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ABSTRACT

The quasilinear equations are derived for an inhomogeneous magnetized plasma without using the random phase approximation and the expansion in the ratio of the Larmor radius to the inhomogeneity scale. The equations take into account both resonant and non-resonant interactions, and possess the necessary conservation properties.

I. INTRODUCTION

In a recent paper [1], it was shown that the quasilinear equations for an unmagnetized inhomogeneous plasma can be derived without making use of the random phase approximation. Instead, the method of correlation functions together with a Fourier-transform technique was applied [2]. Moreover, the authors pointed out inconsistencies arising from the use of the random phase approximation. The equations they obtained comprise new terms which do not appear in conventional derivations.

In this work we generalize these equations to the case of a plasma in a uniform magnetic field $\vec{B}_0$ interacting with electrostatic waves. The plan of the paper is as follows. In Sec. II we derive the quasilinear equations for arbitrary ratio of the turbulence spatial scale to the plasma inhomogeneity scale. In Sec. III these equations are simplified using the adiabatic approximation to obtain equations of motion for both the averaged distribution function and the correlation function of the potential. Finally, in Sec. IV, we prove that the equations possess the necessary conservation properties.

II. QUASILINEAR SYSTEM

A. Fluctuating and averaged equations

Within the electrostatic approximation, the distribution function f_σ $(\vec{v},\vec{r},t)$ for species σ obeys the Vlasov equation

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla\right) f_\sigma = \frac{q_\sigma}{m_\sigma}\nabla\phi\cdot\frac{\partial}{\partial\vec{v}} f_\sigma - \frac{q_\sigma}{m_\sigma}(\vec{v}\times\vec{B}_o)\cdot\frac{\partial}{\partial\vec{v}} f_\sigma \,, \quad (1)$$

where $\Phi(\vec{r},t)$ is the fluctuating electrostatic potential and $\vec{B}_0$ is the uniform magnetic field. Following conventional procedure, the distribution function f_σ is split into an averaged part $\bar{f}_\sigma$, and a fluctuating part $\tilde{f}_\sigma$, i.e.,

$$f_\sigma(\vec{v},\vec{r},t) = \bar{f}_\sigma(\vec{v},\vec{r},t) + \tilde{f}_\sigma(\vec{v},\vec{r},t) \,, \quad (2)$$

with

$$\bar{f}_\sigma \equiv \langle f_\sigma \rangle \,, \qquad |\tilde{f}_\sigma| \ll \bar{f}_\sigma \,,$$

where the angular brackets < > indicate an ensemble average. The evolution equation for the averaged distribution function $\bar{f}_\sigma$ is obtained by taking the ensemble average of (1)

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla\right)\bar{f}_\sigma = \frac{q_\sigma}{m_\sigma}\left\langle \nabla\phi\cdot\frac{\partial}{\partial\vec{v}}\tilde{f}_\sigma \right\rangle - \frac{q_\sigma}{m_\sigma c}(\vec{v}\times\vec{B}_o)\cdot\frac{\partial}{\partial\vec{v}}\bar{f}_\sigma . \quad (3)$$

The evolution equation for the fluctuating part is found by subtracting (3) from (1). Thus, $\tilde{f}_\sigma$ obeys

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla\right)\tilde{f}_\sigma = \frac{q_\sigma}{m_\sigma}\nabla\phi\cdot\frac{\partial}{\partial\vec{v}}\bar{f}_\sigma - \frac{q_\sigma}{m_\sigma c}(\vec{v}\times\vec{B}_0)\cdot\frac{\partial}{\partial\vec{v}}\tilde{f}_\sigma$$

$$+ \frac{q_\sigma}{m_\sigma}\left(\nabla\phi\cdot\frac{\partial}{\partial\vec{v}}\tilde{f}_\sigma - \langle\nabla\phi\cdot\frac{\partial}{\partial\vec{v}}\tilde{f}_\sigma\rangle\right). \tag{4}$$

The last two terms on the right-hand side of (4) (i.e., those involving $\nabla\Phi \cdot \partial\tilde{f}_\sigma/\partial\vec{v}$) may be neglected since they describe the (higher order) wave-wave and wave-particle-wave interactions which lie outside the scope of the quasilinear theory. Doing so, and introducing cylindrical coordinates in velocity space, $\vec{v} = (v_\perp, \; v_\parallel, \alpha)$, with the axis parallel to $\vec{B}_0$, (4) becomes

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla\right)\tilde{f}_\sigma = \frac{q_\sigma}{m_\sigma}\nabla\phi\cdot\frac{\partial}{\partial\vec{v}}\bar{f}_\sigma + \omega_{c\sigma}\frac{\partial}{\partial\alpha}\tilde{f}_\sigma \quad , \tag{5}$$

where $\omega_{c\sigma}$ is the cyclotron frequency. Equation (5) will be solved for $\tilde{f}_\sigma$ which, in turn, will be substituted into (3), thus yielding the evolution equation for the averaged distribution function $\bar{f}_\sigma$. Closure is, of course,ensured by the Poisson equation

$$-\nabla^2\phi = 4\pi\sum_\sigma q_\sigma\int\tilde{f}_\sigma\, d^3v\,. \tag{6}$$

B. Solution of the fluctuating equation

It is judicious, for reasons that will be apparent in Sec. IIC., to solve (5) in Fourier space. It becomes, upon Fourier transforming,

$$-i(\omega-\vec{k}\cdot\vec{v})\,\hat{\tilde{f}}_\sigma(\vec{v},\vec{\kappa}) - \omega_{c\sigma}\frac{\partial}{\partial\alpha}\hat{\tilde{f}}_\sigma(\vec{v},\vec{\kappa})$$

$$= i\frac{q_\sigma}{m_\sigma}\int \text{đ}\kappa'\ \phi_{\vec{\kappa}-\vec{\kappa}'}\,(\vec{k}-\vec{k}')\cdot\frac{\partial}{\partial\vec{v}}\hat{\bar{f}}(\vec{v},\vec{\kappa}'),\tag{7}$$

where $\vec{\kappa} = \{\vec{k},\omega\}$ and $\text{đ}\kappa = d^3k\,d\omega/(2\pi)^4$. The circumflex indicates the Fourier transform, and will be dropped to unburden the notation whenever this does not cause confusion. The convolution on the right-hand side of (7) takes into account possible space and time variations of the averaged distribution function $\bar{f}_\sigma$. Equation (7) is an inhomogeneous first-order linear differential equation. Its solution is

$$\tilde{f}_\sigma(v_\perp, v_\parallel, \alpha, \vec{\kappa}) = -i\frac{q_\sigma}{m_\sigma}\int^{\alpha} \exp\left[i\,\xi_{\vec{\kappa},\sigma}(\alpha') - i\,\xi_{\vec{\kappa},\sigma}(\alpha)\right]$$

$$\times\int \phi_{\vec{\kappa}-\vec{\kappa}'}\,(\vec{k}-\vec{k}')\cdot\frac{\partial}{\partial\vec{v}}\bar{f}_\sigma(v_\perp, v_\parallel, \alpha', \vec{\kappa}')\ \text{đ}\kappa'\,d\alpha',\tag{8}$$

with

$$\xi_{\vec{\kappa},\sigma}(\beta) = \frac{1}{\omega_{c\sigma}}\left[(\omega - k_\parallel v_\parallel)(\beta-\varphi) - k_\perp v_\perp \sin(\beta-\varphi)\right],$$

where $\vec{k} = (k_\perp, k_\parallel, \varphi)$. In other words, $\xi_{\vec{k},\sigma}$ is the primitive, with respect to the velocity polar coordinate α, of the free-streaming operator $\partial/\partial t + \vec{v} \cdot \nabla$ in Fourier space. Notice that the integral in (8) is evaluated only at the upper bound. This ensures that $\tilde{f}_\sigma$ is periodic in α.

C. Averaged distribution function

The form of the averaged distribution function will be chosen so that it is compatible with the equilibrium conditions of the plasma, which we assume to occur when the electrostatic fluctuations are absent. Under such conditions the distribution function obeys the equilibrium Vlasov equation

$$\vec{v} \cdot \nabla f_{\sigma 0} + \frac{q_\sigma}{c m_\sigma} (\vec{v} \times \vec{B}_0) \cdot \frac{\partial}{\partial \vec{v}} f_{\sigma 0} = 0 . \tag{9}$$

Equation (9) is satisfied if $f_{\sigma 0}$ has the following form [3]

$$f_{\sigma 0} = f_{\sigma 0}\left(v_\perp, v_\parallel, x + \frac{v_y}{\omega_{c\sigma}}, y - \frac{v_x}{\omega_{c\sigma}}\right), \tag{10}$$

i.e., if $f_{\sigma 0}$ is the distribution function of the guiding centers. We assume that in the presence of electrostatic fluctuations the averaged distribution function has the same form as (10) with a time dependence

$$\bar{f}_\sigma = \bar{f}_\sigma\left(v_\perp, v_\parallel, x + \frac{v_y}{\omega_{c\sigma}}, y - \frac{v_x}{\omega_{c\sigma}}, t\right) . \qquad (11)$$

Notice that the velocity dependent shift of the variables x and y renders the α' integration in (8) difficult. The usual way of overcoming this difficulty is to expand $\bar{f}_\sigma$ around (x,y), invoking the smallness of the Larmor radius $\rho_L = v_\perp/\omega_{c\sigma}$ with respect to the inhomogeneity scale,

$$\bar{f}_\sigma \simeq \bar{f}_\sigma(v_\perp, v_\parallel, x, y, t) - \frac{(\vec{v}_\perp \times \nabla_\perp)_\parallel}{\omega_{c\sigma}} \bar{f}_\sigma(v_\perp, v_\parallel, x, y, t) .$$

This procedure will be avoided. Using the properties of the Fourier transform, one can easily separate the spatial dependence from the velocity dependence in $\bar{f}_\sigma$. Taking the Fourier transform of (11) we find

$$\text{F.T.}\left[\bar{f}_\sigma\left(v_\perp, v_\parallel, x + \frac{v_y}{\omega_{c\sigma}}, y - \frac{v_x}{\omega_{c\sigma}}, t\right)\right] = \bar{f}_\sigma(v_\perp, v_\parallel, \vec{k}) \exp\left[i \frac{(\vec{k}_\perp \times \vec{v}_\perp)_\parallel}{\omega_{c\sigma}}\right] . \qquad (12)$$

The dependence on the velocity polar coordinate α is now explicitly carried by the exponential. Note that since no variation of $\bar{f}_\sigma$ along the axis is assumed, its Fourier transform will always contain a function $\delta(k_\parallel)$. Expression (12) shall now be introduced into (8).

D. Fluctuating distribution function

As a preliminary step, we shall state explicitly the operator $(\vec{k}-\vec{k}')\cdot\partial/\partial\vec{v}$ acting on the Fourier transform of the averaged distribution function in (8). In cylindrical coordinates, it reads

$$(\vec{k}-\vec{k}')\cdot\frac{\partial}{\partial\vec{v}} = (k_{\parallel}-k_{\parallel}')\frac{\partial}{\partial v_{\parallel}} + \left[k_{\perp}\cos(\alpha-\varphi) - k_{\perp}'\cos(\alpha-\varphi')\right]\frac{\partial}{\partial v_{\perp}} - \left[k_{\perp}\sin(\alpha-\varphi) - k_{\perp}'\sin(\alpha-\varphi')\right]\frac{1}{v_{\perp}}\frac{\partial}{\partial\alpha} .$$

On applying this operator on (12) we have

$$(\vec{k}-\vec{k}')\cdot\frac{\partial}{\partial\vec{v}}\left\{\bar{f}_{\sigma}(v_{\perp},v_{\parallel},\vec{k})\exp\left[i\,\frac{(\vec{k}_{\perp}\times\vec{v}_{\perp})_{\parallel}}{\omega_{c\sigma}}\right]\right\} = \exp\left[i\,\frac{(\vec{k}_{\perp}\times\vec{v}_{\perp})_{\parallel}}{\omega_{c\sigma}}\right]$$

$$\times\left\{(k_{\parallel}-k_{\parallel}')\frac{\partial}{\partial v_{\parallel}} + \left[k_{\perp}\cos(\alpha-\varphi) - k_{\perp}'\cos(\alpha-\varphi')\right]\frac{\partial}{\partial v_{\perp}} - i\,\frac{(\vec{k}_{\perp}\times\vec{k}_{\perp}')_{\parallel}}{\omega_{c\sigma}}\right\}\bar{f}_{\sigma}(v_{\perp},v_{\parallel},\vec{k}') .$$

This term now is substituted into (8). Upon systematic use of the identity

$$\exp(i a \sin\theta) = \sum_{n}\exp(i n\theta)\,J_n(a) ,$$

where the J_n are Bessel functions of the first kind, we obtain

$$\tilde{f}_\sigma(\vec{v},\vec{\kappa}) = -\frac{q_\sigma}{m_\sigma}\sum_{\ell,n,j}\frac{\exp[i(\alpha-\varphi)(j-\ell)]}{\omega-k_\parallel v_\parallel-\omega_{c\sigma}} J_j\left(\frac{k_\perp v_\perp}{\omega_{c\sigma}}\right)$$

$$\times J_{\ell+n}\left(\frac{k_\perp v_\perp}{\omega_{c\sigma}}\right)\int d\vec{Q}'\ \phi_{\vec{\kappa}-\vec{Q}'}\exp[in(\varphi-\psi')]\,J_n\left(\frac{q_\perp' v_\perp}{\omega_{c\sigma}}\right) \tag{13}$$

$$\times\left[(k_\parallel-q_\parallel')\frac{\partial}{\partial v_\parallel}+\frac{\ell\omega_{c\sigma}}{v_\perp}\frac{\partial}{\partial v_\perp}-i\frac{(\vec{k}_\perp\times\vec{q}_\perp')_\parallel}{\omega_{c\sigma}}\right]\bar{f}_\sigma(v_\perp,v_\parallel,\vec{Q}') .$$

We have replaced $\{\vec{k},\omega\}$ by $\vec{Q}\equiv\{\vec{q},\ \Omega\}$, where $\vec{q}=\{q_\perp, q_\parallel, \psi\}$, for further convenience. To obtain (13) we have shifted the indices of the Bessel functions, and used their recurrence properties. Equation (13) may be put in a more compact form by using Graf's addition theorem which for our case reads

$$\sum_n J_{\ell+n}\left(\frac{k_\perp v_\perp}{\omega_{c\sigma}}\right) J_n\left(\frac{q_\perp' v_\perp}{\omega_{c\sigma}}\right)\exp[in(\varphi-\psi')]$$

$$= J_\ell\left(|\vec{k}_\perp-\vec{q}_\perp'|\,v_\perp/\omega_{c\sigma}\right)\left\{\frac{k_\perp-q_\perp'\exp[-i(\varphi-\psi')]}{|\vec{k}_\perp-\vec{q}_\perp'|}\right\}^\ell .$$

We obtain

$$\tilde{f}_\sigma(\vec{v},\vec{\kappa}) = -\frac{q_\sigma}{m_\sigma}\sum_{\ell,j}\frac{\exp[i(\alpha-\varphi)(j-\ell)]}{\omega-k_\parallel v_\parallel-\ell\omega_{c\sigma}} J_j\left(\frac{k_\perp v_\perp}{\omega_{c\sigma}}\right)$$

$$\times\int d\vec{Q}'\, J_\ell\left(\frac{|\vec{k}_\perp-\vec{q}_\perp'|\,v_\perp}{\omega_{c\sigma}}\right)\left\{\frac{k_\perp-q_\perp'\exp[-i(\varphi-\psi')]}{|\vec{k}_\perp-\vec{q}_\perp'|}\right\}^\ell \phi_{\vec{\kappa}-\vec{Q}'} \tag{14}$$

$$\times\left[(k_\parallel-q_\parallel')\frac{\partial}{\partial v_\parallel}+\frac{\ell\omega_{c\sigma}}{v_\perp}\frac{\partial}{\partial v_\perp}-i\frac{(\vec{k}_\perp\times\vec{q}_\perp')}{\omega_{c\sigma}}\right]\bar{f}_\sigma(v_\perp,v_\parallel,\vec{Q}') .$$

E. Evolution equation for the averaged function

The fluctuating distribution function, given by (14), is now substituted into the nonlinear term of (3). Transforming (3) into Fourier space, regrouping all the angular velocity terms into the right-hand side, and averaging over the velocity polar coordinate, we obtain

$$-i\Omega \bar{f}_\sigma(v_\perp, v_\parallel, \vec{Q}) = i\,\frac{q_\sigma^2}{m_\sigma^2} \sum_j \int d\vec{K} \int d\vec{Q}'\; I_{\vec{K}-(\vec{Q}+\vec{Q}')/2,\,\vec{Q}-\vec{Q}'}$$

$$\times \left[(k_\parallel - q_\parallel)\frac{\partial}{\partial v_\parallel} + \frac{j\,\omega_{c\sigma}}{v_\perp}\frac{\partial}{\partial v_\perp} - i\,\frac{(\vec{k}_\perp \times \vec{q}_\perp)_\parallel}{\omega_{c\sigma}} \right]$$

$$\times \frac{J_j(|\vec{k}_\perp - \vec{q}_\perp| v_\perp/\omega_{c\sigma})\; J_j(|\vec{k}_\perp - \vec{q}\,'_\perp| v_\perp/\omega_{c\sigma})}{\omega - k_\parallel v_\parallel - j\,\omega_{c\sigma}} \qquad (15)$$

$$\times \left\{ \frac{[k_\perp - q_\perp \exp i(\varphi-\psi)]\,[k_\perp - q'_\perp \exp i(\psi'-\varphi)]}{|\vec{k}_\perp - \vec{q}_\perp|\; |\vec{k}_\perp - \vec{q}\,'_\perp|} \right\}^j$$

$$\times \left[(k_\parallel - q'_\parallel)\frac{\partial}{\partial v_\parallel} + \frac{j\,\omega_{c\sigma}}{v_\perp}\frac{\partial}{\partial v_\perp} - i\,\frac{(\vec{k}_\perp \times \vec{q}\,'_\perp)_\parallel}{\omega_{c\sigma}} \right] \bar{f}_\sigma(v_\perp, v_\parallel, \vec{Q}') \;,$$

where we have again used the properties of the Bessel functions, and the reality condition $\Phi^*_{\vec{K}} = \Phi_{-\vec{K}}$; $I_{\vec{K},\vec{Q}}$ is the correlation function of the potential, defined by

$$I_{\vec{K},\vec{Q}} \equiv \langle \Phi^*_{\vec{K}-\vec{Q}/2}\; \Phi_{\vec{K}+\vec{Q}/2} \rangle \;.$$

F. Closure equation

Transforming (6) into Fourier space, we have

$$k^2 \phi_{\vec{K}} = 4\pi \sum_\sigma q_\sigma \int \tilde{f}_\sigma(\vec{v},\vec{K})\, d^3v \, .$$

Multiplying this equation by $\phi^*_{\vec{K}-\vec{Q}}$, taking the ensemble average, and substituting (14) for $\tilde{f}_\sigma$, we obtain

$$
\begin{aligned}
I_{\vec{K}-\vec{Q}/2,\,\vec{Q}} = & -\sum_\sigma 2\pi \frac{4\pi q_\sigma^2}{m_\sigma k^2} \int dv \sum_j \int d\vec{Q}' \\
& \times \frac{I_{\vec{K}-(\vec{Q}+\vec{Q}')/2,\,\vec{Q}-\vec{Q}'}\; J_j(k_\perp v_\perp/\omega_{c\sigma})\; J_j(|\vec{k}_\perp-\vec{q}'_\perp| v_\perp/\omega_{c\sigma})}{\omega - k_\parallel v_\parallel - j\omega_{c\sigma}} \\
& \times \left\{ \frac{k_\perp - q'_\perp \exp[i(\varphi'-\varphi)]}{|\vec{k}_\perp - \vec{q}'_\perp|} \right\}^j \Big[(k_\parallel - q'_\parallel)\frac{\partial}{\partial v_\parallel} + j\frac{\omega_{c\sigma}}{v_\perp}\frac{\partial}{\partial v_\perp} \\
& - i\frac{(\vec{k}_\perp \times \vec{q}'_\perp)_\parallel}{\omega_{c\sigma}} \Big] \bar{f}_\sigma(v_\perp, v_\parallel, \vec{Q}') \, ,
\end{aligned}
\tag{16}
$$

where $dv \equiv v_\perp\, dv_\perp\, dv_\parallel$. Equations (15) and (16) govern the self-consistent evolution of the averaged distribution function and the correlation function.

III. ADIABATIC APPROXIMATION OF THE QUASILINEAR SYSTEM

We now make the assumption that the spectrum of the averaged quantities is much narrower than the spectrum of the fluctuating quantities. Keeping the notation of the preceding section, this may be expressed as

$$|\vec{K}| \quad >> \quad |\vec{Q}|, \ |\vec{Q}'| \ . \tag{17}$$

Condition (17) ensures the existence of small parameters $|\vec{Q}|/|\vec{K}|$, $|\vec{Q}'|/|\vec{K}|$ in which we can expand (15) and (16).

A. Expansion of the averaged equation

On expanding (15) to first order in $|\vec{Q}|/|\vec{K}|$, $|\vec{Q}'|/|\vec{K}|$, and transforming back to space and time with respect to the variables $\vec{Q}$, $\vec{Q}'$, we obtain

$$\frac{\partial}{\partial t}\bar{f}_\sigma = -\frac{q_\sigma^2}{2m_\sigma^2}\sum_j \int d\kappa\, \hat{O}_\sigma \frac{\left[(\partial J_j^2/\partial \vec{k}_\perp).\nabla_\perp - 2i\,j\,J_j^2\,(\vec{k}_\perp \times \nabla_\perp)_\parallel / k_\perp^2\right]}{\omega - k_\parallel v_\parallel - j\omega_{c\sigma}}$$

$$\times\ I_{\vec{k}}\,\hat{O}_\sigma \bar{f}_\sigma \quad - \frac{q_\sigma^2}{2m_\sigma^2}\sum_j \int d\kappa\, \hat{O}\left[\nabla_\perp\cdot\left(\frac{\partial I_{\vec{k}}}{\partial \vec{k}_\perp}\right) - \frac{\partial}{\partial t}\left(\frac{\partial I_{\vec{k}}}{\partial \omega}\right)\right. \tag{18}$$

$$\left. + \left(\frac{\partial I_{\vec{k}}}{\partial \vec{k}_\perp}\right).\nabla_\perp - \left(\frac{\partial I_{\vec{k}}}{\partial \omega}\right)\frac{\partial}{\partial t}\right] \frac{J_j^2\,\hat{O}_\sigma \bar{f}_\sigma}{\omega - k_\parallel v_\parallel - j\omega_{c\sigma}} \quad - \frac{q_\sigma^2}{2m_\sigma^2}\sum_j \int d\kappa$$

$$\times\ \hat{O}_\sigma\, I_{\vec{k}} \left\{ \frac{(\partial J_j^2/\partial \vec{k}_\perp).\nabla_\perp - 2i\,J_j^2\left[1 - j(\vec{k}_\perp \times \nabla_\perp)_\parallel / k_\perp^2\right]}{\omega - k_\parallel v_\parallel - j\omega_{c\sigma}}\right\} \hat{O}_\sigma \bar{f}_\sigma \ ,$$

where the operator $\hat{O}_\sigma$ is defined by

$$\hat{O}_\sigma \equiv k_\parallel \frac{\partial}{\partial v_\parallel} + \frac{j\,\omega_{c\sigma}}{v_\perp}\frac{\partial}{\partial v_\perp} - \frac{(\vec{k}_\perp \times \nabla_\perp)_\parallel}{\omega_{c\sigma}} ,$$

and $I_{\vec{K}} = I_{\vec{K}}(\vec{R})$, $J_j = J_j\,(k_\perp v_\perp/\omega_{c\sigma})$, $\bar{f}_\sigma = \bar{f}_\sigma\ (v_\perp, v_\parallel, \vec{R})$, and $\vec{R} \equiv \{\vec{r},t\}$. Notice that (18) has been obtained without any assumption on the magnitude of the Larmor radius.

Separating the resonant and the nonresonant terms, one obtains

$$\begin{aligned}
\frac{\partial}{\partial t}\bar{f}_\sigma = &-\frac{q_\sigma^2}{2m_\sigma^2}\sum_j \mathcal{P}\int d\kappa\ \hat{O}_\sigma \frac{(\partial J_j^2/\partial \vec{k}_\perp)\cdot\nabla_\perp I_{\vec{K}}}{\omega - k_\parallel v_\parallel - j\omega_{c\sigma}}\,\hat{O}_\sigma \bar{f}_\sigma \\
&-\frac{q_\sigma^2}{2m_\sigma^2}\sum_j \mathcal{P}\int d\kappa\ \hat{O}_\sigma \left[\nabla_\perp\cdot\left(\frac{\partial I_{\vec{K}}}{\partial \vec{k}_\perp}\right) - \frac{\partial}{\partial t}\left(\frac{\partial I_{\vec{K}}}{\partial \omega}\right) + \left(\frac{\partial I_{\vec{K}}}{\partial \vec{k}_\perp}\right)\cdot\nabla_\perp - \left(\frac{\partial I_{\vec{K}}}{\partial \omega}\right)\frac{\partial}{\partial t}\right] \qquad (19)\\
&\times \frac{J_j^2\,\hat{O}_\sigma \bar{f}_\sigma}{\omega - k_\parallel v_\parallel - j\omega_{c\sigma}} + \pi\frac{q_\sigma^2}{m_\sigma^2}\sum_j \int d\kappa\ \hat{O}_\sigma I_{\vec{K}} J_j^2 \\
&\times \delta(\omega - k_\parallel v_\parallel - j\omega_{c\sigma})\,\hat{O}_\sigma \bar{f}_\sigma - \frac{q_\sigma^2}{2m_\sigma^2}\sum_j \mathcal{P}\int d\kappa\ \hat{O}_\sigma \frac{I_{\vec{K}}\,(\partial J_j^2/\partial \vec{k}_\perp)\cdot\nabla_\perp}{\omega - k_\parallel v_\parallel - j\omega_{c\sigma}}\,\hat{O}_\sigma \bar{f}_\sigma .
\end{aligned}$$

Notice that all the terms in (19) are real, since all imaginary parts vanish (due to the parity of the integrand).

B. Expansion of the closure equation

Expanding (16) to first order in $|\vec{Q}|/|\vec{K}|$, $|\vec{Q}'|/|\vec{K}|$ and transforming back into space and time with respect to the variables $\vec{Q}$, $\vec{Q}'$, we obtain

$$
\begin{aligned}
I_{\vec{K}} &+ \frac{i}{2}\left(\frac{\partial}{\partial \vec{k}_\perp}\cdot\nabla_\perp - \frac{\partial}{\partial t}\frac{\partial}{\partial \omega}\right) I_{\vec{K}} = -i \sum_\sigma 2\pi \frac{4\pi q_\sigma^2}{2 m_\sigma k^2} \sum_j \int dv \\
&\times \left[\nabla\cdot\left(\frac{\partial I_{\vec{K}}}{\partial \vec{k}_\perp}\right) - \frac{\partial}{\partial t}\left(\frac{\partial I_{\vec{K}}}{\partial \omega}\right) + \left(\frac{\partial I_{\vec{K}}}{\partial \vec{k}_\perp}\right)\cdot\nabla_\perp - \left(\frac{\partial I_{\vec{K}}}{\partial \omega}\right)\frac{\partial}{\partial t}\right] \\
&\times \frac{J_j^2\, \hat{O}_\sigma \bar{f}_\sigma}{\omega - k_\parallel v_\parallel - j\omega_{c\sigma}} \qquad - \sum_\sigma 2\pi \frac{4\pi q_\sigma^2}{2 m_\sigma k^2} \sum_j dv \qquad\qquad (20) \\
&\times \frac{I_{\vec{K}}}{\omega - k_\parallel v_\parallel - j\omega_{c\sigma}} \left\{ i\left(\frac{\partial J_j^2}{\partial \vec{k}_\perp}\right)\cdot\nabla_\perp + 2J_j^2\left[1 - j\,\frac{(\vec{k}_\perp \times \nabla_\perp)_\parallel}{k_\perp^2}\right]\right\} \hat{O}_\sigma \bar{f}_\sigma \; .
\end{aligned}
$$

C. Dispersion relation

To the lowest order, the real part of (20) yields

$$
I_{\vec{K}}\left(1 + \sum_\sigma 2\pi \frac{4\pi q_\sigma^2}{m_\sigma k^2} \sum_j \mathcal{P}\int dv \, \frac{J_j^2\, \hat{O}_\sigma \bar{f}_\sigma}{\omega - k_\parallel v_\parallel - j\omega_{c\sigma}}\right) = 0 \; . \qquad (21)
$$

From (21), we have the dispersion relation

$$1 + \chi_r = 0 , \tag{22}$$

where χ_r is the real part of the susceptibility defined by

$$\chi_r(\vec{\kappa},\vec{R}) = \sum_\sigma 2\pi \frac{4\pi q_\sigma^2}{m_\sigma k^2} \sum_j \mathcal{P}\int dv \frac{J_j^2 \hat{O}_\sigma \bar{f}_\sigma}{\omega - k_\parallel v_\parallel - j\omega_{c\sigma}} . \tag{23}$$

It is worth noting that some authors[3,4] include within the definition (23) some higher-order terms. We do not keep them in order to be consistent with our ordering.

D. Equation of motion for the correlation function

To the lowest order in the expansion parameters, the imaginary part of the closure equation reads

$$\left(\nabla_\perp \cdot \frac{\partial}{\partial \vec{k}_\perp} - \frac{\partial}{\partial t}\frac{\partial}{\partial \omega}\right) I_{\vec{\kappa}} = -\nabla_\perp \cdot \left(\chi_r \frac{\partial I_{\vec{\kappa}}}{\partial \vec{k}_\perp}\right)$$

$$+ \frac{\partial}{\partial t}\left(\chi_r \frac{\partial I_{\vec{\kappa}}}{\partial \omega}\right) - \left(\frac{\partial I_{\vec{\kappa}}}{\partial \vec{k}_\perp}\right)\cdot \nabla_\perp \chi_r + \left(\frac{\partial I_{\vec{\kappa}}}{\partial t}\right)\frac{\partial \chi_r}{\partial \omega} \tag{24}$$

$$- 2\chi_i I_{\vec{\kappa}} - \frac{I_{\vec{\kappa}}}{k^2} \nabla_\perp \cdot \frac{\partial}{\partial \vec{k}_\perp}(k^2 \chi_r) ,$$

where χ_i is the imaginary part of the susceptibility, defined by

$$\chi_i(\vec{k},\vec{R}) = -\pi \sum_\sigma 2\pi$$

$$\times \frac{4\pi q_\sigma^2}{m_\sigma k^2} \sum_j \int d\mathbf{v}\, J_j^2\, \delta(\omega - k_\parallel v_\parallel - j\omega_{c\sigma})\, \hat{O}_\sigma \bar{f}_\sigma . \tag{25}$$

Equation (24) becomes, after some algebra and using (21),

$$-2\chi_i I_{\vec{k}} + \left(\frac{\partial \chi_r}{\partial \vec{k}_\perp}\right)\cdot\nabla_\perp I_{\vec{k}} - \left(\frac{\partial \chi_r}{\partial \omega}\right)\frac{\partial I_{\vec{k}}}{\partial t} - (\nabla_\perp \chi_r)\cdot\frac{\partial I_{\vec{k}}}{\partial \vec{k}_\perp}$$

$$+\left(\frac{\partial \chi_r}{\partial t}\right)\frac{\partial I_{\vec{k}}}{\partial \omega} + I_{\vec{k}}\left(\frac{\partial}{\partial \vec{k}_\perp}\cdot\nabla_\perp - \frac{\partial}{\partial \omega}\frac{\partial}{\partial t}\right)\chi_r = \frac{I_{\vec{k}}}{k^2}\nabla_\perp\cdot\frac{\partial}{\partial \vec{k}_\perp}\left(k^2\chi_r\right). \tag{26}$$

Equation (26) governs the evolution of the correlation function. Together with (19), it forms a closed system describing the evolution of the averaged distribution function (the plasma) and the correlation function (the fluctuations).

IV. CONSERVATION LAWS

Equations (1) and (6) constitute a conservative system. One should verify that the equations derived therefrom (i.e., (19) and (26)) possess the same conservation properties, namely those related to the densities of particles, momentum, and energy.

A. Particles

One can show that (19) conserves the number of particles of each species locally. To do so, one only need to integrate (19) over all velocities. This yields the equation of continuity

$$\frac{\partial}{\partial t} n_\sigma + \nabla \cdot (n_\sigma \vec{v}_\sigma) = 0 , \tag{27}$$

where n_σ and $\vec{v}_\sigma$ are the particle density and average velocity, respectively. The former is defined by

$$n_\sigma = 2\pi \int dv \, \bar{f}_\sigma , \tag{28}$$

whereas $\vec{v}_\sigma$ is given by

$$\begin{aligned}
\vec{v}_\sigma = & - \frac{2\pi q_\sigma^2}{\omega_{c\sigma} n_\sigma m_\sigma^2} \int dk \, (\vec{e}_\parallel \times \vec{k}) \sum_j \Big\{ \\
& - I_{\vec{k}} \pi \int dv \, J_j^2 \, \delta(\omega - k_\parallel v_\parallel - j\omega_{c\sigma}) \, \hat{O}_\sigma \bar{f}_\sigma \\
& + \frac{1}{2} P \int dv \, \frac{(\partial J_j^2 / \partial \vec{k}_\perp) \cdot \nabla_\perp I_{\vec{k}}}{\omega - k_\parallel v_\parallel - j\omega_{c\sigma}} \hat{O}_\sigma \bar{f}_\sigma \\
& + \frac{1}{2} I_{\vec{k}} P \int dv \, \frac{(\partial J_j^2 / \partial \vec{k}_\perp) \cdot \nabla_\perp \hat{O}_\sigma}{\omega - k_\parallel v_\parallel - j\omega_{c\sigma}} \bar{f}_\sigma \\
& + \frac{1}{2} P \int dv \left[\nabla_\perp \cdot \left(\frac{\partial I_{\vec{k}}}{\partial \vec{k}_\perp} \right) - \frac{\partial}{\partial t} \left(\frac{\partial I_{\vec{k}}}{\partial \omega} \right) + \left(\frac{\partial I_{\vec{k}}}{\partial \vec{k}_\perp} \right) \cdot \nabla_\perp - \left(\frac{\partial I_{\vec{k}}}{\partial \omega} \right) \cdot \frac{\partial}{\partial t} \right] \\
& \times \frac{J_j^2 \, \hat{O}_\sigma \bar{f}_\sigma}{\omega - k_\parallel v_\parallel - j\omega_{c\sigma}} \Big\} ,
\end{aligned} \tag{29}$$

where $\vec{e}_{\parallel}$ is the unit vector parallel to the axis. Notice that $\vec{v}_{\sigma}$ is perpendicular to the magnetic field. Multiplying (27) by q_{σ} (the particle charge), and summing over species, one obtains the equation of continuity for the total electric charge,

$$\frac{\partial}{\partial t} \rho + \nabla \cdot \vec{j} = 0, \tag{30}$$

where the charge density ρ is defined by

$$\rho \equiv \sum_{\sigma} q_{\sigma} \, 2\pi \int dv \, \bar{f}_{\sigma} , \tag{31}$$

and the current density $\vec{j}$ is given by

$$\vec{j} = \frac{c}{8\pi B_0} \int d\kappa \, (\vec{e}_{\parallel} \times \vec{k}) \, k^2 \Big\{ \nabla_{\perp} \cdot \frac{\partial I_{\vec{k}}}{\partial \vec{k}_{\perp}} - \frac{1}{k^2} \nabla_{\perp} \cdot \Big[I_{\vec{k}} \frac{\partial}{\partial \vec{k}_{\perp}} (k^2 \chi_r) \Big] \Big\} . \tag{32}$$

To obtain the expression for $\vec{j}$, use was made of (24). Notice that $\vec{j}$ is a first order quantity and is therefore small. Global conservation is guaranteed since local conservation implies global conservation whenever boundary effects are neglected (as is the case here).

B. Energy

Multiplying (19) by $m_\sigma/2\ (v_\perp^2 + v_\parallel^2)$ one obtains

$$\frac{\partial}{\partial t} W_\sigma + \nabla \cdot \vec{S}_\sigma = \nu_\sigma\ , \tag{33}$$

where W_σ is the energy density and $\vec{S}_\sigma$ is the energy flux of species σ. The latter is given by

$$\begin{aligned} \vec{S}_\sigma = & - \frac{2\pi c q_\sigma}{4 B_0} \int d\kappa \, (\vec{e}_\parallel^{\,0} \times \vec{k}) \sum_j \Big\{ \mathcal{P} \int dv \, (v_\perp^2 + v_\parallel^2) \\ & \times \frac{(\partial J_j^2 / \partial \vec{k}_\perp) \cdot \nabla_\perp I_{\vec{k}} \, \hat{O}_\sigma \bar{f}_\sigma}{\omega - k_\parallel v_\parallel - j\omega_{c\sigma}} + \mathcal{P} \int dv \, (v_\perp^2 + v_\parallel^2) \Big[\nabla_\perp \cdot \Big(\frac{\partial I_{\vec{k}}}{\partial \vec{k}_\perp} \Big) \\ & - \frac{\partial}{\partial t} \Big(\frac{\partial I_{\vec{k}}}{\partial \omega} \Big) + \Big(\frac{\partial I_{\vec{k}}}{\partial \vec{k}_\perp} \Big) \cdot \nabla_\perp - \Big(\frac{\partial I_{\vec{k}}}{\partial \omega} \Big) \frac{\partial}{\partial t} \Big] \frac{J_j^2 \, \hat{O}_\sigma \bar{f}_\sigma}{\omega - k_\parallel v_\parallel - j\omega_{c\sigma}} \\ & - 2\pi \, I_{\vec{k}} \int dv \, (v_\perp^2 + v_\parallel^2) \, J_j^2 \, \delta(\omega - k_\parallel v_\parallel - j\omega_{c\sigma}) \, \hat{O}_\sigma \bar{f}_\sigma \\ & + I_{\vec{k}} \, \mathcal{P} \int dv \, \frac{(v_\perp^2 + v_\parallel^2)(\partial J_j^2 / \partial \vec{k}_\perp) \cdot \nabla_\perp}{\omega - k_\parallel v_\parallel - j\omega_{c\sigma}} \, \hat{O}_\sigma \bar{f}_\sigma \Big\} , \end{aligned} \tag{34}$$

and the former is by definition

$$W_\sigma \equiv 2\pi \frac{m_\sigma}{2} \int dv \, (v_\perp^2 + v_\parallel^2) \bar{f}_\sigma \ . \tag{35}$$

The term on the right-hand side of (33), ν_σ, may be interpreted as a source term, arising from the action of the other species of particles. Such a term did not appear when particle conservation was considered since even though particles of different species may exchange momentum and energy, via their interaction with the waves, there is no creation or annihilation of particles. Its explicit form is given by

$$\nu_\sigma = 2\pi \frac{q_\sigma^2}{2 m_\sigma^2} \int d\kappa\, \omega \sum_j \Bigg\{ P\int dv \frac{(\partial J_j^2/\partial \vec{k}_\perp)\cdot \nabla_\perp I_{\vec{k}}\ \hat{O}_\sigma \bar{f}_\sigma}{\omega - k_\parallel v_\parallel - j\omega_{c\sigma}}$$

$$+ P\int dv \left[\nabla_\perp \cdot \left(\frac{\partial I_{\vec{k}}}{\partial \vec{k}_\perp}\right) - \frac{\partial}{\partial t}\left(\frac{\partial I_{\vec{k}}}{\partial \omega}\right) + \left(\frac{\partial I_{\vec{k}}}{\partial \vec{k}_\perp}\right)\cdot \nabla_\perp - \left(\frac{\partial I_{\vec{k}}}{\partial \omega}\right)\frac{\partial}{\partial t}\right]$$

$$\times \frac{J_j^2\ \hat{O}_\sigma \bar{f}_\sigma}{\omega - k_\parallel v_\parallel - j\omega_{c\sigma}} - 2\pi I_{\vec{k}} \int dv\, J_j^2\ \delta(\omega - k_\parallel v_\parallel - j\omega_{c\sigma})\ \hat{O}_\sigma \bar{f}_\sigma \tag{36}$$

$$+ I_{\vec{k}}\, P\int dv \frac{(\partial J_j^2/\partial \vec{k}_\perp)\cdot\nabla_\perp\ \hat{O}_\sigma \bar{f}_\sigma}{\omega - k_\parallel v_\parallel - j\omega_{c\sigma}} \Bigg\} .$$

If we sum the quantity ν_σ over all species, substitute (24) and integrate by parts with respect to ω, we obtain

$$\nu = \nabla_\perp \cdot \frac{1}{8\pi} \int d\kappa\, \omega\, k^2 \left[-\frac{\partial I_{\vec{k}}}{\partial \vec{k}_\perp} + \frac{I_{\vec{k}}}{k^2} \frac{\partial}{\partial \vec{k}_\perp} k^2 \chi_r \right]$$

$$- \frac{\partial}{\partial t} \frac{1}{8\pi} \int d\kappa\, k^2 I_{\vec{k}} , \tag{37}$$

where

$$\nu = \sum_{\sigma} \nu_{\sigma} \quad . \tag{38}$$

One can see that ν consists of the sum of a time derivative term and a divergence term. Thus if (33) is summed over all species, it can be written in a completely conservative form,

$$\frac{\partial}{\partial t}\left(W + W_F \right) + \nabla \cdot \left(\vec{S} + \vec{S}_F \right) = 0 \ , \tag{39}$$

where

$$W = \sum_{\sigma} W_{\sigma} \ , \tag{40}$$

$$\vec{S} = \sum_{\sigma} \vec{S}_{\sigma} \ . \tag{41}$$

W_F and $\vec{S}_F$ are the electrostatic energy density and energy flux, and are given by

$$W_F = \frac{1}{8\pi} \int d\kappa \, k^2 I_{\vec{k}} \quad , \tag{42}$$

and

$$\vec{S}_F = \frac{1}{8\pi} \int d\kappa \, k^2 \omega \left[\frac{I_{\vec{k}}}{k^2} \frac{\partial}{\partial \vec{k}_1}\left(k^2 \chi_r \right) - \frac{\partial}{\partial \vec{k}_1} I_{\vec{k}} \right] . \tag{43}$$

C. Momentum

Due to the presence of the magnetic field it is the momentum $\vec{p}_\sigma$ defined by

$$\overline{\vec{p}_\sigma = m_\sigma \left(\vec{v} + \omega_{c\sigma} \vec{e}_\parallel \times \vec{x} \right)} \tag{44}$$

that is conserved for each particles. The momentum density $\vec{P}_\sigma$ is obtained by averaging (44) over all velocities,

$$\overline{\vec{P}_\sigma = \int \vec{p}_\sigma \bar{f}_\sigma \, d^3v} \tag{45}$$

We therefore have to multiply (19) by $\vec{p}_\sigma$, integrate over velocities, and see whether or not we obtain an equation of the continuity type. Let us consider first the direction parallel to the magnetic field. We obtain

$$\overline{\frac{\partial}{\partial t} P_{\sigma\parallel} + \nabla \cdot \vec{\eta}_\sigma = \beta_\sigma} \tag{46}$$

The quantity $\vec{\eta}_\sigma$ may be interpreted as a part of the pressure tensor, and is given by

$$\vec{\eta}_\sigma = \frac{q_\sigma c}{2 B_0} \, 2\pi \int d\kappa \, (\vec{e}_\parallel \times \vec{k}) \sum_j \Big\{ - \mathcal{P} \int dv$$

$$\times \frac{v_\parallel (\partial J_j^2 / \partial \vec{k}_\perp) \cdot \nabla_\perp I_{\vec{k}} \, \hat{O}_\sigma \bar{f}_\sigma}{\omega - k_\parallel v_\parallel - j\omega_{c\sigma}} - \mathcal{P} \int dv \, v_\parallel \Big[\nabla_\perp \cdot \Big(\frac{\partial I_{\vec{k}}}{\partial \vec{k}_\perp}\Big)$$

$$- \frac{\partial}{\partial t} \Big(\frac{\partial I_{\vec{k}}}{\partial \omega}\Big) + \Big(\frac{\partial I_{\vec{k}}}{\partial \vec{k}}\Big) \cdot \nabla_\perp - \Big(\frac{\partial I_{\vec{k}}}{\partial \omega}\Big) \frac{\partial}{\partial t} \Big] \frac{J_j^2 \, \hat{O}_\sigma \bar{f}_\sigma}{\omega - k_\parallel v_\parallel - j\omega_{c\sigma}} \tag{47}$$

$$+ 2\pi I_{\vec{k}} \int dv \, v_\parallel \, J_j^2 \, \delta(\omega - k_\parallel v_\parallel - j\omega_{c\sigma}) \, \hat{O}_\sigma \bar{f}_\sigma$$

$$- I_{\vec{k}} \mathcal{P} \int dv \, v_\parallel \, \frac{(\partial J_j^2 / \partial \vec{k}_\perp) \cdot \nabla_\perp}{\omega - k_\parallel v_\parallel - j\omega_{c\sigma}} \, \hat{O}_\sigma \bar{f}_\sigma \Big\},$$

while β_σ may be interpreted as a source term arising from the interaction of the other particles with the particles of species σ, via the waves. It reads

$$\beta_\sigma = -\frac{2\pi q_\sigma^2}{2m_\sigma} \int d\vec{k}\; k_\parallel \sum_j \Bigg\{ -\mathcal{P}\int dv \frac{(\partial J_j^2/\partial \vec{k}_\perp)\cdot\nabla_\perp I_{\vec{k}}\; \hat{O}_\sigma \bar{f}_\sigma}{\omega - k_\parallel v_\parallel - j\omega_{c\sigma}}$$

$$-\mathcal{P}\int dv \left[\nabla_\perp\cdot\left(\frac{\partial I_{\vec{k}}}{\partial \vec{k}_\perp}\right) - \frac{\partial}{\partial t}\left(\frac{\partial I_{\vec{k}}}{\partial \omega}\right) + \left(\frac{\partial I_{\vec{k}}}{\partial \vec{k}_\perp}\right)\cdot\nabla_\perp - \left(\frac{\partial I_{\vec{k}}}{\partial \omega}\right)\frac{\partial}{\partial t}\right] \tag{48}$$

$$\times \frac{J_j^2\; \hat{O}_\sigma \bar{f}_\sigma}{\omega - k_\parallel v_\parallel - j\omega_{c\sigma}} - I_{\vec{k}}\,\mathcal{P}\int dv \frac{(\partial J_j^2/\partial \vec{k}_\perp)\cdot\nabla_\perp \hat{O}_\sigma \bar{f}_\sigma}{\omega - k_\parallel v_\parallel - j\omega_{c\sigma}}$$

$$+ 2\pi I_{\vec{k}} \int dv\, J_j^2\, \delta(\omega - k_\parallel v_\parallel - j\omega_{c\sigma})\, \hat{O}_\sigma \bar{f}_\sigma \Bigg\} .$$

As in the case of energy, the source term simplifies when it is summed over all species, and when the equation of motion (24) is used. We find

$$\sum_\sigma \beta_\sigma = -\nabla\cdot\vec{\eta}_F , \tag{49}$$

where $\vec{\eta}_F$ is given by

$$\vec{\eta}_F = \frac{1}{8\pi} \int d\kappa \, k_{\parallel} \left[k^2 \frac{\partial I_{\vec{\kappa}}}{\partial \vec{k}_{\perp}} - I_{\vec{\kappa}} \frac{\partial}{\partial \vec{k}_{\perp}} (k^2 \chi_r) \right] . \quad (50)$$

Notice that there is no time derivative term in the expression (50). This is due to the fact that the Poynting vector is zero since we work within the electrostatic approximation. If we sum (46) over species, we find

$$\frac{\partial}{\partial t} P_{\parallel} + \nabla \cdot (\vec{\eta} + \vec{\eta}_F) = 0 , \quad (51)$$

where

$$\vec{\eta} = \sum_{\sigma} \vec{\eta}_{\sigma} . \quad (52)$$

Equation (51) shows that the total parallel momentum is locally conserved.

We now turn our attention to the components perpendicular to the magnetic field. Multiplying (19) by $\vec{p}_{\sigma\perp}$ and integrating over velocities, we obtain

$$\frac{\partial}{\partial t} \vec{P}_{\sigma\perp} + m_{\sigma} \omega_{c\sigma} (\vec{e}_{\parallel} \times \vec{\chi}) \, \nabla \cdot (n_{\sigma} \vec{V}_{\sigma}) = 0 . \quad (53)$$

Commuting the ∇ operator with $(\vec{e}_{\parallel} \times \vec{x})$ yields

$$\frac{\partial}{\partial t} \vec{P}_{\sigma\perp} + m_\sigma \omega_{c\sigma} (\nabla \cdot n_\sigma \vec{v}_\sigma) \, \vec{e}_{\parallel} \times \vec{x} = m_\sigma \omega_{c\sigma} \, \vec{e}_{\parallel} \times (n_\sigma \vec{v}_\sigma) , \tag{54}$$

where the ∇ operator acts on both $n_\sigma \vec{v}_\sigma$ and $(\vec{e}_{\parallel} \times \vec{x})$. The three terms of (54) are, from left to right, the time derivative of the perpendicular mementum density, the perpendicular component of the divergence of the pressure tensor, and a source term. It can easily be shown that the source term of (54), which we designate $\vec{\mu}_\sigma$, reduces to a gradient term once it is summed over particle species,

$$\vec{\mu} \equiv \sum_\sigma \vec{\mu}_\sigma = \frac{-1}{8\pi} \int d\kappa \, \vec{k}_\perp k^2 \left\{ \nabla_\perp \cdot \frac{\partial I_{\vec{k}}}{\partial \vec{k}_\perp} - \frac{1}{k^2} \nabla_\perp \cdot \left[I_{\vec{k}} \frac{\partial}{\partial \vec{k}_\perp} (k^2 \chi_r) \right] \right\} , \tag{55}$$

where (24) and (29) have been used. Thus, if (54) is summed over species, we find

$$\frac{\partial}{\partial t} \vec{P}_\perp + \sum_\sigma m_\sigma \omega_{c\sigma} (\nabla \cdot n_\sigma \vec{v}_\sigma) \, \vec{e}_{\parallel} \times \vec{x} - \vec{\mu} = 0 , \tag{56}$$

where

$$\vec{P}_\perp = \sum_\sigma \vec{P}_{\sigma\perp} . \tag{57}$$

In view of (55), the perpendicular momentum is also locally conserved.

V. CONCLUSION

We have derived the quasilinear equations for an inhomogeneous magnetized plasma interacting with electrostatic turbulence. These equations were obtained without using the expansion in the ratio of the Larmor radius to the inhomogeneity scale, and without introducing the random phase approximation. They locally conserve particle number, momentum, and energy and simultaneously take into account both the resonant and nonresonant interactions. To the best of our knowledge such equations have not yet appeared in the literature.

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