analytical theory. High convergence rates are found (N^{-4}, N^{-5}, N^{-6}) with cubic elements. For the general case, $\alpha_1 \neq 0$, the cubic elements clearly perform much better than the linear elements which lead to quadratic convergence only. With two mesh points per wavelength the cubic elements produce an acceptable result.

In Ref. 2 we have given an a posteriori justification of our choice of hybrid finite elements[7] for the 2D ion-cyclotron code LION [4]. Working in 1D essentially, we have shown there that in the case of the cold plasma model (α_1 = 0 in the present 1D model) none of the mentioned approximations yields really satisfactory results on meshes of a size typical for a 2D calculation (500 meshpoints). Playing with non-equidistant meshes, however, can help. While being not really satisfactory the results obtained with the finite hybrid elements are nevertheless somewhat better than those obtained with cubic elements and, before all, the approximation with hybrid elements is by far the most efficient one with respect to computer memory and calculation time.

3. Quasilinear Theory

3.1 Motivation

The problem of the last chapter, where we have treated the propagation of small-amplitude waves, has led us to an equation with a structure which fairly often occurs in physics. In the present second example we shall encounter equations with a quite peculiar structure. As they are time-dependent and non-linearly coupled it is more difficult to make general statements of a purely mathematical nature. All the 1D and 2D codes we have produced in this domain [8-13] have to be run with a certain physical intuition although unbiased convergence tests are possible and must be done at different levels of the code development [8, 12, 13]. But even convergence tests are usually performed on physically important quantities and not on abstract mathematical norms. Without the pertinent physical background it is therefore difficult to judge the performance of these codes. Moreover, the physicist usually applies codes of this nature to extreme situations where the mesh resolution is just barely acceptable for the "new" physics he is looking for, and the results are certainly not converged in a strict mathematical sense. For all these reasons, in this chapter we cannot offer any general method which would work for a whole class of equations. We shall show, however, that one can easily apply the finite element method to equations which an unexperienced person would hardly dare to discretize with finite differencies. This chapter is meant as an encouragement to the reader who hesitates to attack his physical problem by means of a numerical method.

3.2 Kinetic equations

Under certain assumptions the electron distribution function f in a homogeneous plasma is governed by an equation of the schematic form

$$\frac{\partial f}{\partial t} = \vec{E}\cdot\frac{\partial f}{\partial \vec{v}} + \frac{\partial}{\partial \vec{v}}\cdot\vec{\vec{D}}\cdot\frac{\partial f}{\partial \vec{v}} + \sum_{\alpha} C_{e\alpha}(f) , \qquad (16)$$

where t and $\vec{v}$ stand for time and velocity, respectively, $\vec{E}$ for an external force (electric field) and $\vec{\vec{D}}$ is the quasilinear diffusion coefficient on which we shall comment shortly. $C_{e\alpha}(f)$ is a collision operator describing collisions between electrons and charged particles α (including the electrons themselves).

Different linear and nonlinear forms of $C_{e\alpha}(f)$ have been used in the literature [14]. For physical reasons one usually prefers the linear operator. For charged particle collisions it has the form of a Fokker-Planck operator. Apart from being Fokker-Planck, the operator has another pecularity which has its origin in the $1/w^4$ dependence of the cross-section for collisions between charged particles with a relative velocity w. If $\vec{\vec{D}} = 0$ there is a $v_{crit} > 0$ depending on $\vec{E}$ such that eq. (16) is advection dominated for $|\vec{v}| > v_{crit}$. Particles in this velocity domain, which are accelerated by the force $\vec{E}$, are called "runaways" because the collisions cannot prevent them from running away. It is this advection dominated region in conjunction with the quasilinear interaction mediated by $\vec{\vec{D}}$ which sometimes makes eq. (16) difficult to solve numerically. On the other hand, the equation is easy to solve if the collisions are the dominating physics.

The quasilinear interaction describes the "collisions" between particles and waves. A plasma always carries the most different kinds of electrostatic and electromagnetic waves and fluctuations which are due to thermal excitation and instabilities or may be injected. These waves can exchange momentum and energy with the particles. In idealized cases the effect of the waves on the particles formally turns out to be a diffusion in velocity space as indicated in eq. (16).

The structure of the quasilinear wave-particle interaction can best be demonstrated on a simplified 1D example [8]

$$\frac{\partial f(v,t)}{\partial t} = \frac{\partial}{\partial v}\int dk\, W(k,t)\ \delta(1-kv)\frac{\partial f}{\partial v}\ , \tag{17}$$

$$\frac{\partial W(k,t)}{\partial t} = W(k,t)\int \frac{dv}{k}\ \delta(1-kv)\frac{\partial f}{\partial v}\ . \tag{18}$$

Here W(k,t) is the (spectral) distribution in wavenumber (k) space of the waves interacting with the particles. The quasilinear diffusion tensor $\vec{\vec{D}}$, introduced in eq. (16) is a scalar here and has obviously the form

$$D(v,t) = \int dk\, W(k,t)\ \delta(1-kv)\ . \tag{19}$$

The pecularity of the interaction between the two "populations" f and W is mediated by the δ-function. Equations (17) and (18) satisfy particle, momentum and energy conservation laws (under the physical assumption that $f \to 0$, $\partial f/\partial v \to 0$ for $|v| \to \infty$)

$$\begin{aligned} &\int f\,dv &&= const, \\ &\int v f\,dv + \int kW\,dk &&= const, \\ &\int \frac{v^2}{2} f\,dv + \int W\,dk &&= const. \end{aligned} \tag{20}$$

In a numerical approximation these conservation laws should be satisfied as well as possible. This means that the expression $\delta(1-kv)\partial f/\partial v$ should be treated in the same way in both eqs. (17) and (18). In a multidimensional problem it is not clear a priori how this could be done because in eq. (17) the δ-function appears under a k-integral whereas in eq. (18) under a v-integral.

The situation changes once we are willing to use a finite element discretization. The starting point is a weak variational form. Let g(v) and Y(k) be test functions of sufficient regularity and construct the weak forms corresponding to eqs. (17) and (18) :

$$\int g \frac{\partial f}{\partial t} dv = - \int dk \int dv \frac{\partial g}{\partial v} W \delta(1-kv) \frac{\partial f}{\partial v} ,$$

$$\int Y \frac{\partial W}{\partial t} dk = \int dk \int dv \frac{1}{k} Y W \delta(1-kv) \frac{\partial f}{\partial v} . \qquad (21)$$

Now both δ-functions can be evaluated under a velocity integral and we can hope to preserve the symmetric structure of eqs. (17) and (18) in the numerical approximation.

3.3 Discretization in v and k space

When discretizing eqs. (17) or (21), respectively, we advantageously restrict ourselves to finite domains in v and k. In order to transform eqs. (17) and (18) into eq. (21) we have to assume $W\partial f/\partial v=0$

on the boundary of the v-space. This condition means that there is no particle flux across the boundary and the conservation laws, eq. (20), still hold. At t=0 the particles are usually distributed according to $\exp(-v^2/2)$ or at least they are concentrated in a finite domain. The domain is chosen such that at any time in the evolution f is negligible on its boundary. This is always possible for a pure quasilinear evolution as described by eq. (21) because the wavenumber domain is finite. A problem arises, however, in the case with an electric field, like eq. (16), because the field can accelerate particles up to the domain boundary.[12]

Let us now choose a non-equidistant velocity mesh,

$$-\infty < v_0 < v_1 < \ldots < v_i < \ldots < v_N = \infty \ . \qquad (22)$$

The most obvious and convenient finite element approximation of f(v,t) is given by linear elements :

$$f(v,t) = \sum_{j=0}^{N} f_j(t)\, \sigma_j(v) \qquad (23)$$

where $f_j(t)$ are the time-dependent expansion coefficients and $\sigma_j(v)$ are the linear basis functions[1] which take the value one at v_j and 0 on all the other mesh points.

As there are no $\partial/\partial k$ operators acting on the spectral distribution $W(k,t)$ a piecewise constant approximation is sufficient. On a mesh

$$k_0 < k_1 < \ldots < k_M \tag{24}$$

we can write the approximation

$$W(k,t) = \sum_{\ell=0}^{M-1} W_{\ell+\frac{1}{2}}(t)\, \gamma_{\ell+\frac{1}{2}}(k)\,, \tag{25}$$

where $\gamma_{\ell+\frac{1}{2}}(k)=1$ between k_ℓ and $k_{\ell+1}$ and 0 elsewhere. Had we only to deal with the most simple system of equations, eqs. (17) and (18), it is clear that the k-mesh, eq. (24), would be chosen such that $k_\ell=1/v_{i_0-1}$ with an appropriately chosen i_0. The resulting discretized eq. (21) would be extremely simple and transparent. In physically interesting situations, however, which are either 2D with a delta function of the form $\delta(1-\vec{k}\cdot\vec{v})$ or 1D but including more than one quasilinear interaction, say $\delta(1-kv)$ and $\delta(3-kv)$, it is not possible to establish a one-to-one correspondence between the meshes in k and v. We therefore do not assume such a correspondence here.

Choosing as test functions $g(v)=\sigma_i(v)$ and $Y(k)=k\,\gamma_{\ell+\frac{1}{2}}(k)$ one derives a discretized form of eq. (21)

$$\begin{aligned} \sum_j A_{ij}\dot{f}_j &= \sum_j B_{ij} f_j\,, \\ \dot{W}_{\ell+\frac{1}{2}} &= \Gamma_{\ell+\frac{1}{2}} W_{\ell+\frac{1}{2}}\,, \end{aligned} \tag{26}$$

where

$$A_{ij} = \int \sigma_i \, \sigma_j \, dv,$$

$$B_{ij} = \sum_\ell W_{\ell+\frac{1}{2}} B_{ij}^{\ell+\frac{1}{2}} \; ; \quad B_{ij}^{\ell+\frac{1}{2}} = - \int_{k_\ell}^{k_{\ell+1}} \frac{dk}{k} \left(\sigma_i' \sigma_j' \right) \Big|_{v=\frac{1}{k}} , \tag{27}$$

$$\Gamma_{\ell+\frac{1}{2}} = \sum_j \dot{\Gamma}_{\ell+\frac{1}{2}}^{j} f_j \; ; \quad \Gamma_{\ell+\frac{1}{2}}^{j} = \frac{2}{k_{\ell+1}^2 - k_\ell^2} \int_{k_\ell}^{k_{\ell+1}} \frac{dk}{k} \left(\dot{\sigma}_j' \right) \Big|_{v=\frac{1}{k}}$$

Here the dot $^{\bullet}$ and the prime $'$ denote derivatives with respect to time and velocity, respectively.

The eq. (26) exactly conserves particles and, due to the trick of using $k \, \gamma_{\ell+\frac{1}{2}}$ as a test function, even momentum. Let us prove this statement.

In accord with eq. (20), the definitions of the approximate particle number $\mathcal{N}$ and of the total momentum $\mathcal{P}$ are [8]

$$\mathcal{N} = \sum_{ij} A_{ij} f_j ,$$

$$\mathcal{P} = \sum_{ij} v_i A_{ij} f_j + \sum_\ell \frac{k_{\ell+1}^2 - k_\ell^2}{2} W_{\ell+\frac{1}{2}} . \tag{28}$$

Here, use has been made of

$$1 = \sum_j \sigma_j(v) ,$$

$$v = \sum_j v_j \, \sigma_j(v) . \tag{29}$$

With the help of eqs. (26) and (27) and with $\sum v_j \sigma_j' = 1$ one proves $\dot{\mathcal{N}} = \dot{\mathcal{P}} = 0$.

Energy is not exactly conserved by the present numerical scheme. Conservation could be achieved with higher order elements.

3.4 Time discretization

In the past we have used both a four-level[8] and a two-level[12] integration formula. Using indices 1, 2, 3 and 4 to denote times t_1, t_2, t_3 and t_4, respectively, we can schematically write the formulae as

$$\begin{aligned} \vec{\vec{A}} \cdot \frac{\vec{f}_3 - \vec{f}_1}{t_3 - t_1} &= \vec{\vec{B}}_2 \cdot \frac{\vec{f}_3 + \vec{f}_1}{2} , \\ \frac{W_4 - W_2}{t_4 - t_2} &= \Gamma_3 \, \frac{W_4 + W_2}{2} , \end{aligned} \tag{30}$$

and

$$\begin{aligned} \vec{\vec{A}} \cdot \frac{\vec{f}_2 - \vec{f}_1}{t_2 - t_1} &= \theta \, \vec{\vec{B}}_2 \cdot \vec{f}_2 + (1-\theta) \, \vec{\vec{B}}_1 \cdot \vec{f}_1 , \\ \frac{W_2 - W_1}{t_2 - t_1} &= \theta \, \Gamma_2 \, W_2 + (1-\theta) \, \Gamma_1 \, W_1 . \end{aligned} \tag{31}$$

Here θ is a number between 0.5 and 1, typically $\theta = 0.55$. Formula (30) is linear in the unknowns and for this reason much simpler to solve then formula (31). It has, however, not been working sufficiently well in situations where the time step had to be changed rapidly.[11, 13] The conservation laws were not well verified. We therefore had to resort to eq. (31) which necessitates a Picard iteraction at each time step. Note that the conservation laws $\mathcal{N}_2=\mathcal{N}_1$ and $\mathcal{P}_2=\mathcal{P}_1$ exactly hold for eq. (31).

3.5 Two-dimensional problems

The ideas outlined in Sect. 3.3 and 3.4 can be applied to more-dimensional situations. In a non-magnetized plasma, for instance, the quasilinear interaction is mediated by a δ-function of the form $\delta(1-\vec{k}\cdot\vec{v})$ or, in general

$$\delta(\omega(\vec{k}) - \vec{k}\cdot\vec{v}) \tag{32}$$

where $\omega(k)$ is determined by the nature of the waves in question. In a magnetized plasma, on the other hand, it takes the form

$$\delta(\omega(\vec{k}) - k_{\parallel} v_{\parallel} - n\omega_c) \tag{33}$$

where $k_{\parallel}$ and $v_{\parallel}$ are the components of $\vec{k}$ and $\vec{v}$ parallel to the ambient magnetic field (the other components are $k_{\perp}$ and $v_{\perp}$), ω_c is the cyclotron frequency of the particle and n is an integer taking the values 0 and -1 in the examples we have treated in the past.[11,13]

For the dicretization of the k-space we have used so far 3 different methods. Originally[8] we used equally spaced discrete waves as commonly used in theoretical physics ("waves in a box"). This method eventually had to be rejected because it made inefficient use of the computer memory. The information i, j, $\ell+{}^1/_2$, $B_{i,j}{}^{\ell+1}\neq 0$ and $\Gamma^{j}_{\ell+\frac{1}{2}} \neq 0$ had to be stored for too many waves ℓ.

Subsequently a piecewise-constant finite-element approach on a non-equidistant rectangular mesh has been adopted. This is a straightforward generalization of eq. (25). The integrals $\int d^2k$ usually have been evaluated with a 9-point formula using equal weights. The integration points then play the role of the discrete waves. There is, however, much less information to be stored on the whole because $B_{i,j}{}^{\ell+1/2}$ and $\Gamma^{j}_{\ell+\frac{1}{2}}$ contain the information of 9 points. As the piecewise constant approach leads to a diffusion equation like eq. (17) with a piecewise-constant diffusion coefficient the lowest order integration scheme is appropriate. But as the two discretized "populations" f and w should be influenced by each other in all the cells where analytically they do we settled for a 9-point formula with equal weights.

In the hope to gain in the smoothness of the solution we recently made experiments[2] with a piecewise-linear finite-element ansatz for $W(\vec{k})$ on a rectangular mesh. The additional regularity of the solution led to overshoots which could not be tolerated on the coarse meshes imposed to us by the limited memory space. In the context of our codes we conclude therefore that the piecewise-constant ansatz for $W(\vec{k})$ is the best.

If we now discuss the discretization in v-space it must be born in mind that in general the kinetic equation governing the evolution of f has the form of eq. (16), and therefore the quality of the discretization might strongly depend on the parameters (or the physics) one looks for. We do not dare to declare one method as better than the other. All we might have are certain unpublishable feelings. For this reason we just mention the three different methods we have used with some success in the past.

For a non-magnetized pure quasilinear situation[8] we have used a standard piecewise-linear approximation[15] on a triangularized mesh.

In the magnetized case we were able to do reasonable physics with what we called a $1^1/_2$D approximation[12]

$$f(v_\parallel, v_\perp, t) = \sum_{j=0}^{N} f_j(t)\, \sigma_j(v_\parallel) \frac{1}{2\pi\, T_j(t)} \exp\left[-\frac{v_\perp^2}{2\, T_j(t)}\right], \tag{34}$$

with unknown $f_j(t)$, $T_j(t)$. Two nonlinear equations for f_j and T_j, respectively, can be obtained by constructing weak forms using $\sigma_i(v_\parallel)$ and $v_\perp^2\sigma_i(v_\parallel)$ as test functions. Evidently, this approach has its origin in the physical intuition which says that the detailed form of f is only needed in the $v_\parallel$-direction because of eq. (33). In many situations the ansatz indeed works quite well. In other situations the nonlinearity introduced by eq. (34) in conjuction with numerical overshoots, mostly due to advection by E, leads to code failure.

For this latter reason we have recently undertaken the construction of a new 2D code[13] based this time on a piecewise-linear approximation for both $v_{\parallel}$ and $v_{\perp}$ on a domain with rectangular subdivisions. We can, however, already now say that we shall be quite limited in resolution.

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FIGURE CAPTIONS

Fig. 1 Configuration of the slab model. The plasma is situated between two vacuum regions (I + II and III) which are delimited by ideally conducting walls. An idealized antenna is situated in one of the vacuum regions.

Fig. 2 Transition of T or α_1, respectively, from a finite value to zero at the plasma edge, $x=x_{pr}+\sigma$. The boundary conditions at $x=x_{pr}-\sigma$, eq. (12), are derived under the assumption that $\sigma \to 0$.

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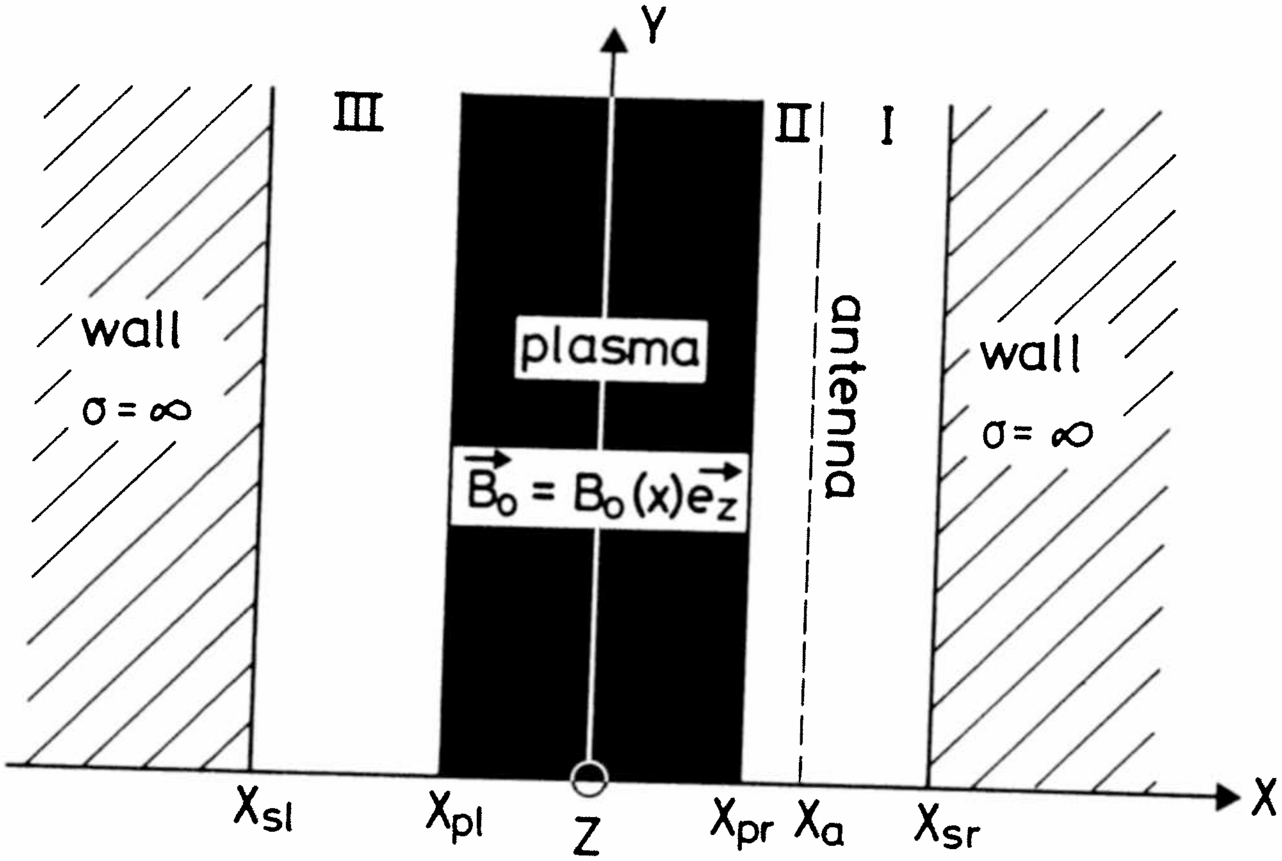
Y
III
II
I
wall
σ = ∞
plasma
antenna
B₀ = B₀(x)e_z
wall
σ = ∞
X_sl
X_pl
Z
X_pr
X_a
X_sr
X

Fig. 1

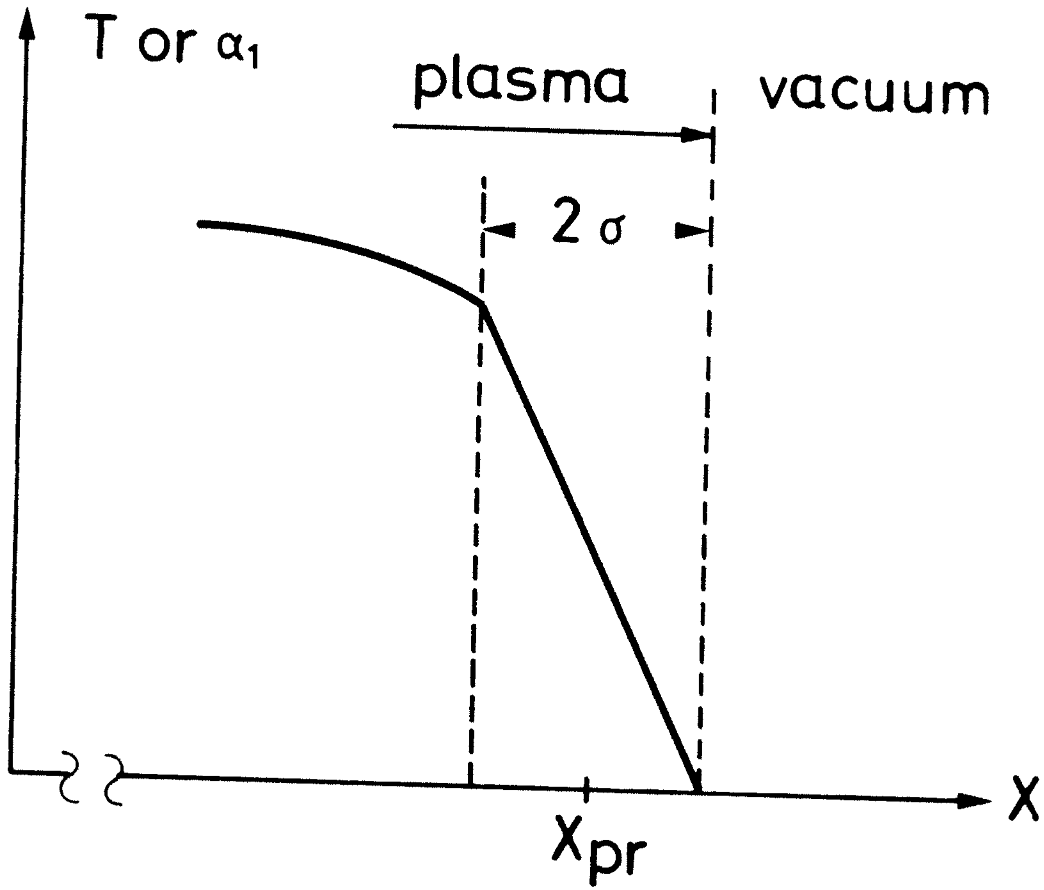

Fig. 2