

August 1984

LRP 246/84

THE PONDEROMOTIVE FORCE OF AN ELECTROMAGNETIC WAVE
IN A COLLISIONAL PLASMA

M.L. SAWLEY

Centre de Recherches en Physique des Plasmas
Association Euratom - Confédération Suisse
Ecole Polytechnique Fédérale de Lausanne
21, av. des Bains, CH-1007 Lausanne/Switzerland

THE PONDEROMOTIVE FORCE OF AN ELECTROMAGNETIC WAVE
IN A COLLISIONAL PLASMA

M.L. SAWLEY

Centre de Recherches en Physique des Plasmas
Association Euratom - Confédération Suisse
Ecole Polytechnique Fédérale de Lausanne
21, av. des Bains, CH-1007 Lausanne/Switzerland

ABSTRACT

The nonlinear propagation of a circularly polarized, electromagnetic wave in a collisional, infinite, magnetized plasma is considered. The presence of collisions leads to spatial variation in the amplitude of the wavefield which gives rise to a time-independent ponderomotive force. The ponderomotive potential for a left (right) circularly polarized wave attains a maximum at the ion (electron) cyclotron frequency. In the vicinity of the cyclotron frequency it is shown to be always positive. A decrease in both the particle density and the real and imaginary parts of the complex wavenumber is shown to result from the effect of the ponderomotive force.

1. INTRODUCTION

It is well known that spatial gradients in the amplitude of an oscillating electric field can give rise to a time-independent, ponderomotive force. This force plays a fundamental role in nonlinear plasma dynamics due to its influence on the equilibrium plasma parameters. Considerable effort has been made in the past to obtain a general expression for the ponderomotive force in a collisionless magnetized plasma. (An extensive review of this work is given by Statham and ter Haar (1983)).

In a bounded plasma, electric field gradients may be externally imposed by a localized antenna. Numerous authors have considered the effect of the ponderomotive force for such a non self-consistent electromagnetic field on, for example, the acceleration of charged particles (Consoli and Hall, 1963), and the radio-frequency plugging (Eubank, 1969; Fader et al. 1981) and low frequency mode stabilization (Ferron et al. 1983; Yasaka and Itatani, 1984) of open-ended mirror devices.

For a collisionless, infinite, magnetized plasma, Festeau-Barrio and Weibel (1980) have investigated the ponderomotive force due to an ion cyclotron wave which, for spatial gradients in the amplitude of the self-consistent electric field to exist, must be either a standing wave or a wave beyond cut-off. An expression for the ponderomotive force, exerted by a circularly polarized wave with gradients in the direction of the steady magnetic field, was obtained in terms of the gradient of a quasi-potential. This ponderomotive potential is zero, in the laboratory frame, for an undamped wave propagating in an infinite plasma (although, as discussed by Roberts and Buchsbaum (1964), a travelling wave can create a quasi-potential in the wave frame).

In the present paper, we consider a self-consistent treatment of the propagation of a circularly polarized, electromagnetic wave along the direction of a steady magnetic field in a collisional, infinite plasma. The presence of collisions leads to a spatial decay of the electric field amplitude, the decay being exponential for a wave of sufficiently small amplitude. We show that this spatial variation of the wavefield gives rise to a non-zero, time-independent ponderomotive force. Solving the nonlinear equations to second order in electric field strength, an expression for the ponderomotive potential is obtained. The modification of the particle densities and the complex wavenumber (that is, the phase velocity and attenuation length) that results from the influence of the ponderomotive force is calculated.

2. BASIC EQUATIONS

The nonlinear propagation of an electromagnetic wave in a collisional, warm plasma may be adequately described by the self-consistent solution of the following set of multi-fluid equations:

the equation of motion for species σ ,

$$n_{\sigma} m_{\sigma} \left(\frac{\partial \underline{u}_{\sigma}}{\partial t} + \underline{u}_{\sigma} \cdot \nabla \underline{u}_{\sigma} \right) = n_{\sigma} q_{\sigma} (\underline{E} + \underline{u}_{\sigma} \times \underline{B}) - \nabla p_{\sigma} - n_{\sigma} m_{\sigma} \nu_{\sigma} \underline{u}_{\sigma} ; \quad (1)$$

the equation of continuity,

$$\frac{\partial n_{\sigma}}{\partial t} + \nabla \cdot (n_{\sigma} \underline{u}_{\sigma}) = 0 ; \quad (2)$$

Maxwell's equations,

$$\nabla \times \underline{E} = - \frac{\partial \underline{B}}{\partial t} , \quad (3)$$

$$\nabla \times \underline{B} = \mu_0 \sum_{\sigma} n_{\sigma} q_{\sigma} \underline{u}_{\sigma} + \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} ; \quad (4)$$

and the equation of state,

$$p_{\sigma} = n_{\sigma} T_{\sigma} . \quad (5)$$

In (1), a simple form for the collision term, namely $-n_{\sigma} m_{\sigma} \nu_{\sigma} \underline{u}_{\sigma}$, has been used to avoid unnecessary mathematical complications. This collision term is appropriate, for example, to describe charged particle-neutral collisions in a weakly ionized plasma. We shall assume that the temperature T_{σ} of each species is constant, and unaffected by the presence of the wave.

The equilibrium quantities, in the absence of the wave, are

$$\underline{B} = B_0 \hat{z} , \quad n_{\sigma} = \bar{n}_{\sigma} , \quad (6)$$

where B_0 and \bar{n}_{σ} are constants in space and time. The equilibrium values of \underline{E} and \underline{u}_{σ} are assumed to be zero.

We shall consider the propagation in the positive \hat{z} direction (i.e., parallel to B_0) of a circularly polarized electromagnetic wave with an electric wavefield of the form:

$$\underline{E}_{\omega}(z, t) = E_0 (1, \pm i, 0) \exp [i(Kz - \omega t)] , \quad (7)$$

where

$$K = k(z) + i \gamma(z) , \quad (8)$$

$$(k, \gamma \text{ real})$$

and E_0 , the amplitude at $z = 0$, is constant in time. In (7), the upper (lower) sign refers to the right (left) circularly polarized wave.

Using (3) and (7), the magnetic field component of the wave may be obtained:

$$\underline{B}_\omega(z, t) = \frac{1}{\omega} \left(K + \frac{dK}{dz} z \right) E_0 (\mp i, 1, 0) \exp[i(Kz - \omega t)] . \quad (9)$$

Equation (1) can then be split into the perpendicular and parallel components:

$$n_\sigma m_\sigma \frac{\partial \underline{u}_\sigma}{\partial t} = n_\sigma q_\sigma (\underline{E}_\omega + \underline{u}_\sigma \times \underline{B}_0) - n_\sigma m_\sigma \nu_\sigma \underline{u}_\sigma , \quad (10)$$

$$n_\sigma q_\sigma (E_z + \underline{u}_\sigma \times \underline{B}_\omega)_z - \frac{\partial p_\sigma}{\partial z} = 0 . \quad (11)$$

Equation (10) is linear in the wave quantities. It yields the following fluid velocities for species σ :

$$\underline{u}_\sigma(z, t) = \frac{q_\sigma E_0}{m_\sigma (\omega \pm \Omega_\sigma + i\nu_\sigma)} (i, \mp 1, 0) \exp[i(Kz - \omega t)] , \quad (12)$$

where $\Omega_\sigma = q_\sigma B_0 / m_\sigma$ is the cyclotron frequency for species σ .

The force balance along the direction of the magnetic field \underline{B}_0 , is given by (11). Taking the real part of (9) and (12), it may be shown that for a circularly polarized wave propagating in a collisional plasma, the nonlinear (ponderomotive) force exerted by the

wave on each species is independent of time and given by

$$q_{\sigma} (\underline{u}_{\sigma} \times \underline{B}_{\omega}) = \hat{z} \frac{q_{\sigma}^2 E_0^2 e^{-2\gamma z}}{m_{\sigma} \omega [(\omega \pm \Omega_{\sigma})^2 + \nu_{\sigma}^2]} \frac{d}{dz} \left[\nu_{\sigma} k z + (\omega \pm \Omega_{\sigma}) \gamma z \right]. \quad (13)$$

In steady state, the ponderomotive force is balanced by axial pressure gradients and the time-independent electrostatic field E_z , which arise from the spatial separation of the different species.

The electric wavefield given by (7) must satisfy the wave equation obtained from (3) and (4):

$$\frac{\partial^2 \underline{E}_{\omega}}{\partial z^2} = \mu_0 \sum_{\sigma} n_{\sigma} q_{\sigma} \frac{\partial \underline{u}_{\sigma}}{\partial t} + \frac{1}{c^2} \frac{\partial^2 \underline{E}_{\omega}}{\partial t^2}.$$

Substituting from (7) and (12) yields the nonlinear dispersion relation

$$\left(\frac{dKz}{dz} \right)^2 - i \frac{d^2 Kz}{dz^2} - \frac{\omega^2}{c^2} \left[1 - \sum_{\sigma} \frac{\omega_{p\sigma}^2}{\omega(\omega \pm \Omega_{\sigma} + i\nu_{\sigma})} \right] = 0, \quad (14)$$

where $\omega_{p\sigma} = (n_{\sigma} q_{\sigma}^2 / m_{\sigma} \epsilon_0)^{1/2}$ is the plasma frequency for species σ .

3. THE LINEAR WAVENUMBER

If we assume that the wave amplitude is sufficiently small to neglect the nonlinear interaction, the wavenumber (denoted as $K_0 = k_0 + i\gamma_0$ for the linear solution) is independent of z and given, using (14), by

$$K_0^2 - \frac{\omega^2}{c^2} \left[1 - \sum_{\sigma} \frac{\overline{\omega_{p\sigma}^2}}{\omega(\omega \pm \Omega_{\sigma} + i\nu_{\sigma})} \right] = 0. \quad (15)$$

This is the usual form of the linear dispersion relation for an electromagnetic wave propagating in a collisional plasma (Ginzburg, 1961).

Defining

$$A = 1 - \sum_{\sigma} \frac{\bar{\omega}_{p\sigma}^2 (\omega \pm \Omega_{\sigma})}{\omega [(\omega \pm \Omega_{\sigma})^2 + \nu_{\sigma}^2]}, \quad (16)$$

and

$$B = \sum_{\sigma} \frac{\bar{\omega}_{p\sigma}^2 \nu_{\sigma}}{\omega [(\omega \pm \Omega_{\sigma})^2 + \nu_{\sigma}^2]},$$

the real and imaginary parts of the complex wavenumber may be expressed as

$$\begin{aligned} k_0^2 &= \frac{1}{2} \frac{\omega^2}{c^2} \left[A + (A^2 + B^2)^{\frac{1}{2}} \right], \\ \gamma_0^2 &= \frac{1}{2} \frac{\omega^2}{c^2} \left[-A + (A^2 + B^2)^{\frac{1}{2}} \right]. \end{aligned} \quad (17)$$

In the presence of collisions, the wave does not suffer from the effects of resonance and cut-off; for all frequencies the wave possesses a non-zero, finite value of k_0 . However, in the frequency ranges where the undamped wave is cut-off, $\gamma_0 \gg k_0$, that is, the wave is heavily damped.

For future reference, we note that the ratio of real and imaginary parts of the complex wavenumber may be written as

$$\frac{k_0}{\gamma_0} = \frac{A}{B} + \left(1 + \frac{A^2}{B^2} \right)^{\frac{1}{2}}. \quad (18)$$

If the collision frequency ν_{σ} is sufficiently small, then for $\omega \approx \pm \Omega_{\sigma}$ we may approximate A and B by the contribution due to the

species σ alone. Thus

$$A \approx - \frac{\overline{\omega_{p\sigma}^2} (\omega \pm \Omega_\sigma)}{\omega [(\omega \pm \Omega_\sigma)^2 + \nu_\sigma^2]},$$

$$B \approx \frac{\overline{\omega_{p\sigma}^2} \nu_\sigma}{\omega [(\omega \pm \Omega_\sigma)^2 + \nu_\sigma^2]},$$

and, substituting into (18),

$$\frac{\nu_\sigma k_0}{\gamma_0} \approx - (\omega \pm \Omega_\sigma) + [(\omega \pm \Omega_\sigma)^2 + \nu_\sigma^2]^{\frac{1}{2}}. \quad (19)$$

4. THE NONLINEAR SOLUTION

The axial force balance equation (11) may be solved to second order in electric field strength by substituting the wavenumber to first order, calculated in section 3, into the expression (13) for the ponderomotive force. We then obtain

$$q_\sigma (\underline{u}_\sigma \times \underline{B}_0) = - \frac{d \Phi_\sigma}{d z} \hat{z},$$

where the ponderomotive potential for species σ is given, for $\gamma_0 \neq 0$, by

$$\Phi_\sigma = \frac{q_\sigma^2 E_0^2 e^{-2\gamma_0 z}}{2 m_\sigma [(\omega \pm \Omega_\sigma)^2 + \nu_\sigma^2]} \left[(\omega \pm \Omega_\sigma) + \frac{\nu_\sigma k_0}{\gamma_0} \right]. \quad (20)$$

Writing the axial electrostatic field as the gradient of a scalar potential,

$$E_z = - \frac{d U}{d z},$$

we obtain from (11), using (5),

$$- q_{\sigma} \frac{dU}{dz} - \frac{d\Phi_{\sigma}}{dz} - \frac{T_{\sigma}}{n_{\sigma}} \frac{dn_{\sigma}}{dz} = 0 .$$

The solution to second order in electric field strength is

$$n_{\sigma}(z) = \bar{n}_{\sigma} \left[1 - \frac{(q_{\sigma} U + \Phi_{\sigma})}{T_{\sigma}} \right] . \quad (21)$$

where we have used the boundary condition, $n_{\sigma}(z \rightarrow \infty) = \bar{n}_{\sigma}$.

The electrostatic potential U may be obtained by imposing that the plasma is quasi-neutral, that is,

$$\sum_{\sigma} n_{\sigma} q_{\sigma} = 0 .$$

Substituting from (21) yields

$$U = - \frac{\sum_{\sigma} \bar{n}_{\sigma} q_{\sigma} \Phi_{\sigma} / T_{\sigma}}{\sum_{\sigma} \bar{n}_{\sigma} q_{\sigma}^2 / T_{\sigma}} .$$

Defining

$$\epsilon_{\sigma} = \frac{q_{\sigma}^2 [(\omega \pm \Omega_{\sigma}) + \nu_{\sigma} k_0 / \gamma_0]}{2 m_{\sigma} \omega [(\omega \pm \Omega_{\sigma})^2 + \nu_{\sigma}^2]} , \quad (22)$$

and

$$U = - \frac{\sum_{\sigma} \bar{n}_{\sigma} q_{\sigma} \epsilon_{\sigma} / T_{\sigma}}{\sum_{\sigma} \bar{n}_{\sigma} q_{\sigma}^2 / T_{\sigma}} , \quad (23)$$

the density of species σ , to second order in electric field strength,

may be written as

$$n_{\sigma}(z) = \bar{n}_{\sigma} \left[1 - \frac{(q_{\sigma} u + \varepsilon_{\sigma})}{T_{\sigma}} E_0^2 e^{-2\gamma_0 z} \right] . \quad (24)$$

The electric field therefore produces a nonlinear modification of the density of each particle species. The change in density causes, in turn, a change in the wavenumber. Substituting (24) into (14), the modified wavenumber satisfies the following nonlinear dispersion relation

$$\left(\frac{dKz}{dz} \right)^2 - i \frac{d^2 Kz}{dz^2} - K_0^2 (1 + \alpha E_0^2 e^{-2\gamma_0 z}) = 0 , \quad (25)$$

where

$$\alpha = \frac{\omega^2}{K_0^2 c^2} \sum_{\sigma} \frac{\bar{\omega}_{p\sigma}^2}{\omega(\omega \pm \Omega_{\sigma} + i\nu_{\sigma})} \frac{(q_{\sigma} u + \varepsilon_{\sigma})}{T_{\sigma}} . \quad (26)$$

The solution of (25) that satisfies the boundary conditions

$$K(z=0) \quad \text{finite} ,$$

$$K(z \rightarrow \infty) = K_0 ,$$

is, to second order in electric field strength,

$$K(z) = K_0 + \frac{\alpha K_0^2 E_0^2}{4(K_0 + i\gamma_0)} \frac{(1 - e^{-2\gamma_0 z})}{\gamma_0 z} . \quad (27)$$

5. PONDEROMOTIVE EFFECT FOR $\omega \approx \pm \Omega_{\sigma}$

It was shown in section 4 that the nonlinear force exerted by an electromagnetic wave in a collisional plasma may be expressed, for each particle species σ , as the gradient of a ponderomotive potential,

Φ_σ . This potential, in addition to exhibiting the usual dependence on $|E_\omega|^2$, has also, in general, a dependence on the wave properties through the parameter $v_\sigma k_0/\gamma_0$. However, if v_σ is sufficiently small, then for $\omega \approx \pm\Omega_\sigma$ we may substitute into (20) the approximate value of this parameter given by (19). We then obtain

$$\Phi_\sigma \approx \frac{q_\sigma^2 E_0^2 e^{-2\gamma_0 z}}{2 m_\sigma \Omega_\sigma^2} \Psi_\sigma^\pm \left(\frac{\omega}{\Omega_\sigma}, \frac{v_\sigma}{\Omega_\sigma} \right), \quad (28)$$

where

$$\Psi_\sigma^\pm \left(\frac{\omega}{\Omega_\sigma}, \frac{v_\sigma}{\Omega_\sigma} \right) = \frac{1}{\frac{\omega}{\Omega_\sigma} \left[\left(1 \pm \frac{\omega}{\Omega_\sigma} \right)^2 + \frac{v_\sigma^2}{\Omega_\sigma^2} \right]^{1/2}}.$$

The ponderomotive potential for each particle species, in the presence of a wave with appropriate polarization, attains a maximum at the cyclotron frequency of that species. In addition, (28) shows that it is always positive in the vicinity of the cyclotron frequency. We note that this is contrary to the case of an electromagnetic wave in a collisionless plasma. The standard expression for the ponderomotive potential (Motz and Watson, 1967),

$$\Phi_\sigma = \frac{q_\sigma^2 |E(z)|^2}{2 m_\sigma \omega (\omega \pm \Omega_\sigma)}, \quad (29)$$

reveals a change of sign of the left (right) circularly polarized wave as the frequency crosses the ion (electron) cyclotron frequency. However, care must be taken when applying expression (29) if the electric field $E(z)$ is calculated self-consistently. For a collisionless plasma, the wave is propagatory for $\omega < |\Omega_\sigma|$ but

evanescent for $\omega \gtrsim |\Omega_\sigma|$. Therefore the wave must satisfy different boundary conditions in the different frequency regimes (Festeau-Barrioz and Sawley, 1984). Hence, expression (29) can not be applied to a self-consistent wave in a continuous fashion as the wave frequency crosses the cyclotron frequency.

If we consider a plasma for which $|q_\sigma|$ and T_σ are the same for all particle species, then for $\omega \approx \pm \Omega_\sigma$ we may approximate (23) as

$$u \approx - \frac{\bar{n}_\sigma \varepsilon_\sigma}{2 \bar{n}_e q_\sigma} .$$

Therefore

$$\frac{q_\sigma u + \varepsilon_\sigma}{T_\sigma} \approx \frac{q_\sigma^2}{2 m_\sigma \Omega_\sigma^2 T_\sigma} \left(1 - \frac{\bar{n}_\sigma}{2 \bar{n}_e} \right) \Psi_\sigma^\pm > 0 , \quad (30)$$

and from (24) it may be seen that the ponderomotive force causes a decrease in the particle density of species σ .

Writing

$$\begin{aligned} k &= k_0 (1 + \Delta k) , \\ \gamma &= \gamma_0 (1 + \Delta \gamma) , \end{aligned}$$

then from (27) we may obtain approximate expressions for the modification of the real and imaginary parts of the complex wavenumber for $\omega \approx \pm \Omega_\sigma$:

$$\begin{aligned} \Delta k &\approx -\beta \left\{ (\omega \pm \Omega_\sigma) + 2 \left[(\omega \pm \Omega_\sigma)^2 + \gamma_\sigma^2 \right]^{\frac{1}{2}} \right\} E_0^2 \frac{(1 - e^{-2\gamma_0 z})}{\gamma_0 z} , \\ \Delta \gamma &\approx -\beta \left\{ (\omega \pm \Omega_\sigma) + \left[(\omega \pm \Omega_\sigma)^2 + \gamma_\sigma^2 \right]^{\frac{1}{2}} \right\} E_0^2 \frac{(1 - e^{-2\gamma_0 z})}{\gamma_0 z} , \end{aligned}$$

where

$$\beta = \frac{\omega \overline{\omega_{ps}^2}}{4(k_o^2 + 4\gamma_o^2)c^2} \frac{1}{[(\omega \pm \Omega_o)^2 + \nu_o^2]} \frac{(q_o u + \epsilon_o)}{T_o} .$$

From (30), $\beta > 0$, and therefore the decrease in particle density that results from the effect of the ponderomotive force for $\omega \approx \pm \Omega_o$, causes a decrease in both the real and imaginary parts of the complex wavenumber. Thus, as a result of the ponderomotive force, both the phase velocity (ω/k) and the attenuation length ($1/\gamma$) are increased. The nonlinear density modification caused by the ponderomotive force therefore acts to oppose its source, that is, the spatial attenuation of the wave.

6. CONCLUSIONS

The spatial decay of a circularly polarized, electromagnetic wave in a collisional plasma has been shown to give rise to a time-independent, ponderomotive force. An expansion to second order in the electric field strength has been used to write this force in terms of the gradient of a ponderomotive potential. This potential attains a maximum at the cyclotron frequency of the resonant particle species. In the vicinity of the cyclotron frequency, the ponderomotive potential is always positive, and causes a decrease in the particle density which results in an increase in the phase velocity and attenuation length of the wave.

REFERENCES

- CONSOLI, T. and HALL, R.B. 1963 Nucl. Fusion, 3, 237.
- EUBANK, H.P. 1969 Phys. Fluids, 12, 234.
- FADER, W.J., JONG, R.A., STUFFLEBEAM, J.H. and SZIKLAS, E.A. 1981
Phys. Rev. Lett. 46, 999.
- FERRON, J.R., HERSHKOWITZ, N., BREUN, R.A., GOLVATO, S.N. and
GOULDING, R. 1983 Phys. Rev. Lett. 51, 1955.
- FESTEUAU-BARRIOZ, M.C. and SAWLEY, M.L. 1984 Lausanne Report LRP 245/84
(to be published).
- FESTEUAU-BARRIOZ, M.C. and WEIBEL, E.S. 1980 Phys. Fluids, 23, 2045.
- GINZBURG, V.L. 1961 Propagation of Electromagnetic Waves in Plasma,
Gordon and Breach.
- MOTZ, H. and WATSON, C.J.M. 1967 Adv. Electron. Electron Phys. 23,
153.
- ROBERTS, C.S. and BUCHSBAUM, S.J. 1964 Phys. Rev. A135, 381.
- STATHAM, G. and TER HAAR, D. 1983 Plasma Phys. 25, 681.
- YASAKA, Y. and ITATANI, R. 1984 Nucl. Fusion, 24, 445.