MARCH 1981 LRP 180/81

Ponderomotive Effect in a Two Ion Species Magnetized and Bounded Plasma

E.S. Weibel and M.C. Festeau-Barrioz

Ponderomotive Effect in a Two Ion Species Magnetized and Bounded Plasma

E.S. Weibel and M.C. Festeau-Barrioz

Centre de Recherches en Physique des Plasmas
Association Confédération Suisse - Euratom
Ecole Polytechnique Federale de Lausanne

Abstract

The non-linear wave equation for a magnetized two ion species plasma enclosed in a cylindrical cavity can be obtained from a Lagrangian integral. The Lagrangian contains the description of the density modifications due to the ponderomotive force.

I INTRODUCTION AND RESULTS

Large amplitude waves whose frequencies lie close to the ion cyclotron resonance can exert forces which tend to spatially separate ions of different charge to mass ratio. Hidekuma et al.[1] and S. Hiroe et al.[2] have observed this effect in magnetic cusps. J.M. Dawson et al.[3] give experimental evidence for such separation in a uniform magnetic field. Much earlier, T. Consoli et al.[4] used the ponderomotive effect of electron cyclotron waves to accelerate plasmas. The waves involved were electrostatic. The ponderomotive force due to electromagnetic waves can also produce isotope separation. As we have shown recently[5,6] a left circularly polarized wave propagating in a two ion species plasma tends to spatially separate them provided that the frequency of the wave lies between the two ion Larmor frequencies. Unfortunately this result cannot be checked experimentally since it is impossible to set up a pure left circularly polarized wave in a finite volume. It is even difficult to create an approximately pure wave of this type. Boundary conditions always cause the presence of oblique waves which are generally much stronger than the wanted wave, as we show in the appendix. We therefore have extended our work to the case of waves resonating within a cylindrical cavity, of radius a and height h.

We consider a plasma consisting of two kinds of ions and of electrons whose masses, charges, densities, and temperatures are designated by m_σ, q_σ, n_σ, T_σ, where $\sigma = 1, 2, e$. A constant magnetic field $\vec{B}_0$ is present. We use a cylindrical coordinate system whose z-axis is parallel to $\vec{B}_0$. We assume the plasma to be cold as far as the wave propagation is concerned, that is, we assume

$$\omega \gg k v_{T\sigma}$$

$$|\omega - \Omega_\sigma| \gg k v_{T\sigma} \tag{1}$$

where k is the largest wave number and ω the frequency of the wave while $\Omega_\sigma = q_\sigma B_0 / m_\sigma$ and $v_{T\sigma} = (T_\sigma / m_\sigma)^{1/2}$. The temperatures cannot be neglected when we determine the densities from the equilibrium of the pressure gradients and the ponderomotive force. The plasma is assumed to be collisionless, that is

$$\tau_\sigma |\omega - \Omega_\sigma| \gg 1 \tag{2}$$

where τ_σ is the collision frequency of the ion. These are stringent conditions, but they can always be met by sufficiently reducing the plasma density. They are graphically represented in Fig. 1 for singly ionized neon assuming an ionization degree of one percent. The inequality (2) has been evaluated for coulomb collisions. The condition for ion-neutral collisions is less severe than (1) and (2) combined.

The description of cavity modes necessarily leads to a non-linear partial differential equation for the electric field vector. Numerical solutions have been obtained by means of a variational formulation of the problem. A particular solution is presented in the Figures 2, 3, 4 and 5. They pertain to a singly ionized neon plasma containing both isotopes, Ne^{20} and Ne^{22}, in their natural concentrations : $\beta_1 = 0.0882$ (Ne^{22}) and $\beta_2 = 0.9118$ (Ne^{20}). The calculations were made using the dimensionless quantities given in section V. Therefore a single numerical solution represents a family of physical situations. The relevant input parameters are

$$h/a = 10.0$$

$$\alpha = \frac{m_1 - m_2}{m_1 + m_2} = 0.04761$$

$$\varepsilon = \frac{2 m_1 m_2 \omega}{e B_0 (m_1 + m_2)} = 1.0$$

$$\eta = \frac{2 m_1 m_2 n_{e0}}{\varepsilon_0 (m_1 + m_2) B_0^2} = 39.56$$

These values are obtained, for instance, if one takes $B_0 = 1.0$ Tesla $n_0 = 10^{16}\ m^{-3}$ and $\omega = 4.56$ MHz .

In the Figures 2 and 3 the electric fields are measured in units of $(T_i/m)^{1/2} B_0$ where $m = 2m_1 m_2/(m_1 + m_2)$ while the coordinates are measured in units of the radius.

Figures 4 and 5 show respectively the density distributions of the two isotopes. The largest density modification occurs at $r = 0.6\,a$ and $z = 0$. There the density of the minority species is increased by 28 percent while the density of the majority is reduced by 5 percent. The nonlinear eigenvalue belonging to this solution is $\lambda = 2.583*10^{-4}$. It yields a radius of $a = \lambda^{1/2} c\, \omega^{-1}$. For the abovementioned example this becomes a = 1.056m so that h = 10.56m.

The numerical techniques used to obtain these solutions are of sufficient interest to warrant their description together with other solutions, in a forthcoming report. In the following sections natural units, $c = \mu_0 = \varepsilon_0 = 1$, will be used.

II THE NONLINEAR WAVE EQUATION

We write the electric field in the form

$$\mathbf{E}(r,\varphi,z,t) = \sqrt{2}\left[E_r(r,z)\cos\varkappa\,,\ E_\varphi(r,z)\sin(\varkappa),\ E_z(r,z)\cos\varkappa\right] \quad (3)$$

where

$$\varkappa = \omega t - \nu\varphi$$

is the phase and ν an integer. The amplitudes E_r , E_φ , E_z are real. On the conducting walls of the cavity they must satisfy the boundary conditions

$$
\begin{aligned}
E_\varphi(a,z) &= E_z(a,z) = 0 \\
E_\varphi(r,0) &= E_r(r,0) = 0 \qquad (4) \\
E_\varphi(r,h) &= E_r(r,h) = 0
\end{aligned}
$$

where a and h are the radius and height of the cavity. Solutions which are singular at the axis can be excluded by the boundary condition

$$\lim_{r \to 0} r E_\varphi = 0 \qquad (5)$$

It will be convenient to use the real vector

$$\vec{E}(r,z) = (E_r , E_\varphi , E_z) \qquad (6)$$

which must not be confused with $\mathbb{E}$ defined in (3). The wave equation can be written in the form

$$\vec{\nabla}_{rz}\times\left(\vec{\nabla}_{rz}\times\vec{E}\right) - \omega^2\left(1+\overleftrightarrow{\chi}\right)\vec{E} = 0 \tag{7}$$

where $\overleftrightarrow{\chi}$ represents the susceptibility. The differential operator in Eq. (7) has the form

$$\vec{\nabla}_{rz}\times\left(\vec{\nabla}_{rz}\times\vec{E}\right) =$$

$$\Big(\frac{\nu^2}{r^2}E_r - \frac{\partial^2}{\partial z^2}E_r - \frac{\nu}{r^2}\frac{\partial}{\partial r}rE_\varphi + \frac{\partial^2}{\partial r\partial z}E_z\ ,$$

$$-\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}rE_\varphi - \frac{\partial^2}{\partial z^2}E_\varphi + \nu\frac{\partial}{\partial r}\frac{1}{r}E_r + \frac{\nu}{r}\frac{\partial}{\partial z}E_z\ , \tag{8}$$

$$-\frac{1}{r}\frac{\partial}{\partial r}rE_z + \frac{\nu^2}{r^2}E_z - \frac{\nu}{r}\frac{\partial}{\partial z}E_\varphi + \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial z}E_r\Big)$$

The susceptibility

$$\overleftrightarrow{\chi} = \begin{pmatrix} \chi_1 & \chi_2 & 0 \\ \chi_2 & \chi_1 & 0 \\ 0 & 0 & \chi_z \end{pmatrix}$$

which must not be confused with $\mathbb{E}$ defined in (3). The wave equation can be written in the form

$$\vec{\nabla}_{rz}\times\left(\vec{\nabla}_{rz}\times\vec{E}\right) - \omega^2\left(1+\overleftrightarrow{\chi}\right)\vec{E} = 0 \tag{7}$$

where $\overleftrightarrow{\chi}$ represents the susceptibility. The differential operator in Eq. (7) has the form

$$\begin{aligned}
&\vec{\nabla}_{rz}\times\left(\vec{\nabla}_{rz}\times\vec{E}\right) = \\
&\left(\frac{\nu^2}{r^2}E_r - \frac{\partial^2}{\partial z^2}E_r - \frac{\nu}{r^2}\frac{\partial}{\partial r}rE_\varphi + \frac{\partial^2}{\partial r\partial z}E_z\right., \\
&-\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}rE_\varphi - \frac{\partial^2}{\partial z^2}E_\varphi + \nu\frac{\partial}{\partial r}\frac{1}{r}E_r + \frac{\nu}{r}\frac{\partial}{\partial z}E_z, \\
&\left.-\frac{1}{r}\frac{\partial}{\partial r}rE_z + \frac{\nu^2}{r^2}E_z - \frac{\nu}{r}\frac{\partial}{\partial z}E_\varphi + \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial z}E_r\right)
\end{aligned} \tag{8}$$

The susceptibility

$$\overleftrightarrow{\chi} = \begin{pmatrix} \chi_1 & \chi_2 & 0 \\ \chi_2 & \chi_1 & 0 \\ 0 & 0 & \chi_z \end{pmatrix}$$

has only real components. They are given by

$$\chi_1 = \tfrac{1}{2}(\chi_R + \chi_L)$$

$$\chi_2 = \tfrac{1}{2}(\chi_R - \chi_L)$$

and

$$\chi_R = -\sum_\sigma \frac{\omega_{p\sigma}^2}{\omega(\omega + \Omega_\sigma)}$$

$$\chi_L = -\sum_\sigma \frac{\omega_{p\sigma}^2}{\omega(\omega - \Omega_\sigma)}$$

$$\chi_z = -\sum_\sigma \frac{\omega_{p\sigma}^2}{\omega^2}$$

where

$$\omega_{p\sigma}^2 = \frac{q_\sigma^2 n_\sigma}{m_\sigma} \qquad (9)$$

$$\Omega_\sigma = \frac{q_\sigma B_0}{m_\sigma}$$

The wave equation (7) is a real equation for the real vector (6).

The ponderomotive force acting on the species σ is the gradient of the potential

$$\phi_\sigma = \frac{q_\sigma^2}{2m_\sigma\omega}\left(\frac{E_L^2}{\omega-\Omega_\sigma} + \frac{E_R^2}{\omega+\Omega_\sigma} + \frac{E_z^2}{\omega}\right)$$

where

$$E_L = (E_r - E_\varphi)/\sqrt{2}$$

$$E_R = (E_r + E_\varphi)/\sqrt{2}$$

represent the right and left circularly polarized components of $\mathbb{E}$. Since the ponderomotive force produced by the electric field is different for the three species, it tends to separate them spatially. Therefore an electric charge density appears which produces an electrostatic potential $\mathcal{U}$ governed by Poisson's equation

$$\nabla^2 \mathcal{U} = -\sum_\sigma q_\sigma n_\sigma \tag{10}$$

In equilibrium the pressure gradient must equal the ponderomotive force and the electrostatic force for each species

$$T_\sigma \vec{\nabla} n_\sigma + n_\sigma \vec{\nabla}\left(\phi_\sigma + q_\sigma \mathcal{U}\right) = 0.$$

Therefore we obtain the Boltzmann distributions

$$n_\sigma = n_{\sigma 0} \exp\left(-\frac{\phi_\sigma + q_\sigma \mathcal{U}}{T_\sigma}\right) \tag{11}$$

for the densities. The constants $n_{\sigma o}$ can be chosen freely. The densities (11) must be introduced into Eqs. (7) and (10) which thus become nonlinear partial differential equations. Our objective is to find solutions of these equations subject to the boundary conditions (4) and (5), and to determine the density variations according to (11). There exists an infinite number of such solutions. We shall restrict our work to a few which seem particularly interesting and relevant in connection with the problem of isotope separation.

We shall assume that the plasma consists of two isotopes of which the lighter one is more abundant. The frequency of the wave will be chosen to lie between the two ion Larmor frequencies.

III VARIATIONAL FORM

Consider the integral

$$I = \int (\vec{\nabla} \times \mathbb{E})^2 \, r\,dr\,dz\,d\varphi = \frac{\pi}{2} \int \left(\vec{\nabla}_{rz} \times \vec{E}\right) r\,dr\,dz$$

where

$$\vec{\nabla}_{rz} \times \vec{E} =$$

$$\left(\frac{\nu}{r} E_z - \frac{\partial}{\partial z} E_\varphi \,,\; \frac{\partial}{\partial z} E_r - \frac{\partial}{\partial r} E_z \,,\; \frac{1}{r} \frac{\partial}{\partial r} r E_\varphi - \frac{\nu}{r} E_r \right) \qquad (12)$$

The integral is evaluated within the volume of the cylindrical cavity. It is easy to see that the variation of I with respect to $\vec{E}$ equals

$$\delta I = \pi \int \left[\vec{\nabla}_{rz} \times (\vec{\nabla}_{rz} \times \vec{E}) \right] \cdot \delta \vec{E}$$

provided the variation $\delta \vec{E}$ satisfies the boundary conditions (4) and (5).

Consider now the total pressure of the plasma

$$\mathcal{P} = \sum_\sigma T_\sigma n_\sigma = \sum T_\sigma n_{\sigma 0} \exp\left[-(\phi_\sigma + q_\sigma \mathcal{U})/T_\sigma \right]$$

If this expression is varied with respect to the electric field $\vec{E}$ and with respect to the potential $\mathcal{U}$ one obtains

$$\delta \mathcal{P} = (\overleftrightarrow{\chi} \vec{E}) \cdot \delta \vec{E} \quad - \quad \sum_\sigma q_\sigma n_\sigma \, \delta \mathcal{U}$$

Consequently the wave equation (7) and Poisson's equation (10) are Euler's equation for minimizing the Lagrangian

$$\mathcal{L} = \pi \int \left[\tfrac{1}{2} (\vec{\nabla}_{rz} \times \vec{E})^2 + \tfrac{1}{2} (\nabla_{rz} \mathcal{U})^2 \right] r\,dr\,dz$$

under the condition

$$\mathcal{A} = \pi \int \left(\tfrac{1}{2} E^2 + \eta \mathcal{P} \right) r\,dr\,dz \tag{13}$$

and

$$\vec{\nabla}_{rz} \mathcal{U} = \left(\frac{\partial \mathcal{U}}{\partial r} , \frac{\nu}{r} \mathcal{U} , \frac{\partial \mathcal{U}}{\partial z} \right)$$

In this form the problem becomes amenable to numerical computation.

IV CHARGE NEUTRALITY

It is possible to reduce the number of unknown functions and of equations from four to three. In all cases in which we are interested, the Debye length will turn out to be much smaller than the scale of spatial variations. Therefore the plasma is nearly perfectly charge neutral, so that we may safely put

$$\sum_{\sigma} q_\sigma n_\sigma = 0 .$$

We determine $\mathcal{U}$ from this equation rather than from Poisson's equation (10). Once $\mathcal{U}$ is so determined, we express the densities in terms of the ponderomotive potentials alone. To simplify matters we assume $T_1 = T_2 = T_i \neq T_e$, $q_1 = q_2 = -q_e = e$, and introduce

$$\tau = T_e / T_i .$$

We obtain

$$\exp\left(-\frac{e\mathcal{U}}{T_i}\right) = \left[n_{eo} Z^{-1} \exp\left(-\frac{\phi_e}{T_e}\right)\right]^{\frac{\tau}{1+\tau}}$$

where

$$Z = \sum_{j=1,2} n_{jo} \exp\left(-\frac{\phi_j}{T_i}\right)$$

The densities become

$$n_j = n_{jo} \left(n_{eo} / Z\right)^{\frac{\tau}{1+\tau}} \exp\left(-\frac{\phi_j}{T_i} - \frac{\phi_e}{T_e+T_i}\right)$$

and

$$n_e = n_{eo}^{\frac{\tau}{1+\tau}} Z^{\frac{1}{1+\tau}} \exp\left(-\frac{\phi_e}{T_e+T_i}\right)$$

It is not difficult to show that the wave equation (7) is now Euler's equation of the Lagrangian

$$\mathcal{L} = \frac{\pi}{2} \int \left(\vec{\nabla}_{rz} \times \vec{E} \right)^2 r\,dr\,dz$$

where the auxiliary condition has the same form as before, Eq. (13), but $\mathcal{P}$ is now given by the expression

$$\mathcal{P} = (T_e + T_i)\, n_{eo}^{\frac{\tau}{1+\tau}} \; z^{\frac{1}{1+\tau}} \; \exp\left(- \frac{\phi_e}{T_e + T_i} \right)$$

We seek solutions of this problem under the following conditions. The plasma composition is initially specified, the temperatures are given, the magnetic field and the aspect ratio of the cavity h/a are prescribed, as well as the frequency ω of the wave. If we scale the linear dimensions of the cavity with radius, then we see that

$$\lambda = a^2 \omega^2$$

becomes the nonlinear eigenvalue to be computed together with the field $\vec{E}$ itself. Thus the eigenvalue determines the size of the cavity.

V DIMENSIONLESS FORM OF EQUATION

We introduce the following dimensionless quantities

$$\alpha = \frac{m_1 - m_2}{m_1 + m_2} > 0 \quad ,$$

$$m = 2\,\frac{m_1 m_2}{m_1 + m_2} \quad ,$$

$$\varepsilon = \frac{m\omega}{e B_0} \quad ,$$

$$\mu = \frac{m_e}{m} \quad ,$$

$$\beta_j = \frac{n_{j0}}{n_{10} + n_{20}} \quad , \qquad j = 1,2 .$$

We assume that the lighter isotope is more abundant $\beta_2 > \beta_1$. The electric field is measured in units of

$$B_0 \,(T_i / m)^{1/2}$$

This permits us to write

$$P = (1+\tau)\, Z^{\frac{1}{1+\tau}} \exp\left(- \frac{A_e}{1+\tau}\right) \tag{14}$$

where

$$Z = \beta_1 e^{-A_1} + \beta_2 e^{-A_2}$$

and

$$A_1 = \frac{1-\alpha}{2\varepsilon}\left(\frac{E_R^2}{\varepsilon+1-\alpha} + \frac{E_L^2}{\varepsilon-1+\alpha} + \frac{E_z^2}{\varepsilon}\right)$$

$$A_2 = \frac{1+\alpha}{2\varepsilon}\left(\frac{E_R^2}{\varepsilon+1+\alpha} + \frac{E_L^2}{\varepsilon-1-\alpha} + \frac{E_z^2}{\varepsilon}\right)$$

$$A_e = \frac{1}{2\varepsilon}\left(\frac{E_R^2}{\mu\varepsilon-1} + \frac{E_L^2}{\mu\varepsilon+1} + \frac{E_z^2}{\mu\varepsilon}\right)$$

To summarize we write the Lagrangian and the auxiliary condition

$$\mathcal{L} = \int \frac{1}{2}\left(\vec{\nabla}_{rz} \times \vec{E}\right)^2 r\,dr\,dz \tag{15}$$

$$\mathcal{A} = \int\left(\frac{1}{2}E^2 + \eta\mathcal{P}\right) r\,dr\,dz \tag{16}$$

where $\mathcal{P}$ is given by (14) and $\vec{\nabla}_{rz} \times \vec{E}$ by (12). The Lagrangian multiplier λ will become the eigenvalue $\lambda = (\alpha\omega)^2$.

VI ITERATION PROCEDURE

We have to minimize the Lagrangian (15) under the auxiliary condition (16), which leads to a nonlinear eigenvalue problem. The eigenvalue λ and the solution $\vec{E}$ depend on the intensity, $I(\vec{E})$, of the field which we define in terms of the minimum value assumed within the cavity by the ponderomotive potential acting on the majority species.

$$I(\vec{E}) = - A_{2min} = - \phi_{2min}/T_i$$

To obtain solutions to this problem, we use a procedure of iteration and progression in intensity. We solve repeatedly a linear eigenvalue problem for $\vec{E}^{k,j}$ in which the auxiliary condition has the form

$$\mathcal{O} = \frac{\pi}{2} \int \left[\delta_{\alpha\beta} + \chi_{\alpha\beta}(\tilde{\vec{E}}^{k,j}) \right] E_{\alpha}^{k,j+1} \, E_{\beta}^{k,j+1} \, r dr dz$$

where

$$\tilde{\vec{E}}^{k,j} = s^{k,j} \left[(1-b) \vec{E}^{k,j} + b \, \vec{E}^{k,j-1} \right]$$

The constant $0 < b < 1$ is sometimes called a "brake". It slows down the iteration to ensure convergence and is determined empirically.

During iterations on j , the intensity is held at a fixed value by choosing at each step the normalising factor λ^{kj} , such that

$$I(\tilde{\vec{E}}^{k,j}) = I^{k}$$

For each normalized linear solution $\tilde{\vec{E}}^{k,j}$ the total number of particles of each species, $N_1^{k,j}$, $N_2^{k,j}$, is computed. For the next step the values of β_1 and β_2 are adjusted by putting

$$\begin{aligned} \beta_1^{k\,j+1} &= N_1 / N^{kj} \\ \beta_2^{k\,j+1} &= N_2 / N^{kj} \end{aligned} \qquad (17)$$

where N_1 and N_2 are the number of particles initially present in the cavity.

As j increases the linear solutions , $\vec{E}^{kj}$ converge to the non-linear solution $\vec{E}^{k}$ having the intensity I^{k} and eigenvalue λ^{k}. The adjustments (17) guarantee that this solution reproduces the initially given particle numbers.

For each level I^k of the intensity the interaction starts with the solution corresponding to the previous value of the intensity

$$\vec{E}^{\,k,0} = \vec{E}^{\,k-1}$$

The entire progression begins at zero intensity with a particular linear solution $\vec{E}^{\,0}$ corresponding to the eigenvalue λ^0 (see Appendix). Any linear solution could be used. However, we restricted our numerical work to $\nu = 1$ and choose the solution for which E_r has one null in the radial direction, while E_z has one null in the axial direction.

ACKNOWLEDGEMENT

The authors wish to thank Dr. Ralf Gruber, who has made available to them his numerical program for solving linear eigenvalue problems.

APPENDIX : The Linear Problem

As explained in the main text, our numerical scheme starts out from a particular solution of the linear wave equation obtained in the limit $\vec{E} \to 0$. Although cylindrical cavity modes of a magnetized plasma have been determined before, we briefly give here our method of solution which is simpler than those we have seen in the literature[7,8,9].

We start with the well known plane wave solution

$$\vec{E} = \left(1,\ i\varepsilon_2\ ,\ \varepsilon_3\right) \exp\left(i\omega\left(n_\perp x + n_\parallel z - t\right)\right) \tag{A1}$$

where $n_\perp$, $n_\parallel$, are the perpendicular and parallel refraction indices. They have to satisfy the dispersion relation[10]

$$S n_\perp^4 + \left[(S+P)\, n_\parallel^2 - PS - PL\right] n_\perp^2 + P\left(R - n_\parallel^2\right)\left(L - n_\parallel^2\right) = 0$$

where $R = 1 + \chi_R$, $L = 1 + \chi_L$, $S = \frac{1}{2}(R+L)$, $P = 1 + \chi_z$.

We assume that the X axis makes an angle α with respect to the direction $\varphi = 0$ and express the fields (A1) in cylindrical coordinates

$$\vec{E}_\alpha = \left(E_r\,, E_\varphi\,, E_z\right) \exp\left[i\omega\left(n_\parallel z - t\right) + i\omega n_\perp r \cos(\varphi - \alpha)\right] \tag{A2}$$

with

$$E_r = \cos(\varphi-\alpha) + i\,\varepsilon_2 \sin(\varphi-\alpha)$$

$$E_\varphi = -\sin(\varphi-\alpha) + i\,\varepsilon_2 \cos(\varphi-\alpha)$$

$$E_z = \varepsilon_3$$

The coefficients ε_2 , ε_3 are given by

$$\varepsilon_2 = \frac{D}{n^2 - S}$$

$$\varepsilon_3 = \frac{n_\parallel n_\perp}{n_\perp^2 - P}$$

with

$$D = \tfrac{1}{2}(R-L) = \tfrac{1}{2}(\chi_R - \chi_L)$$

We now superimpose the plane waves (A2) for all angles α with the weight $\exp(i\nu\alpha)$

$$\vec{E} = \int_0^{2\pi} \vec{E}_\alpha \exp(i\nu\alpha)\, d\alpha$$

The resulting integrals are Bessel functions and we obtain

$$E_r = -i\left[J_\nu'(\rho) + \varepsilon_2 \frac{\nu}{\rho} J_\nu(\rho)\right] e^{i\omega(n_\parallel z - t) + i\nu\varphi}$$

$$E_\varphi = \left[\frac{\nu}{\rho} J_\nu(\rho) + \varepsilon_2 J_\nu'(\rho)\right] e^{i\omega(n_\parallel z - t) + i\nu\varphi} \qquad \text{(A3)}$$

$$E_z = \varepsilon_3 J_\nu(\rho)\, e^{i\omega(n_\parallel z - t) + i\nu\varphi}$$

with

$$\rho = \omega n_{\perp} r$$

The real part of (A3) is not a physical solution. The general solution must be obtained by superposition of such solutions for $\pm\omega$, $\pm n_{\parallel}$, $n_{\perp 1}$ and $n_{\perp 2}$.

Thus, imposing the boundary conditions $E_r = E_\varphi = 0$ at $z = 0$ and $z = h$ we find

$$E_r = A \sin(\omega n_{\parallel} z) \cos(\nu\varphi - \omega t)$$

$$E_\varphi = - B \sin(\omega n_{\parallel} z) \sin(\nu\varphi - \omega t)$$

$$E_z = C \cos(\omega n_{\parallel} z) \cos(\nu\varphi - \omega t)$$

where

$$A = \sum_{i=1,2} a_i \left[J_\nu'(\rho_i) + \varepsilon_2 \frac{\nu}{\rho_i} J_\nu(\rho_i) \right]$$

$$B = \sum_{i=1,2} a_i \left[\frac{\nu}{\rho_i} J_\nu(\rho_i) + \varepsilon_2 J_\nu'(\rho_i) \right]$$

$$C = \sum_{i=1,2} a_i \, \varepsilon_{3i} \, J_\nu(\rho_i)$$

The boundary condition $E_r = 0$ at $r = a$ leads to the equation for the radius

$$\frac{\nu}{\alpha_1} + \varepsilon_2 \frac{J_\nu'(\alpha_1)}{J_\nu(\alpha_1)} = \frac{\nu}{\alpha_2} + \varepsilon_2 \frac{J_\nu'(\alpha_2)}{J_\nu(\alpha_2)} \tag{A4}$$

where

$$\alpha_i = \omega a n_{\perp i}$$

It should be noted that the solutions for opposite signs of ν are not identical. It is a simple matter to find numerical solutions of Eq. (A4), which yield the linear eigenvalue χ^0 .

REFERENCES

1) A. Hidekuma, S. Hiroe, T. Watari, T. Shoji, T. Sato, K. Takayama, Phys.Rev.Lett. 33, 1537 (1974)

2) S. Hiroe, S. Hidekuma, T. Watari, T. Soji, T. Sato, K. Takayama Nuclear Fusion 15, 769 (1975)

3) J.M. Dawson, H.C. Kim, B.D. Fried, R.W. Gould, L.O. Heflinger, C.F. Kennel, T.E. Romesser, R.L. Stenzel, A. Y. Wong, and R.F. Wuerker, Phys.Rev.Lett. 37, 1547 (1976)

4) T. Consoli and R.B. Hall, Fusion Nucléaire 3,237 (1963)

5) E.S. Weibel, Phys.Rev.Lett. 44, 377 (1980)

6) M.C. Festeau-Barrioz, and E.S. Weibel, Phys.Fluids 23, 2045 (1980)

7) U. Sahl, L.M. Walker, Bell Syst.Techn.Journal 33, 579 (1954)

8) D.A. Gore, and U. Lashinsky, Phys.Fluids 22, 2178 (1979)

9) W.P. Allis, S.J. Buchsbaum and A. Bers, Waves in Anisotropic Plasmas, MIT Press, Cambridge, Mass. 1963

10) H.T. Stix, The Theory of Plasma Waves, Mc Graw Hill, 1962

FIGURE CAPTIONS

Figure 1 Domains of the plane B_0 , T for which the inequalities (1) and (2) are satisfied with a margin of 10, assuming $|1-\omega/\Omega| = 0.045$. B_0 is measured in Tesla, T in eV and n in m^{-3}. The logarithms are to the base of 10.

Figure 2 The electric field, E_r , E_φ , E_z as a function of radius. The electric fields are measured in units of $(T_i/m)^{1/2} B_0$.

Figure 3 The electric field, E_r , E_φ , E_z as a function of height. The electric fields are measured in units of $(T_i/m)^{1/2} B_0$.

Figure 4 The density variation of the majority species in fractions of the original density as a function of r and z .

Figure 5 The density variation of the minority species in fractions of the original density as a function of r and z .

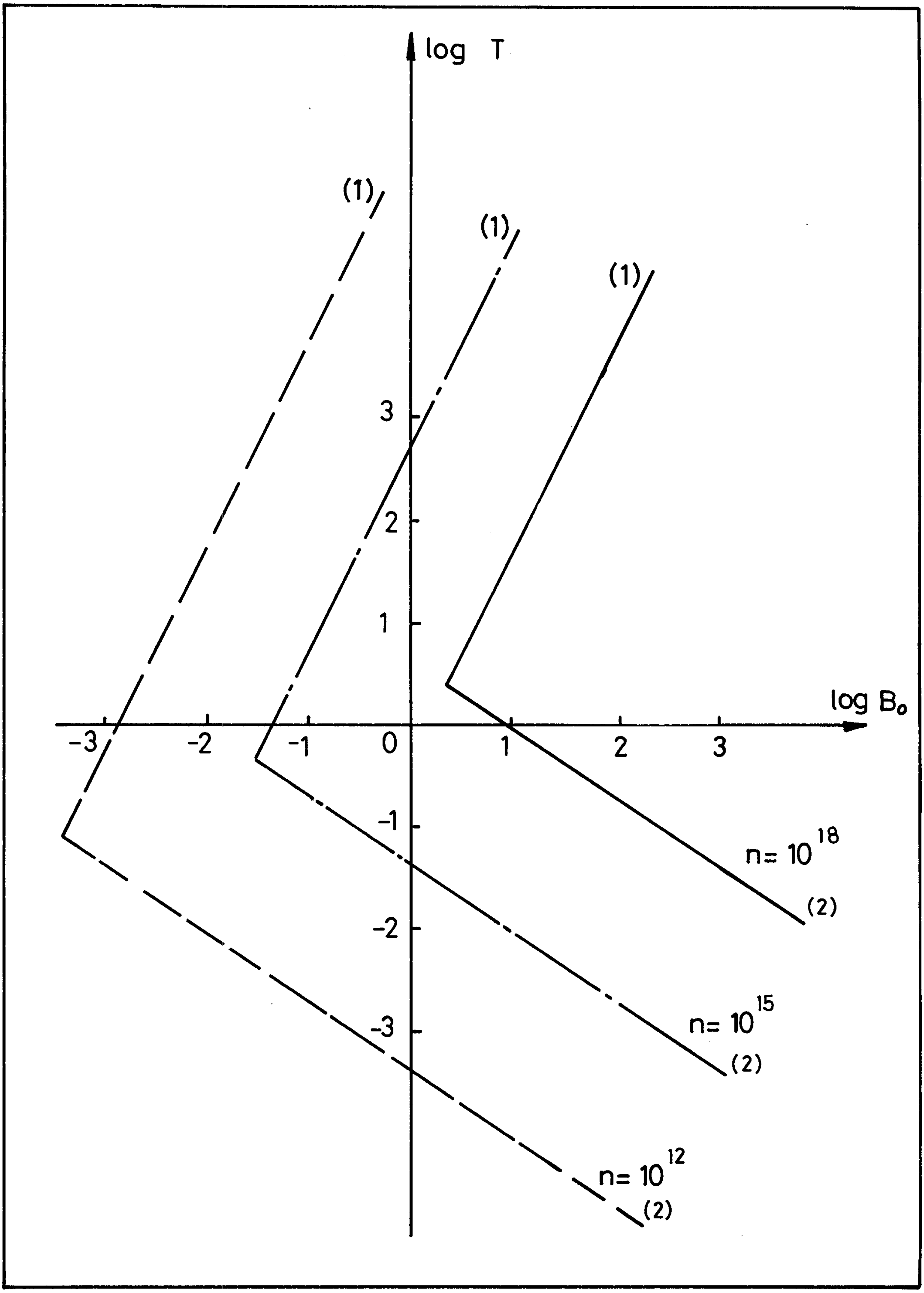

Fig. 1

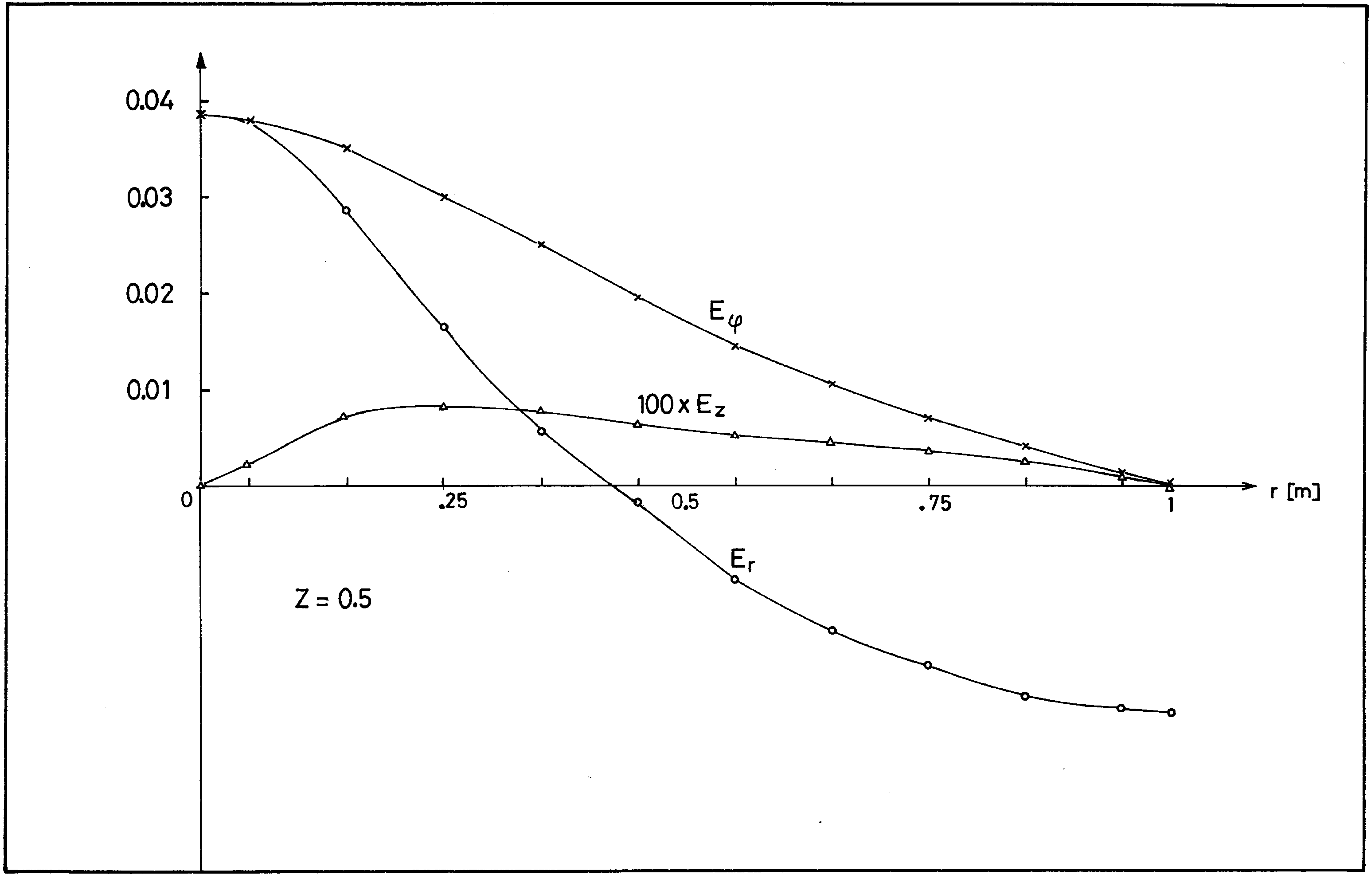

Fig. 2

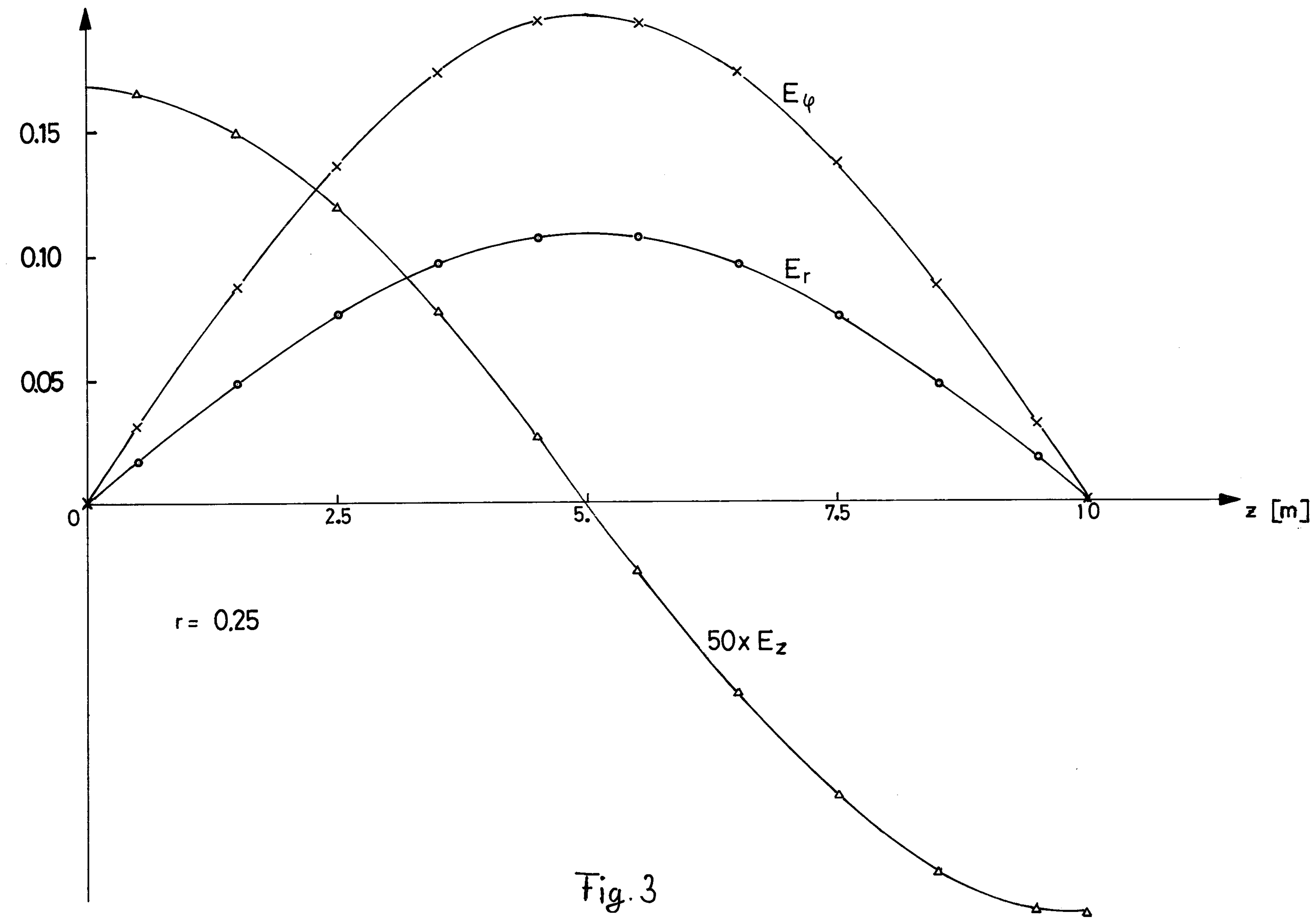

Fig. 3

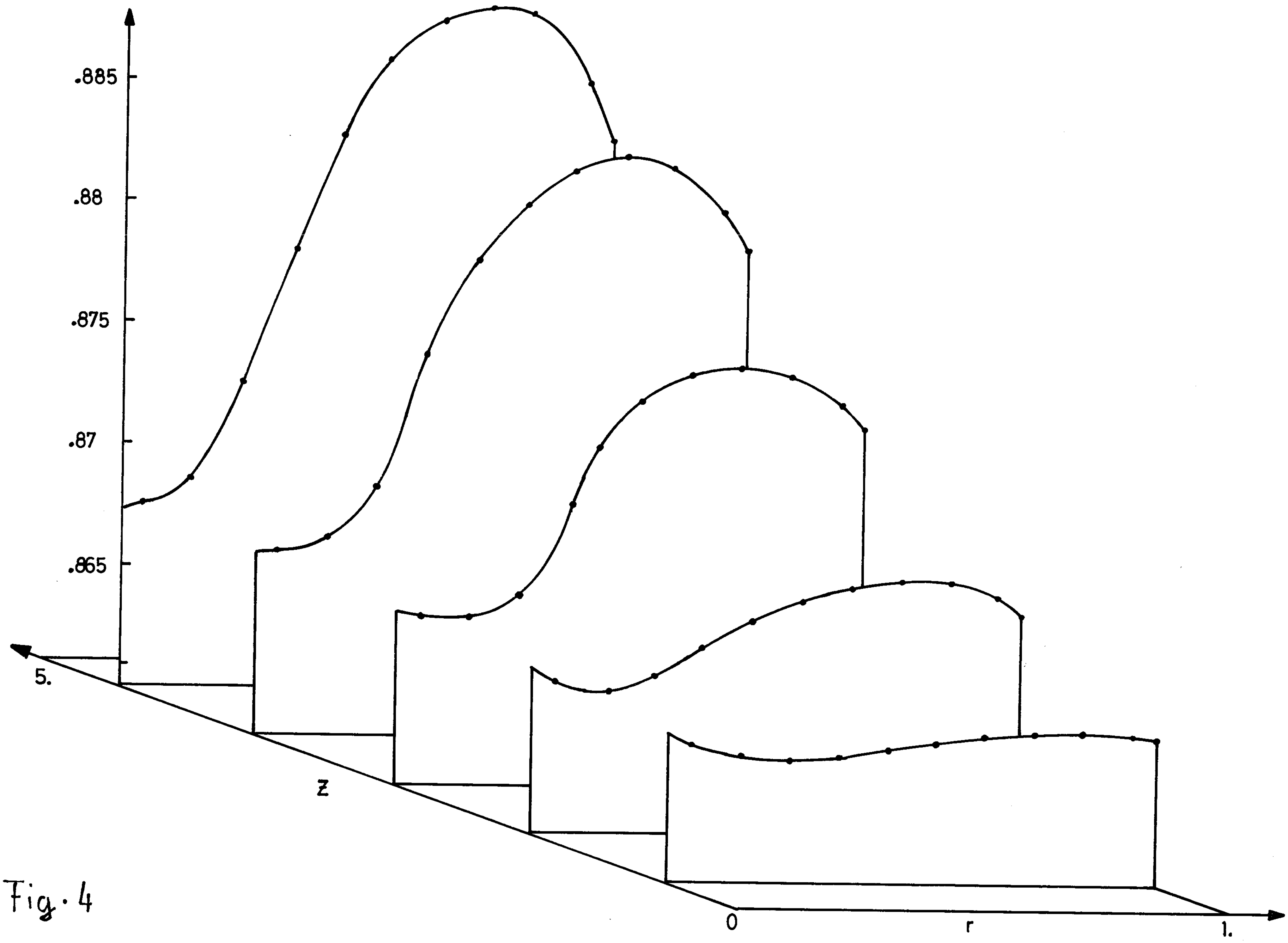

Fig. 4

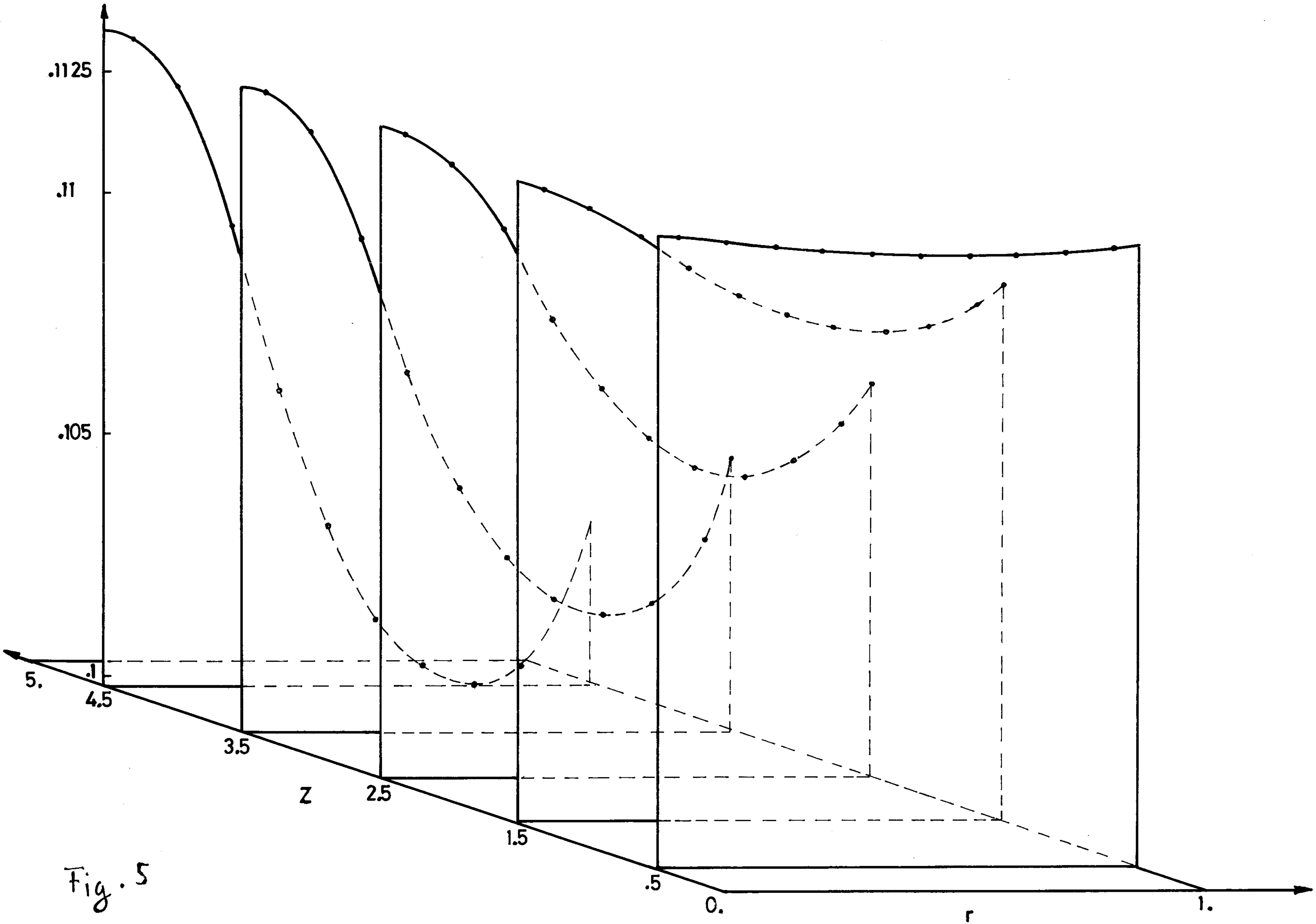

Fig. 5