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ABSTRACT

The spectral code ERATO has been transformed into a stability code based on the energy principle  $\delta W$ . It only gives the stability index, i.e. the "yes or no" answer to the stability of an axisymmetric toroidal plasma equilibrium. Compared to the full ERATO code, the computing time is reduced by a factor of 4. Storage requirements and input/output operations are reduced by a factor of 5.

## I. INTRODUCTION

Presently, spectral codes are the most widely used computational tools to study the ideal MHD stability of axisymmetric toroidal plasmas, such as Tokamaks, stabilized Pinches or Spheromaks. ERATO {1} and PEST {2} are commonly used for that purpose. They both solve a variational form of the linearized set of MHD equations, leading to an eigenvalue problem. In many parameter studies one is only interested in the stability limit and the knowledge of the growth-rates and eigenmodes is of secondary importance. This means that only the stability index, defined as the sign of the lowest eigenvalue, is needed. The energy principle {7} provides such an information. It consists of minimizing the potential energy. The stability index is given by the sign of this minimum. This minimization can be carried out by minimizing successively with respect to 3 components of the displacement  $Y$ ,  $V$  and  $X$ . First, minimizing with respect to the vector component parallel to the magnetic field,  $Y$ , leads to the incompressibility condition,  $\underline{\nabla} \cdot \underline{\xi} = 0$ . The remaining expression contains the two contravariant components of the displacement  $X$  and  $V$ . Since there is no radial derivative on  $V$ , the minimization with respect to  $V$  can be performed independently on each magnetic surface. This allows one to express  $V$  in terms of  $X$  and its derivatives on each flux surface. Replacing  $V$  in terms of  $X$  in the potential energy, we finally obtain a variational problem with only one component  $X$ .

The elimination of the  $V$  component can be done in two different ways. The first consists of minimizing the potential energy  $W$  with respect to  $V$ , construct an Euler equation on each flux surface which is solved by a Green's function technique. This is the scheme implemented in the new version of the Princeton code, PEST 2 {3}. It implies an increase in the complexity of the code, with 3 different discretizations: Double Fourier expansion of the Green's function to eliminate  $V$ , a Fourier expansion of  $X$  in the poloidal direction and a finite element expansion for  $X$  in the radial direction. This means 3 truncations with 3 different convergence laws. The second method consists of discretizing first  $X$  and  $V$ , performing the variations over  $X$  and  $V$  and then carrying out the elimination of  $V$  numerically. This method keeps the quadratic convergence properties. Its main advantage is that we can make use of the existing structure of the discretized potential energy. The method of elimination of  $V$  on each flux surface makes use of the sparseness of the matrix, leading to a very efficient elimination technique. It should be pointed out that the success of this method is due to the particular choice of piecewise constant finite elements for  $V$  in the radial direction. This paper describes the transformations of ERATO into such a  $\delta W$  code.

The difference between the Green's function approach and ours to transform a spectral code into a  $\delta W$  code reflects a basic difference in conception. We believe that equations should be coded in a simple form, without too much complicated analytical pretreatment. The eventual knowledge of properties of the solution should be taken into account by the numerical method.

## II. THE PHYSICAL PROBLEM

The eigenmodes and eigenfrequencies of an axisymmetric toroidal plasma can be obtained by variation of the Lagrangian  $L$  with respect to the displacement vector  $\underline{\xi}(\underline{r})$  :

$$\delta L = \delta W_p + \delta W_v - \omega^2 \delta K = 0,$$

where  $W_p$ ,  $W_v$  and  $-\omega^2 K$  are, respectively, the plasma potential energy, the vacuum potential energy and the kinetic energy associated with the displacement  $\underline{\xi}(\underline{r})$ , assumed to vary as  $e^{i\omega t}$  [1]:

$$W_p = \frac{1}{2} \iint \frac{ds}{s} d\chi \left[ a |I_1|^2 + b |I_2|^2 + c |I_3|^2 + d |I_4|^2 - e |I_5|^2 \right]$$

Plasma

$$W_v = \frac{1}{2} \iint \underline{B} \cdot \underline{\delta B} d\sigma, \quad K = \frac{1}{2} \iint \frac{ds}{s} 2\rho\psi J |\underline{\xi}|^2 .$$

Vacuum Plasma

where

(1)

$$I_1 = F(X) \equiv \frac{1}{q} \frac{\partial X}{\partial \chi} + i n X$$

$$I_2 = r^2 \underline{v}_T \cdot \left( \frac{\underline{\xi}}{r} \right) \equiv \frac{\partial X}{\partial s} + \frac{\partial V}{\partial \chi}$$

$$I_3 = i n q V + \beta \frac{q}{X} F(X) + \frac{q}{T} \frac{\partial \left( \frac{TX}{q} \right)}{\partial s} + \frac{2j \psi r}{|\underline{\nabla} \psi|^2_s} X$$

$$I_4 = \underline{v} \cdot \underline{\xi} \equiv \frac{1}{r^2} \frac{\partial r^2 X}{\partial s} + \frac{1}{r^2} \frac{\partial r^2 V}{\partial \chi} + \frac{q^2}{r^2} F(r^2 Y)$$

$$I_5 = X$$

The 3 components  $\underline{X}$ ,  $\underline{V}$  and  $\underline{Y}$  are related to the true displacement  $\underline{\xi}$ ,

$$\underline{\xi} = rX(\underline{\hat{e}}_{\phi} \times \underline{\nabla}\chi) + \frac{rs}{2\Psi} V(\underline{\nabla}\Psi \times \underline{\hat{e}}_{\phi}) + \frac{r^2 s}{2\Psi} YB,$$

where  $s$  is the radial variable related to the flux function  $\Psi$ ,  $s = (\Psi/\Psi_s)^{1/2}$ ,  $\chi$  the poloidal angle,  $\phi$  the toroidal angle, and all other undefined symbols appearing in the equations are known equilibrium quantities defined in ref. 1.

The eigenvalue of the problem is  $\omega^2$ . An unstable displacement is characterized by  $\omega^2 < 0$ . As long as we are only interested in the stability index, i.e. in the sign of  $\omega^2$ , we only have to consider the sign of the total potential energy  $W$ .

$$\begin{aligned} &> 0 && \text{stable} \\ W = W_p + W_v &= 0 && \text{marginal} \\ &< 0 && \text{unstable.} \end{aligned} \tag{2}$$

In the reduced variational form  $\delta W = 0$ , the variation with respect to the component  $Y$  parallel to the magnetic field is immediate. It leads to

$$I_4 = \underline{\nabla} \cdot \underline{\xi} = 0. \tag{3}$$

In the remaining expression for  $W$  the contravariant tangential component  $V$  appears without any derivative with respect to the radial coordinates. We now make use of this property to eliminate  $V$ .

### III. THE ELIMINATION OF $V$

A finite hybrid element expansion {5} of the reduced potential energy, Eqs. (2) and (3), gives a symmetric matrix  $\underline{\underline{A}}$ . The block diagonal structure of this matrix is shown in Fig. 1. Each block is divided into  $3 \times 3$  subblocks. Each subblock consists of a non-zero band with a bandwidth of 7. The overlaps with neighboring blocks only occur through the first and the last subblocks. The stability index, Eq. (2), is then related to the sign of the reduced potential energy matrix  $\underline{\underline{A}}$  by

$$A = \left\{ \begin{array}{ll} \text{positive definite} & \longrightarrow \text{stable} \\ \text{positive indefinite} & \longrightarrow \text{marginal} \\ \text{not positive} & \longrightarrow \text{unstable} \end{array} \right\} \quad (4)$$

A configuration is stable if all the eigenvalues of  $\underline{\underline{A}}$  are positive, marginal if one or several eigenvalues are zero and all other positive, and unstable if at least one is negative. Practically we proceed by determining the lowest eigenvalue  $\omega^2$ , of the following eigenvalue problem :

$$\underline{\underline{A}} \underline{x} = \omega^2 \underline{\underline{B}} \underline{x}, \quad (5)$$

$\underline{x} = (X, V)$ , and verify its sign. In this eigenvalue problem, the matrix  $\underline{\underline{B}}$  plays the role of a norm. It can be any positive definite matrix. It can be used to displace the stable continuous spectrum {6} associated to the original problem. If we substitute for  $\omega^2$  the discretized kinetic energy  $\omega^2 K$ , Eq. (1), there is a continuous spectrum which may extend to the marginal point  $\omega^2 = 0$ . By omitting from the kinetic energy the dominant contributions containing  $V$  and  $Y$ , this continuous spectrum is pushed to infinity. Another choice of  $\underline{\underline{B}}$ , for example by replacing  $V$  in  $K$  by its approximate expression in terms of  $\partial X / \partial s$ , might leave the lowest edge of the continuous spectrum at  $\omega^2 = 0$ . Changing  $\underline{\underline{B}}$  moves the eigenvalues around, but no eigenvalue can cross the marginal point, as long as  $\underline{\underline{B}}$  remains positive definite. In the  $\delta W$  version of ERATO, we choose for  $\underline{\underline{B}}$  an expression involving only the  $X$  components.

Equation (5) is solved by an inverse vector iteration {8}

$$\underline{\underline{A}} \underline{x}^{k+1} = \underline{u}^k = \underline{\underline{B}} \underline{x}^k. \quad (6)$$

An example with only 3 blocks is given in Fig. (1). It is used to illustrate how the tangential vector component  $V = (V_1, V_2, V_3)$  can be eliminated. The 7 matrix equations of the example are :

$$\begin{aligned} \text{(I)} \quad & \underline{\underline{A}}_1 * \underline{X}_1 + \underline{\underline{A}}_2 * \underline{V}_1 + \underline{\underline{A}}_3 * \underline{X}_2 &= \underline{u}_1 \\ \text{(II)} \quad & \underline{\underline{A}}_2^T * \underline{X}_1 + \underline{\underline{A}}_4 * \underline{V}_1 + \underline{\underline{A}}_5 * \underline{X}_2 &= 0 \\ \text{(III)} \quad & \underline{\underline{A}}_3^T * \underline{X}_1 + \underline{\underline{A}}_5^T * \underline{V}_1 + \underline{\underline{A}}_6 * \underline{X}_2 + \underline{\underline{A}}_7 * \underline{V}_2 + \underline{\underline{A}}_8 * \underline{X}_3 &= \underline{u}_3 \\ \text{(IV)} \quad & \underline{\underline{A}}_7^T * \underline{X}_2 + \underline{\underline{A}}_9 * \underline{V}_2 + \underline{\underline{A}}_{10} * \underline{X}_3 &= 0 \\ \text{(V)} \quad & \underline{\underline{A}}_8^T * \underline{X}_2 + \underline{\underline{A}}_{10}^T * \underline{V}_2 + \underline{\underline{A}}_{11} * \underline{X}_3 + \underline{\underline{A}}_{12} * \underline{V}_3 + \underline{\underline{A}}_{13} * \underline{X}_4 &= \underline{u}_5 \\ \text{(VI)} \quad & \underline{\underline{A}}_{12}^T * \underline{X}_3 + \underline{\underline{A}}_{14} * \underline{V}_3 + \underline{\underline{A}}_{15} * \underline{X}_4 &= 0 \end{aligned} \quad (7)$$



Equations (II), (IV) and (VI), can be used to express the vectors  $\underline{V}_1$ ,  $\underline{V}_2$ , and  $\underline{V}_3$  in terms of the radial component  $X = (\underline{X}_1, \underline{X}_2, \underline{X}_3, \underline{X}_4)$ .

$$\begin{aligned} \underline{V}_1 &= -\underline{A}_{=4}^{-1} * (\underline{A}_{=2}^T * \underline{X}_1 + \underline{A}_{=5} * \underline{X}_2) \\ \underline{V}_2 &= -\underline{A}_{=9}^{-1} * (\underline{A}_{=7}^T * \underline{X}_2 + \underline{A}_{=10} * \underline{X}_3) \\ \underline{V}_3 &= -\underline{A}_{=14}^{-1} * (\underline{A}_{=12}^T * \underline{X}_3 + \underline{A}_{=15} * \underline{X}_4) \end{aligned} \quad (8)$$

The inversion of the symmetric banded matrices  $\underline{A}_{=4}$ ,  $\underline{A}_{=9}$ , and  $\underline{A}_{=14}$  leads to full matrices. Instead of inverting, we rather leave them in decomposed form

$$\underline{A}_{=j} = \underline{L}_{=j} \underline{D}_{=j} \underline{L}_{=j}^T, \quad j = 4, 9, 14 \quad (9)$$

and replace the multiplications in Eq. (8) by a resolution of a system of linear equations.

Let us now replace  $\underline{V}_1$ ,  $\underline{V}_2$ , and  $\underline{V}_3$  in the remaining equations (7):

$$\begin{aligned} \text{(I)} \quad & \underline{A}_{=1}^{\sim} * \underline{X}_1 + \underline{A}_{=3}^{\sim} * \underline{X}_2 = \underline{u}_1 \\ \text{(III)} \quad & \underline{A}_{=3}^{\sim T} * \underline{X}_1 + \underline{A}_{=6}^{\sim} * \underline{X}_2 + \underline{A}_{=8}^{\sim} * \underline{X}_3 = \underline{u}_3 \\ \text{(V)} \quad & \underline{A}_{=8}^{\sim T} * \underline{X}_2 + \underline{A}_{=11}^{\sim} * \underline{X}_3 + \underline{A}_{=13}^{\sim} * \underline{X}_4 = \underline{u}_5 \\ \text{(VII)} \quad & \underline{A}_{=13}^{\sim T} * \underline{X}_3 + \underline{A}_{=16}^{\sim} * \underline{X}_4 = \underline{u}_7 \end{aligned} \quad (10)$$

The transformed matrices become

$$\begin{aligned}
 \underline{\underline{\tilde{A}}}_1 &= \underline{\underline{A}}_1 - \underline{\underline{A}}_2 * \underline{\underline{A}}_4^{-1} * \underline{\underline{A}}_2^T \\
 \underline{\underline{\tilde{A}}}_3 &= \underline{\underline{A}}_3 - \underline{\underline{A}}_2 * \underline{\underline{A}}_4^{-1} * \underline{\underline{A}}_5 \\
 \underline{\underline{\tilde{A}}}_6 &= \underline{\underline{A}}_6 - \underline{\underline{A}}_5^T * \underline{\underline{A}}_4^{-1} * \underline{\underline{A}}_5 \\
 \underline{\underline{\tilde{A}}}_6 &= \underline{\underline{\tilde{A}}}_6 - \underline{\underline{A}}_7 * \underline{\underline{A}}_9^{-1} * \underline{\underline{A}}_7^T \\
 \underline{\underline{\tilde{A}}}_8 &= \underline{\underline{A}}_8 - \underline{\underline{A}}_7 * \underline{\underline{A}}_9^{-1} * \underline{\underline{A}}_{10} \\
 \underline{\underline{\tilde{A}}}_{11} &= \underline{\underline{A}}_{11} - \underline{\underline{A}}_{10}^T * \underline{\underline{A}}_9^{-1} * \underline{\underline{A}}_{10} \\
 \underline{\underline{\tilde{A}}}_{11} &= \underline{\underline{\tilde{A}}}_{11} - \underline{\underline{A}}_{12} * \underline{\underline{A}}_{14}^{-1} * \underline{\underline{A}}_{12}^T \\
 \underline{\underline{\tilde{A}}}_{13} &= \underline{\underline{A}}_{13} - \underline{\underline{A}}_{12} * \underline{\underline{A}}_{14}^{-1} * \underline{\underline{A}}_{15} \\
 \underline{\underline{\tilde{A}}}_{16} &= \underline{\underline{A}}_{16} - \underline{\underline{A}}_{15}^T * \underline{\underline{A}}_{14}^{-1} * \underline{\underline{A}}_{15}
 \end{aligned} \tag{11}$$

All these matrix operations are simply rectangular rules applied on matrices. These rectangular rules always go in the clockwise direction. The reduction of the system of (6+1) matrix equations (7) to the system of (3+1) matrix equations (10) is computationally "cheap". The reduction from 3 unknowns (X,Y,V) to 1 unknown (X) per mesh cell theoretically reduce CPU time by a factor of 9 for  $N \rightarrow \infty$ . In practice, this gain of CPU time is of the order of 4. The core storage and the amount of input-output operations are reduced by a factor of 5. Due to the small bandwidths of the sparse matrices, the

gain in CPU time in a vector machine is smaller, but the gain in core storage and IO operations remain.

#### IV. AN EXAMPLE

As an example, Fig. 2, we study the case of a barely stable internal mode of an analytic Solovév equilibrium [4]. The aspect ratio is 3, elongation is 2, the safety factor on axis  $q_0 = 0.5071$  and the toroidal wave number  $n = 1$ . We first study (A) the growthrate  $\Gamma^2 = -\omega^2$  as a function of the number  $N = N_s = N_x$  of intervals in both directions,  $s$  and  $x$ , by means of the ERATO code, which solves the full problem, Eq. (1). A quadratic extrapolation leads to a barely stable configuration. A very small increase of  $q_0$  to  $q_0 = 0.5072$  would already give an unstable mode. We repeat the calculation (B) with the reduced code to find the stability index. The convergence is again quadratic. Note that the scale for (B) is 50 times bigger than for (A). The (A) and (B) curves cross the marginal point  $\Gamma^2 = -\omega^2 = 0$  for the same value of  $N$ . In all the cases studied we have verified that the number of negative eigenvalues was the same for (A) and (B). The CPU time was one order of magnitude smaller for (B) than for (A).

The convergence from below, i.e. from the unstable side, is due to the more general finite hybrid element approach [5].

We conclude that the elimination of the two components Y and V makes life much easier as long as one is only interested in the stability index. After having performed the trivial elimination of Y, the elimination of V can easily be done through sparse matrix operations. The quadratic convergence properties are kept in this method. The number of negative eigenvalues is untouched.

ACKNOWLEDGMENTS

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FIGURE CAPTIONS

Figure 1 : The reduced sparse matrix problem. Each subblock  $\underline{A}_j$  has a band structure with a bandwidth of 7. The rang of a subblock is related to the number of angular intervals  $N_\chi$ . The number of big matrix blocks is related to the number of radial intervals  $N_s$  (in the example given,  $N_s = 3$ ).

Figure 2 : Eigenvalue squared as a function of number of intervals  $N$  plotted in a  $N^{-2}$  scale. The full case (A) as well as the reduced case (B) cross  $\omega^2 = 0$  at the same point. The different scales for the cases (A) and (B) are due to the different norms. For case (B), the continuous spectrum has been pushed towards very high values of  $\omega^2$ .



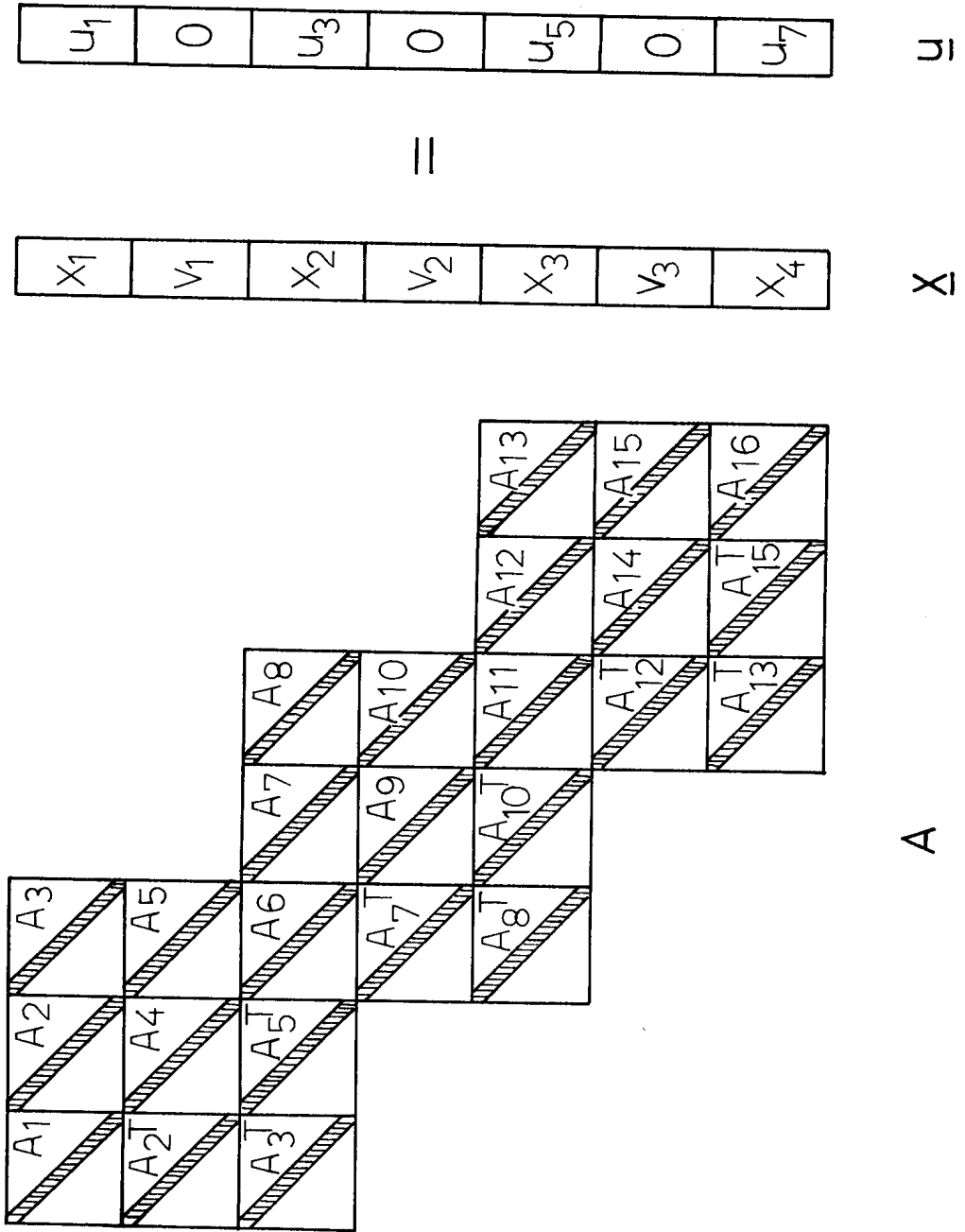


FIG. 1

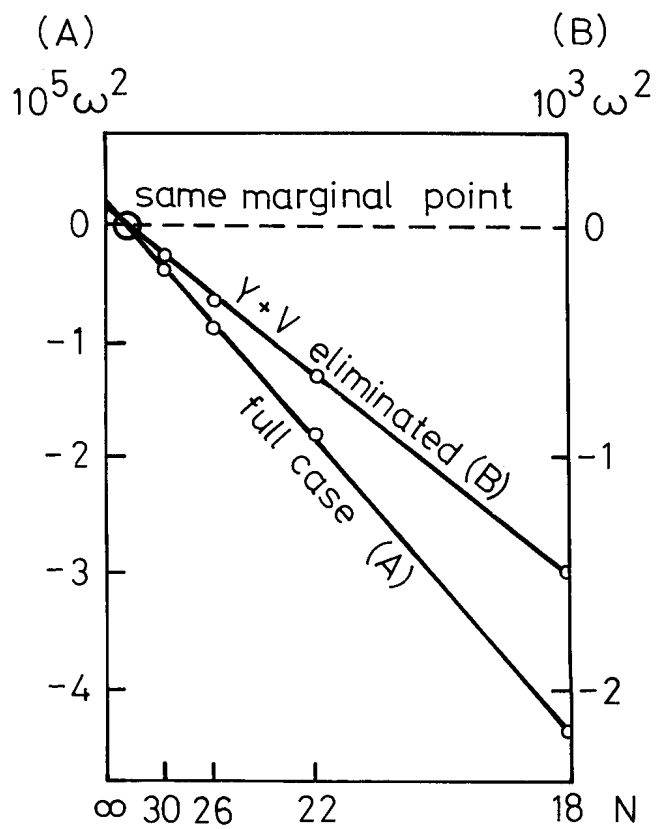


FIG. 2