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ABSTRACT

The motion of a single particle under the influence of the ponderomotive force directed perpendicular to the external magnetostatic field is analyzed. By solving the exact equation of motion for a specific applied electromagnetic field, the resultant ponderomotive drift is compared with the prediction of oscillation center theory. The regime of validity of oscillation center theory is discussed. It is shown that for certain values of the amplitude and frequency of the electromagnetic field, the particle motion is unstable and therefore the concept of a ponderomotive force is meaningless.

I. INTRODUCTION

It is well known that a high frequency electromagnetic field may exert on a charged particle a time averaged (ponderomotive) force. The resultant particle motion is generally analyzed by means of oscillation center theory,[1] in which the position of the particle is split into slowly and rapidly varying parts, $\vec{x} = \vec{x}_s + \vec{x}_f$. Here $\vec{x}_s$ describes the time averaged motion of the oscillating center, and $\vec{x}_f$ the fast motion about the oscillating center. Assuming that $|\vec{x}_f| \ll |\vec{x}_s|$, the equation of motion can be separated according to slow and fast time scales. Solving the equation of motion on the slow time scale yields the time averaged motion of the particle under the influence of the ponderomotive force. In the presence of an external magnetostatic field $\vec{B}_o$, the ponderomotive force parallel to $\vec{B}_o$ gives rise to acceleration in the direction along $\vec{B}_o$, while the perpendicular ponderomotive force leads to a mutually perpendicular particle drift.

The concept of the ponderomotive force has received much attention in the field of plasma physics due to its application to the confinement and acceleration of plasmas.[1] Recent studies have investigated, for example, rf plugging[2,3] and low frequency mode stabilization[4,5] in mirror devices. Since the collective nature of a plasma under the influence of an rf field may play an important role in determining the plasma dynamics, a single particle treatment may not, however, always be appropriate. Indeed, it has been shown that the ponderomotive force obtained from a fluid description of the plasma[6,7] does not agree with that obtained from the oscillation center theory described above. A detailed discussion of the origin of this fundamental difference between the fluid and single particle ponderomotive forces may be found in Ref. 8.

It may be concluded from a recent paper[9] that the fluid result for the slow drift caused by the perpendicular ponderomotive force can be obtained from single particle theory. Using the oscillation center approximation, a drift in addition to that determined by other workers[1] was apparently derived, thereby rationalizing the difference between fluid and single particle theory. That this additional drift may not be obtained from single particle theory can be shown most convincingly by comparing the fluid result with the solution of the exact equation of motion for a single particle (i.e., without using the oscillation center approximation). In the present paper, a specific example of single particle motion under the influence of a perpendicular ponderomotive force is considered. By numerical integration of the equation of motion, we compare the exact solution with that obtained from the correct application of oscillation center theory. We are thus able to show that single particle theory is not capable of yielding the fluid result.

The use of oscillation center theory to yield the slow motion of the particle tacitly assumes that the amplitude of the fast motion is small. However, as this paper will show, the fast motion can in fact become unstable for certain values of frequency and amplitude of the rf field. Under these conditions, the concept of a ponderomotive force is, of course, meaningless.

II. MATHEMATICAL FORMULATION

A. Equation of motion

In this paper we shall consider the motion of a single particle of charge q and mass m under the combined influence of an electromagnetic field and a uniform magnetostatic field ($\vec{B}_0 = \hat{e}_z B_0$). For the sake of simplicity we shall consider the specific example for which the total electric and magnetic fields may be written as

$$\vec{E}(x,t) = -\hat{e}_y B_0 \dot{\tilde{b}} x \quad ; \quad \vec{B}(t) = \hat{e}_z B_0 (1 + \tilde{b}) , \qquad (1)$$

where $\tilde{b}$ = b cos ωt and b is constant. We shall ignore the self-consistent contribution of the total fields due to the motion of the charged particle.

The equation of motion for the particle in the above fields is

$$\ddot{x} = \Omega (1 + \tilde{b}) \dot{y} , \qquad (2)$$

$$\ddot{y} = -\Omega \dot{\tilde{b}} x - \Omega (1 + \tilde{b}) \dot{x} , \qquad (3)$$

$$\ddot{z} = 0 , \qquad (4)$$

where the cyclotron frequency $\Omega = qB_0/m$. The particle motion perpendicular to $\vec{B}_0$ is thus described mathematically in terms of two coupled differential equations. It should be noted that these equations are nonlinear in the rf field amplitude but, for the present choice of fields, linear in the particle position.

In the following, we shall normalize quantities by adopting Ω^{-1} and p Ω^{-1} as time and distance units. The most appropriate choice for the normalizing velocity p will become apparent in Sec. II C. We shall use the normalized quantities:

$$X = \Omega x/p \; ; \quad Y = \Omega y/p \; ; \quad T = \Omega t \quad \text{and} \quad \nu = \omega/\Omega \, . \tag{5}$$

B. Oscillation center approximation

Splitting the particle motion perpendicular to $\vec{B}_0$ into slowly and rapidly varying parts,

$$\vec{X}_\perp = \hat{e}_x (X_s + X_f) + \hat{e}_y (Y_s + Y_f) \, , \tag{6}$$

Eqs. (2) and (3) may be separated according to slow and fast time scales, as is usual in oscillation center theory.[1] The fast time scale equation is solved to O(b) to yield

$$\left. \begin{aligned} X_f &= - \frac{2 b X_s}{(1-\nu^2)} \cos \frac{(1-\nu)T}{2} \cos \frac{(1+\nu)T}{2} \, , \\ Y_f &= \frac{2 \nu b X_s}{(1-\nu^2)} \cos \frac{(1-\nu)T}{2} \sin \frac{(1+\nu)T}{2} \, . \end{aligned} \right\} \tag{7}$$

In general, a linear combination of sin T and cos T must be added to solutions (7) in order to satisfy the initial conditions for the particle fast motion. However, this yields no contribution to the slow (time averaged) motion, which is given to $O(b^2)$ by

$$\frac{dX_s}{dT} = 0 \quad ; \quad \frac{dY_s}{dT} = -\frac{\nu^2 b^2}{2(1-\nu^2)} X_s \quad . \tag{8}$$

From Eq. (8) it can be seen that the ponderomotive force perpendicular to $\vec{B}_0$ gives rise to a time averaged drift in the y direction. It may be readily shown that this drift can be written, as is usual, in terms of the single particle ponderomotive potential:[1]

$$\frac{dY_s}{dT} = \frac{d\phi_p}{dX_s} \quad \text{where} \quad \phi_p = -\frac{|\vec{E}(X_s,T)/B_0|^2}{4(1-\nu^2)} \quad . \tag{9}$$

C. Exact solution

Equations (2) and (3) may be numerically integrated to yield the exact solution for the particle motion perpendicular to $\vec{B}_0$. However, using the Hamiltonian formulation, since y is an ignorable coordinate the y component of the canonical momentum, $P_y \equiv m\, p$, is a constant of motion. It is straightforward to show that Eqs. (2) and (3) may therefore be written, in normalized form:

$$\frac{d^2X}{dT^2} = (1+\tilde{b}) - (1+\tilde{b})^2 X \quad , \tag{10}$$

$$\frac{dY}{dT} = 1 - (1+\tilde{b}) X \quad . \tag{11}$$

Equation (10), which describes the x component of the particle motion, may be recognized as an inhomogeneous Hill equation.

Equations (10) and (11) must be supplemented by an appropriate set of initial conditions. For definiteness, we consider

$$X(0) = 1 \quad ; \quad \frac{dX(0)}{dT} = 0 \quad \text{and} \quad Y(0) = 0 \quad . \tag{12}$$

This initial value problem may be solved by standard Runge-Kutta numerical integration techniques, to yield the exact nonlinear motion of the particle under the influence of the electromagnetic field.

III. RESULTS

A. Particle motion

The initial value problem defined by Eqs. (10) - (12) has been numerically solved for a number of different values of ν and b. The solution for a particular example ($\nu = 0.9$, $b = 0.1$) is shown in Fig. 1. Under the influence of the fields, the particle exhibits a fast orbital motion that is amplitude modulated at an intermediate frequency ν_m. The slow time scale (ponderomotive) drift of the particle in the (negative) y direction is also apparent.

The effect of changing the frequency ν of the electromagnetic field is demonstrated in Fig. 2. For the six cases shown, the amplitude of the field is b = 0.1. It may be observed that the particle motion is strongly modified by a change of frequency. For certain of the values shown (ν = 0.5, 1.0, 2.0), the motion is unstable; the amplitude of the fast component increasing without bound. Comparing the cases ν = 0.9 and 1.1, it may be seen that the direction of the ponderomotive drift changes sign as the frequency crosses the cyclotron frequency (ν = 1). For both of these cases the amplitudes of the fast motion and ponderomotive drift are larger than for the case ν = 0.2 for which the frequency is further from the cyclotron frequency.

Figure 3 illustrates the effect on the particle motion of a change in the field amplitude while keeping the frequency constant (ν = 0.9). As the field amplitude is increased, the amplitude of both the fast motion and the ponderomotive drift increases. For b = 0.6, the motion of the particle is no longer stable. Also evident from Fig. 3 is that as b is increased from 0.1 to 0.5, the frequency ν_m at which the fast motion is modulated decreases.

B. Comparison with oscillation center theory

Except for particular values of field frequency ν, the numerically calculated particle motion agrees qualitatively with that predicted by oscillation center theory provided that the field amplitude b is sufficiently small. A more quantitative comparison can be obtained from Figs. 4 - 6.

In Fig. 4 is plotted, for three values of ν, the modulation frequency ν_m, normalized to the values $(1 - \nu)/2$ determined from oscillation center theory, as a function of b. For $b \ll 1$, $\nu_m \simeq (1 - \nu)/2$ as is expected. However, as b is increased, a sharp drop in ν_m is observed, being most pronounced for ν close to 1. The decrease of ν_m to zero results from approaching the instability boundary: further increase in the field amplitude increases the growth rate of the unstable motion.

The effect of the field amplitude on the magnitude of the ponderomotive drift velocity is shown in Fig. 5. In this figure are plotted values of the ponderomotive drift velocity dY_s/dT, for three values of ν, obtained from both numerical integration and oscillation center theory, Eq. (8). Again it may be seen that for $b \ll 1$ the two calculations yield similar results. However, as b is increased so does the discrepancy between the results. When the particle motion is unstable, the concept of a ponderomotive drift becomes meaningless.

Figure 6 shows a plot, for b = 0.2, of the frequency dependence of the value of the ponderomotive drift velocity obtained from numerical integration, normalized to that calculated from oscillation center theory. The points plotted in this figure are for values of ν and b for which the fast motion of the particle is stable. It is found that unstable orbits occur in the vicinity of

$$\nu = \frac{2}{m} \qquad \text{for integer m.} \qquad (13)$$

The width of these regions of instability increases with the value of b, and is largest for small values of m.

From Fig. 6 it may also be seen that the values of the ponderomotive drift velocity obtained by the two methods diverge if

$$\nu = \frac{1}{n} \qquad \text{for integer n.} \tag{14}$$

Clearly for these values of frequency the application of oscillation center theory is not valid. Note that the ponderomotive drift velocity arising from stable orbits near the frequencies given by Eq. (13), with m odd, are well approximated by oscillation center theory.

The origin of the regions of instability can be appreciated by noticing that for b sufficiently small, Eq. (10) becomes an inhomogeneous form of the Mathieu equation:[10]

$$\frac{d^2X}{dz^2} + (a - 2q\cos 2z)X = 0 , \tag{15}$$

where $z = \frac{\nu T}{2}$, $a = \frac{4}{\nu^2}$ and $q = -\frac{4b}{\nu^2}$.

The well known stability diagram for the Mathieu equation reveals unstable regions which, for small q, are in the vicinity of $a = m^2$ for integer m. This condition can be seen to be equivalent to Eq. (13).

In oscillation center theory, the equation of motion is solved to second order in b to obtain the ponderomotive drift. However, if b is non-negligible, higher order terms may yield a significant contribution. These higher order contributions to the ponderomotive drift are particularly important if the frequency of the applied electromagnetic field is a sub-harmonic of the cyclotron frequency. The nonlinearity of the equation of motion leads, in this case, to oscillatory motion at the cyclotron frequency, resulting in a resonant ponderomotive contribution. Thus if Eq. (14) is approximately satisfied, oscillation center theory will yield an inaccurate calculation of the full nonlinear ponderomotive force.

IV. CONCLUSIONS

By analyzing in detail a particular example, the motion of a single particle under the influence of an electromagnetic field has been investigated. It has been shown that the particle motion is strongly modified by a change in the electromagnetic field amplitude (if sufficiently large) or the wave frequency (if in the vicinity of the cyclotron frequency).

Comparison has been made of the ponderomotive drift obtained from oscillation center theory and from numerical integration of the equation of motion. Within the regime of validity of such a perturbation approach, the oscillation center theory yields a good approximation to the exact solution. However, it has been shown that even for a modest field amplitude, the particle motion is unstable for frequencies in the vicinity of twice the cyclotron frequency and its sub-harmonics. In addition, the ponderomotive drift velocity calculated from numerical integration diverges not only if the applied frequency equals the cyclotron frequency, but also at the sub-harmonics of the cyclotron frequency. In the vicinity of these frequencies, the application of oscillation center theory is therefore clearly inappropriate.

The results of the present study reinforce the assertion of Ref. 8 that the single particle and fluid (collective) ponderomotive forces are fundamentally different. It has been shown that oscillation center theory is complete (within the limits of its application) in describing single particle behavior under the influence of the perpendicular ponderomotive force. It is not possible to obtain the fluid result[6,7] from single particle theory (whether oscillation center or exact solution).

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Figure Captions:

FIG. 1: Particle motion for $\nu = 0.9$ and $b = 0.1$.

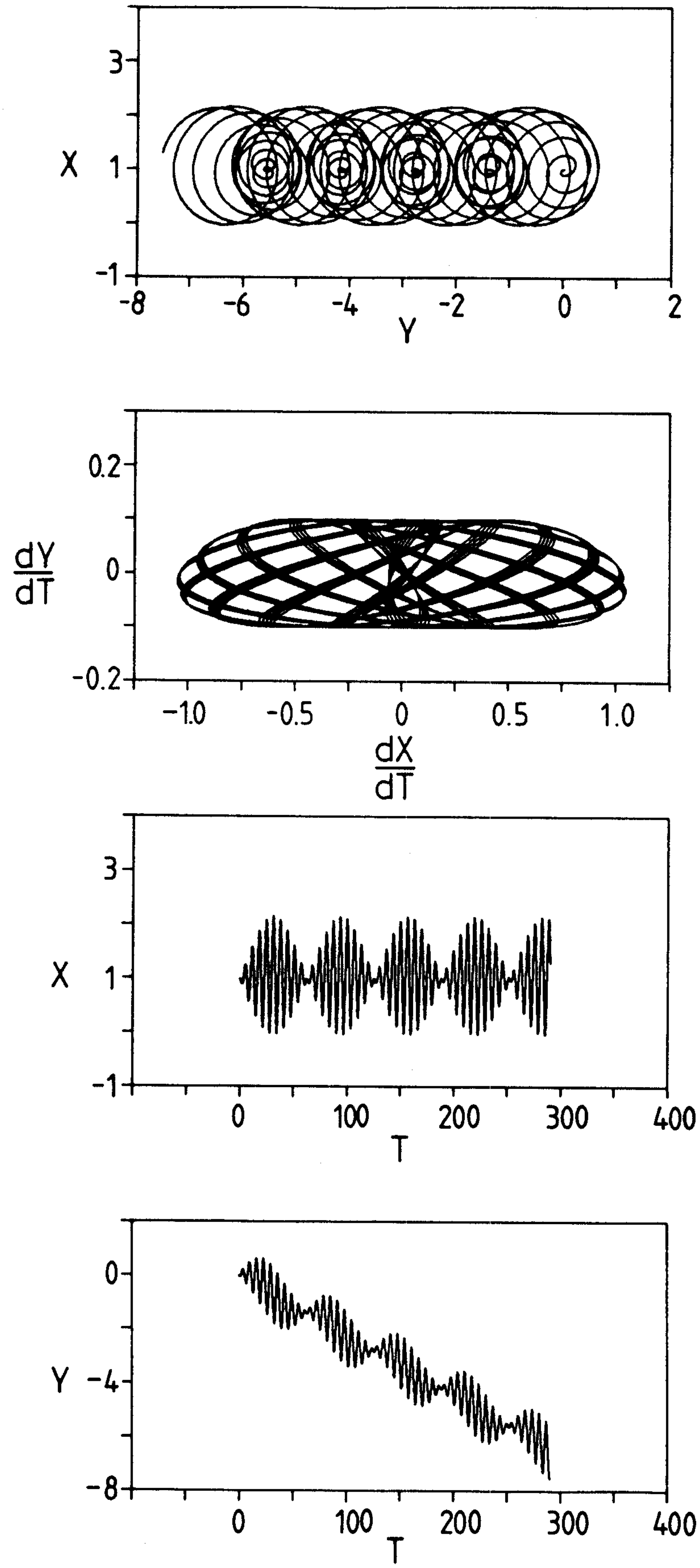
FIG. 2: Particle motion for $b = 0.1$ and for the six indicated values of ν. The three examples on the left represent stable orbits, while the three on the right are unstable. The total time is the same ($T = 290$) for all examples.

FIG. 3: Particle motion for $\nu = 0.9$ and for the four indicated values of b. The total time is the same ($T = 290$) for all examples.

FIG. 4: Normalized modulation frequency as a function of b, for three values of ν: o ,$\nu = 0.85$; × ,$\nu = 0.9$; + ,$\nu = 0.95$.

FIG. 5: Ponderomotive drift velocity as a function of b, for three values of ν: o ,$\nu = 0.7$; × ,$\nu = 0.8$; + ,$\nu = 0.9$. The solid lines are the corresponding curves obtained from oscillation center theory, Eq. (8).

FIG. 6: Normalized ponderomotive drift velocity as a function of ν for $b = 0.2$. The vertical dashed lines indicate values of ν satisfying Eq. (14), while the vertical solid lines indicate values of ν satisfying Eq. (13) for odd m.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X 1
-1
-8 -6 -4 -2 0 2
Y
0.2
dY/dT 0
-0.2
-1.0 -0.5 0 0.5 1.0
dX/dT
3
X 1
-1
0 100 200 300 400
T
0
Y -4
-8
0 100 200 300 400
T

Figure 1

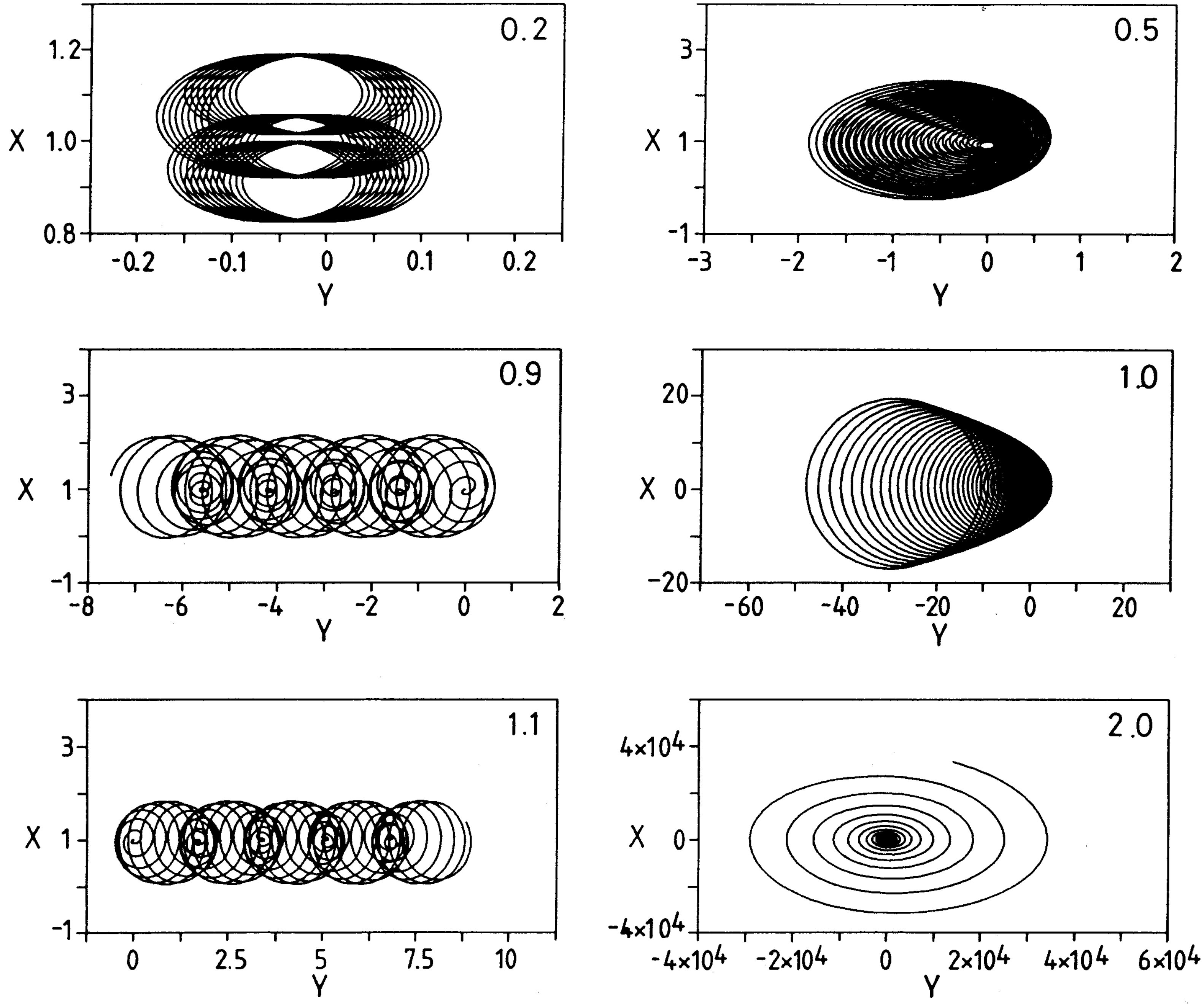
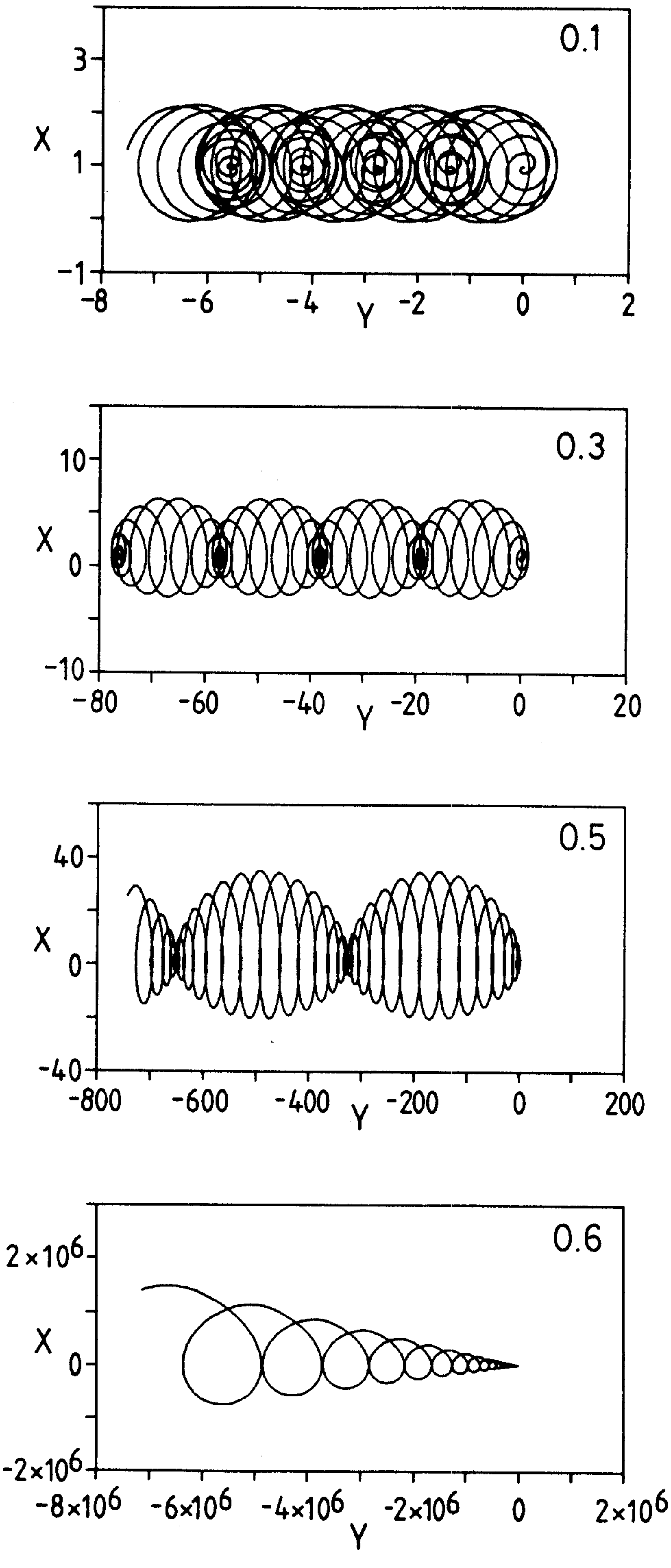

0.2
0.5
0.9
1.0
1.1
2.0
X
Y
1.2
1.0
0.8
-0.2
-0.1
0
0.1
0.2
3
1
-1
-3
-2
2
-8
-6
-4
20
-20
-60
-40
2.5
5
7.5
10
4x10^4
-4x10^4
-2x10^4
2x10^4
6x10^4

Figure 2



0.1
3
X
1
-1
-8
-6
-4
Y
-2
0
2
0.3
10
X
0
-10
-80
-60
-40
Y
-20
0
20
0.5
40
X
0
-40
-800
-600
-400
Y
-200
0
200
0.6
2×10^6
X
0
-2×10^6
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-4×10^6
Y
-2×10^6
0
2×10^6

Figure 3

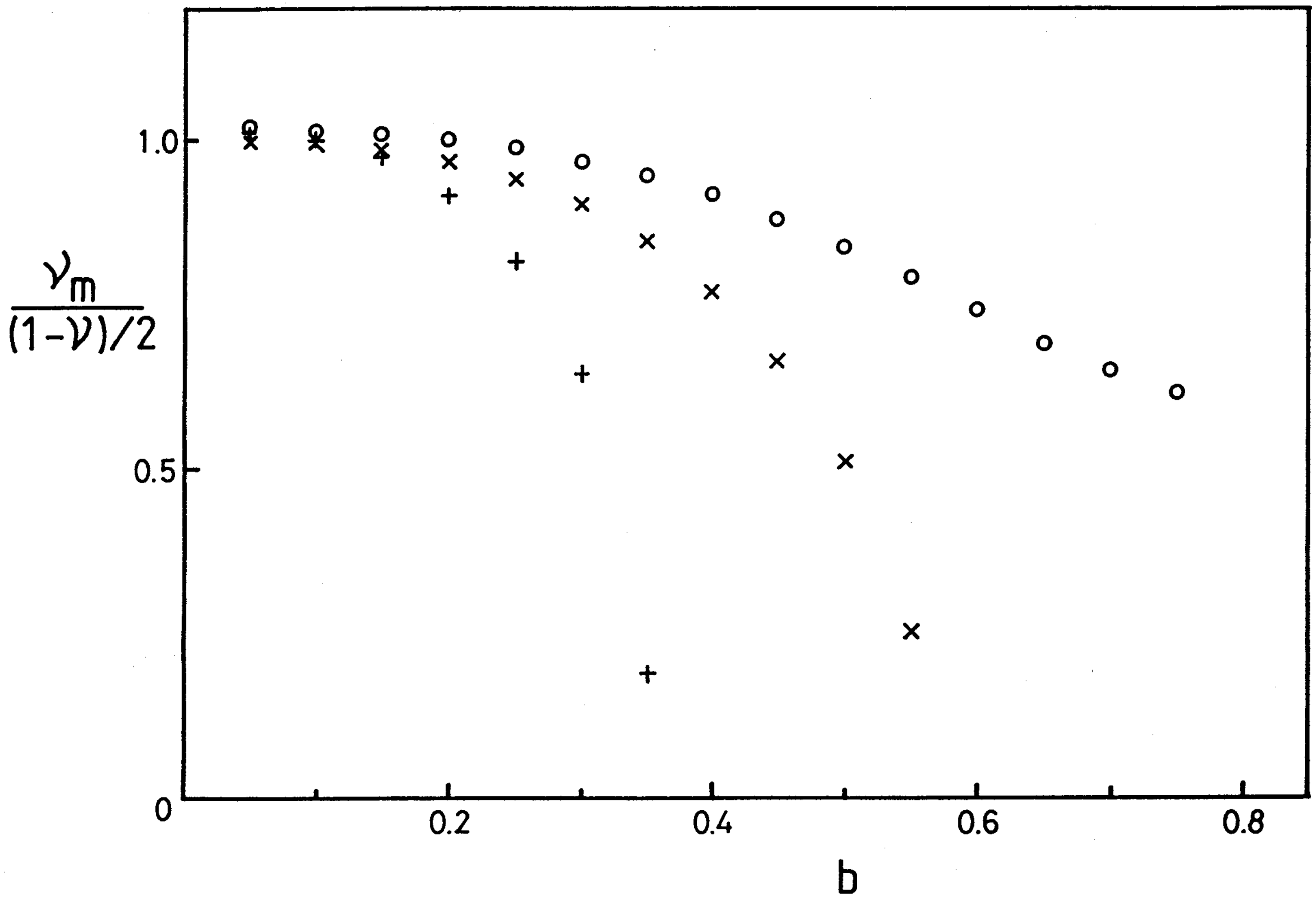

Figure 4

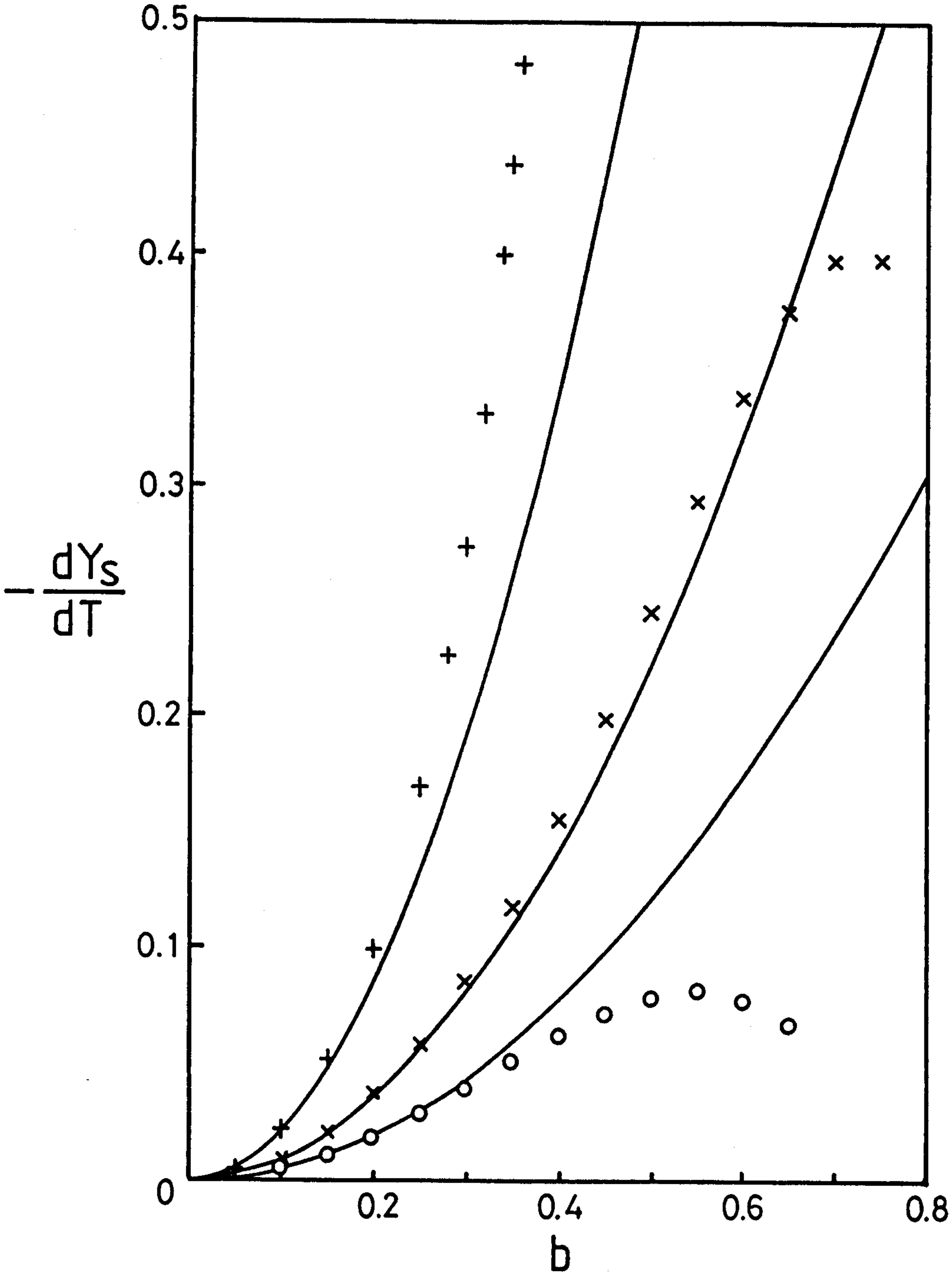

Figure 5

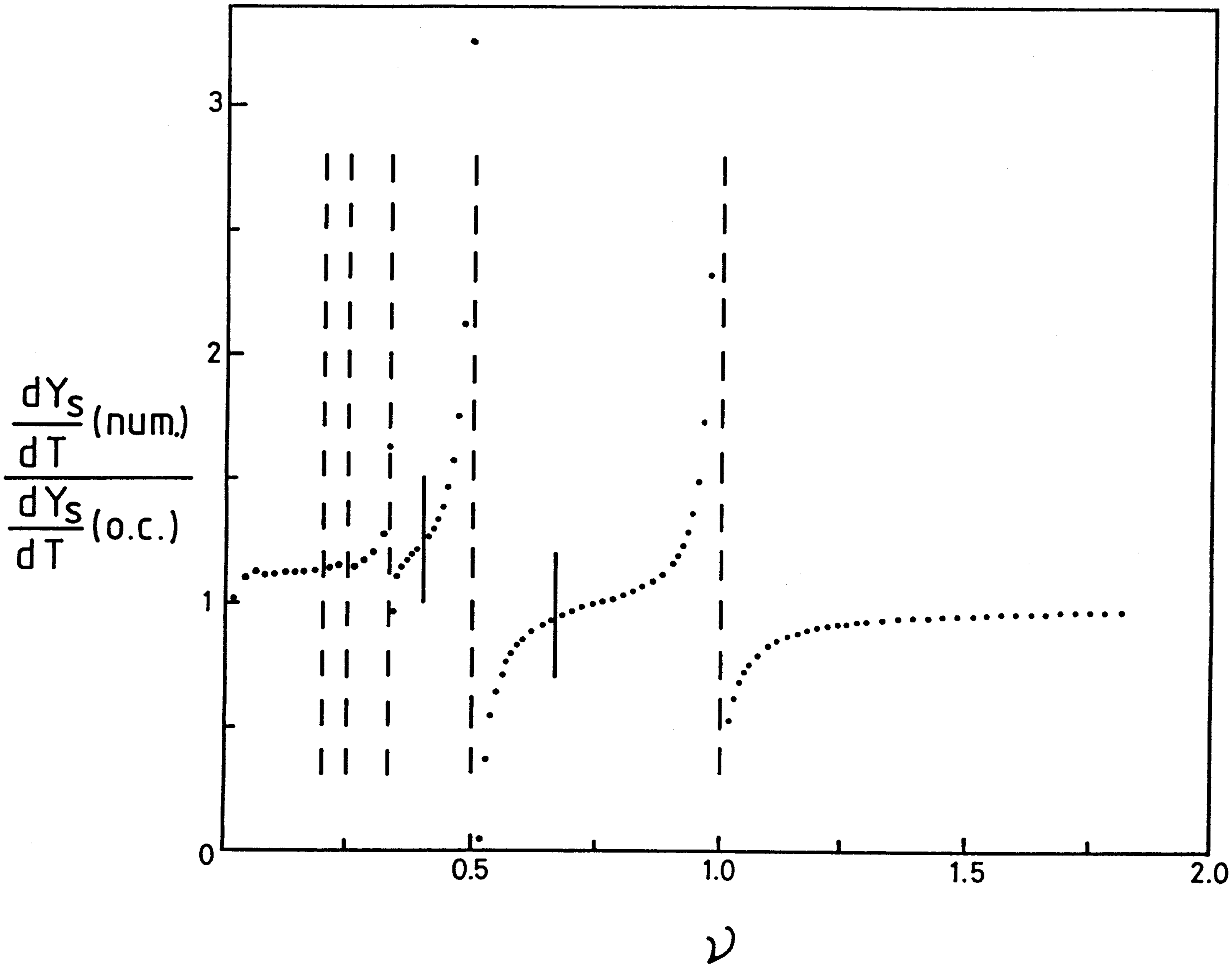

Figure 6