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PONDEROMOTIVE FORCE

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## ABSTRACT

The ponderomotive force experienced by an ion in a left circularly polarized wave changes sign as the frequency increases from below to above the ion cyclotron frequency. If the wave frequency is chosen to lie between the ion cyclotron frequencies of two different isotopes then the ponderomotive force tends to separate the two species in space. As a result the index of refraction is modified. Self-consistent solutions of the non-linear wave equation taking into account both effects are obtained analytically and numerically. The ratio of the densities of the two ions is proportional to  $\exp\left[m E^2(z) / 4 \gamma (1-\gamma) \propto B_0^2 k T_i\right]$  where  $E(z)$  is the amplitude of the wave,  $B_0$  the magnetic field and  $T_i$  the temperature of the ions while  $m = 2 m_1 m_2 / (m_1 + m_2)$ ,  $\alpha = (m_1 - m_2) / (m_1 + m_2)$  and  $\gamma = (\omega - \Omega_2) / (\Omega_1 - \Omega_2)$ .

## 1. INTRODUCTION

A method for isotope separation using the ioncyclotron resonance in a magnetized plasma has been described by J.M. Dawson<sup>1</sup> et al. In previous reports<sup>2,3</sup> we have proposed a different method which uses the ponderomotive force produced by a left circularly polarized wave. In the present paper we present a complete analysis of the equations in one dimension describing the electromagnetic field and the density variations which it produces. In a few cases these equations are also numerically integrated. We use the natural system of units throughout the analysis; however, experimentally important quantities are also given in the international MKSA system<sup>4</sup>.

## 2. THE NONLINEAR WAVE EQUATION

Consider a magnetized plasma consisting of several species of particles with masses  $m_\sigma$  and charges  $q_\sigma$ . In such a plasma a plane left circularly polarized wave exerts on each type of particle a ponderomotive force<sup>5</sup>

$$F_\sigma = - \frac{\partial}{\partial z} \frac{q_\sigma^2 [E(z)]^2}{2 m_\sigma \omega (\omega - \Omega_\sigma)} \quad (1)$$

where  $E(z)$  is the amplitude of the wave and  $\Omega_\sigma = q_\sigma B_0 / m_\sigma$  the cyclotron frequency of species  $\sigma$ . If the frequency  $\omega$  is chosen to lie between two cyclotron frequencies,  $\Omega_1 < \omega < \Omega_2$  the ponderomotive force acts in opposite directions on the species 1 and 2. It

should be possible to use this effect to separate isotopes, or more generally, ions of different charge to mass ratio.

The ponderomotive force (1) is usually obtained by means of a quasilinear approximation as the time average taken over the period of the oscillation. However, if the electric field has the form

$$E(z) \left\{ \cos \omega t, \quad -\sin \omega t, \quad 0 \right\} \quad (2)$$

the force  $F_r$  is simply the Lorentz force which is constant in time, as we shall now show.

The fluid equation of motion for each species has the form

$$\dot{\underline{u}} + \underline{\Omega} \times \underline{u} = \frac{q}{m} \underline{E} + \frac{q}{m} \underline{u} \times \underline{B} - \nabla p - (\underline{u} \cdot \nabla) \underline{u}$$

where  $\underline{B}$  is the oscillating magnetic field associated with the wave.

If the electric field has the form (2) this equation of motion allows a solution with  $u_z = 0$  and splits exactly into two parts :

$$\dot{\underline{u}}_1 + \underline{\Omega} \times \underline{u}_1 = \frac{q}{m} \underline{E}_1 \quad (3)$$

and

$$\frac{q}{m} (\underline{E}_1 + \underline{u}_1 \times \underline{B}) - \frac{1}{n} \nabla_{\parallel} p = 0. \quad (4)$$

Eq. (3) is linear because the term  $(\underline{u} \cdot \nabla) \underline{u}$  vanishes. Integration of (3), using (2) yields

$$\underline{u}_\perp = \frac{q}{m(\omega - \Omega)} E(z) \left\{ \sin \omega t, \cos \omega t, 0 \right\}. \quad (5)$$

From Maxwell's equations one finds

$$\underline{B} = \frac{1}{\omega} \frac{d}{dz} E(z) \left\{ \cos \omega t, -\sin \omega t, 0 \right\}.$$

The crossproduct  $\underline{u}_\perp \times \underline{B}$  is constant in time and can be expressed in the form

$$\frac{q}{m} \underline{u}_\perp \times \underline{B} = -\underline{e}_z \frac{\partial}{\partial z} \frac{1}{m} \phi$$

where, for each species

$$\phi_\sigma = \frac{q_\sigma^2}{2m_\sigma} \frac{[E(z)]^2}{\omega(\omega - \Omega_\sigma)}.$$

The electric field  $\underline{E}_z$  can be derived from an electrostatic potential

$$\underline{E}_z = - \frac{\partial \mathcal{U}}{\partial z}.$$

Equation (4), therefore, takes the form

$$\frac{\partial}{\partial z} (q_\sigma \mathcal{U} + \phi_\sigma) + \frac{1}{n_\sigma} \frac{\partial}{\partial z} p_\sigma = 0.$$

Assuming for each species the pressure  $p_\sigma = T_\sigma n_\sigma$ , we find for the densities the Boltzman distributions

$$n_\sigma = n_{\sigma 0} \exp \left[ (q_\sigma \mathcal{U} + \phi_\sigma) / T_\sigma \right]. \quad (6)$$

We now return to Eq. ( 5 ) to obtain the susceptibility<sup>6</sup> of the left circularly polarized wave

$$\chi_L = - \sum_{\sigma} \frac{\omega_{p\sigma}^2}{\Omega_{\sigma}(\omega - \Omega_{\sigma})} \quad (7)$$

where  $\omega_{p\sigma}^2 = q_{\sigma}^2 n_{\sigma} / m_{\sigma}$  . The conventional expression of  $\chi_L$  seems to contain a pole at  $\omega=0$ , whose residue, however, vanishes as a consequence of charge neutrality. In (7), this spurious pole has been eliminated.

We note that in the present case the susceptibility is exactly linear, so that there is no condition on the amplitude of the wave. This susceptibility does not depend on the wave vector since  $\nabla \cdot \underline{u} = 0$ . Kinetic effects play no role either since the phase velocity of the wave will turn out to be much larger than the thermal particle velocities.

Thus we obtain the exact wave equation

$$\frac{\partial^2 E}{\partial z^2} + \omega^2 \left[ 1 - \sum_{\sigma} \frac{\omega_{p\sigma}^2}{\Omega_{\sigma}(\omega - \Omega_{\sigma})} \right] E = 0 \quad (8)$$

in which  $\omega_{p\sigma}$  is a function of  $z$  . Indeed, according to (6), we have

$$\omega_{p\sigma}^2 = \frac{q_{\sigma}^2 n_{\sigma 0}}{m_{\sigma}} \exp \left[ (q_{\sigma} U + \phi_{\sigma}) / T_{\sigma} \right] .$$

The electrostatic potential  $U$  satisfies Poisson's equation

$$\frac{\partial^2 U}{\partial z^2} = - \sum q_{\sigma} n_{\sigma} . \quad (9)$$

### 3. CHARGE NEUTRALITY

A system of non-linear wave equations of the same type as (8) and (9) and also involving the ponderative force has been previously discussed and solved by one of the authors.<sup>7</sup> It was shown that the direct integration of the system of equations (8), (10) leads to numerical difficulties since it contains two vastly different characteristic lengths, namely the Debye length and the wave length. But it was also demonstrated that the small ratio of these lengths assures near perfect charge neutrality. It is therefore possible to replace (9) by the condition

$$\sum_{\sigma} q_{\sigma} n_{\sigma} = 0$$

and use it to eliminate the potential U from the equations (6).

Consider now a plasma composed of only two species of ions, having the same temperature:  $\sigma = 1, 2, e$ ;  $T_1 = T_2 = T_i \neq T_e$ ,  $m_1 > m_2$ . In this case the elimination of U yields

$$n_1 = n_{10} n_{e0}^{\frac{\tau}{1+\tau}} \frac{\exp \left[ -\frac{\phi_1}{T_i} - \frac{\phi_e}{T_e + T_i} \right]}{\left[ n_{10} e^{-\frac{\phi_1}{T_i}} + n_{20} e^{-\frac{\phi_2}{T_i}} \right]^{\frac{\tau}{1+\tau}}} \quad (10)$$

$$n_2 = n_{20} n_{e0}^{\frac{\tau}{1+\tau}} \frac{\exp \left[ -\frac{\phi_2}{T_i} - \frac{\phi_e}{T_e + T_i} \right]}{\left[ n_{10} e^{-\frac{\phi_1}{T_i}} + n_{20} e^{-\frac{\phi_2}{T_i}} \right]^{\frac{\tau}{1+\tau}}} \quad (11)$$



Charge neutrality is now guaranteed no matter what values are chosen for  $n_{10}$ ,  $n_{20}$ , and  $n_{e0}$ . The formulas (10) and (11) are used to express the plasma frequencies  $\omega_{pi}$  which are then substituted into Eq. (8). This is then the second order non-linear wave equation which we have to solve for  $E(z)$ .

We now proceed to reduce (8) to a nondimensional form. We introduce the quantities

$$m = 2m_1m_2 / (m_1 + m_2) ,$$

$$\alpha = (m_1 - m_2) / (m_1 + m_2) ,$$

$$\beta_i = n_{i0} / n_{e0} , \quad i = 1, 2 ,$$

and

$$\gamma = (\omega - \Omega_2) / (\Omega_1 - \Omega_2) .$$

Obviously  $0 < \alpha < 1$  and  $0 < \gamma < 1$ . The electric field amplitude and the distance along the  $z$ -axis will be measured, respectively in units of  $\mathcal{E}$  and of  $\mathcal{L}$ , which are given by

$$\mathcal{E} = \left( \frac{\alpha T_i}{m} \right)^{1/2} B_0 , \tag{12}$$

$$\mathcal{L} = \frac{\alpha^{1/2}}{1 + \alpha - 2\alpha\gamma} \left( \frac{m}{e^2 n_{e0}} \right)^{1/2}. \quad (13)$$

With these conventions the densities can be written in the form

$$n_j = n_{e0} \psi_j, \quad j = 1, 2, \quad (14)$$

$$n_e = n_1 + n_2,$$

where

$$\psi_1 = D^{-1} \beta_1 \exp \left( - \frac{E^2}{4(1-\gamma)} \right), \quad (15)$$

$$\psi_2 = D^{-1} \beta_2 \exp \left( \frac{E^2}{4\gamma} \right), \quad (16)$$

and

$$D = \left[ \beta_1 \exp \left( - \frac{E^2}{4(1-\gamma)} \right) + \beta_2 \exp \left( \frac{E^2}{4\gamma} \right) \right]^{\frac{\hat{\tau}}{1+\hat{\tau}}}. \quad (17)$$

The wave equation (8) assumes the form

$$E'' + \left( \gamma - \frac{\psi_1}{2(\gamma-1)} + \frac{\psi_2}{2\gamma} \right) E = 0 \quad (18)$$

where

$$\eta = \frac{\alpha B_0^2}{m n_{e0}} \quad (19)$$

In equation (18) the term corresponding to  $\omega_{pe}^2 / \omega_{ce} (\omega + \omega_{ce})$  in (7) has been neglected. Its contribution is less than  $m_e / m$  times the remaining terms.

#### 4. ANALYTICAL INTEGRATION OF THE DIFFERENTIAL EQUATION

The differential Eq. (18) has the following first integral

$$(E')^2 - F(E^2) = K \quad (20)$$

where  $K$  is a constant and

$$F(E^2) = 2(1+\epsilon) \left\{ 1 - \left[ \beta_1 e^{-\frac{E^2}{4(1-\epsilon)}} + \beta_2 e^{-\frac{E^2}{4\delta}} \right]^{\frac{1}{1+\epsilon}} \right\} - \eta E^2$$

$F(E^2)$  has two zeros one of which is  $E^2=0$ . Moreover

$$\left. \frac{dF}{d(E^2)} \right|_{E=0} = \eta_0 - \eta$$

where

$$\eta_0 = \frac{1}{2} \left( \frac{\beta_1}{1-\gamma} - \frac{\beta_2}{\gamma} \right) .$$

Depending on whether  $\eta \geq \eta_0$  the second zero of  $F$  appears for positive or negative values of  $E^2$

$$\eta > \eta_0 \quad , \quad F(E_a^2) = 0 \quad , \quad E_a > 0 \quad , \quad (21)$$

$$\eta < \eta_0 \quad , \quad F(-E_c^2) = 0 \quad , \quad E_c > 0 \quad . \quad (22)$$

The function  $F(E^2)$  is shown schematically in Fig. 1.

## 5. BOUNDARY CONDITIONS

We choose the boundary condition at  $z=0$  such that the natural concentration of the two species  $n_{i0}/n_{e0}$  is maintained at  $z=0$  and that the solution proceeds with increasing  $z$  towards enriched mixtures. As a physical realization of this boundary condition, we imagine a grid in the plane  $z=0$  which is transparent to the plasma but not to the wave. For  $z<0$  we assume a large reservoir of plasma composed of the isotopes in their natural proportions. It is necessary, here, to distinguish two cases, depending on whether the minority ions are lighter or heavier than the majority.

In case A the minority isotope is the lighter one  $n_{10} > n_{20}$  :

We choose  $E_0 = 0$  ,  $E'_0 \neq 0$  ,  $\beta_1 + \beta_2 = 1$  and  $\beta_1/\beta_2 = n_{10}/n_{20}$ .

Thus  $K = (E'_0)^2$ , and the first integral (20) becomes

$$(E')^2 = (E'_0)^2 + F(E^2) .$$

The solution can be written implicitly as an integral

$$x = \int_0^E \frac{dE}{\sqrt{(E'_0)^2 + F(E^2)}} .$$

These solutions are always periodic and  $E(z)$  is symmetric about the z-axis. According to (14), (15) and (16) the ratio of the density of the two species is given at any point by

$$\frac{n_2}{n_1} = \frac{n_{20}}{n_{10}} \exp \left[ \frac{E^2}{4\gamma(1-\gamma)} \right] .$$

Thus the maximum of enrichment coincides with the maximum of the electric field amplitude. The segregation of the isotopes occurs over a quarter wavelength of the standing wave.

The amplitude  $E_{max}$  is determined by the choice of  $E'_0$  through the equation

$$(E'_0)^2 + F(E_{max}^2) = 0 .$$

This maximum occurs by definition when  $z$  equals a quarter wave-length.

If  $\eta < \eta_0$  then this maximum is always larger than  $E_a$  defined by (21). This value is attained asymptotically as  $E'_0$  tends to zero.

If  $\eta \gg \eta_0$  the maximum  $E_{max}$  tends to zero with  $E'_0$ .  $E_{max}$  can be evaluated approximately in two limits. For

$$\eta \gg \eta_0, \quad E_{max} = \eta^{-1/2} E'_0.$$

In this case the plasma density is so low that the wave propagates nearly as in free space. For

$$\eta \ll \eta_0, \quad E_{max} \approx 48(1+\tau) \lg \left[ \frac{1}{\beta_2} + \frac{(E'_0)^2}{2\beta_2(1+\tau)} \right].$$

At the point at which this maximum is attained the density  $n_2$  will be much larger than  $n_1$  which means that here the minority ions have become the majority. This seems too good to be true. It might well be that the wave becomes unstable at large amplitudes. If an instability does limit the field to values below some  $E^*$  then it would no longer be possible to establish the field over a full quarter length. Rather the field would resemble a cut-off wave penetrating from the plane of excitation into the plasma.

In case B the minority isotope is the heavier one,  $n_{10} < n_{20}$ .

We choose  $E_0 \neq 0$ ,  $E'_0 = 0$  and replace  $\beta_1$  and  $\beta_2$  respectively by

$$\beta_1 \exp \left[ \frac{E_0^2}{4(1-\gamma)} \right] , \quad (23)$$

$$\beta_2 \exp \left[ - \frac{E_0^2}{4\gamma} \right] , \quad (24)$$

where  $\beta_1 + \beta_2 = 1$  and  $\beta_1 / \beta_2 = n_{10} / n_{20}$ . As a consequence, Eq. (20) can be written in the form

$$(E')^2 = F(E^2 - E_0^2)$$

whose integral is

$$X = - \int_{E_0}^E \frac{dE}{\sqrt{F(E^2 - E_0^2)}}$$

For  $E < E_c$  defined by (22), these solutions are periodic and symmetric about the z-axis. For  $E_0 = E$  the solution is aperiodic and bell-shaped, while for  $E > E_c$  it is again periodic but does not change sign and

$$E_0^2 - E_{min}^2 = E_c^2 .$$

The ratio of the densities of the two isotopes is still given by

(14), (15) and (16) but with  $\beta_1$  and  $\beta_2$  replaced by (23) and (24)

$$\frac{n_1}{n_2} = \frac{n_{10}}{n_{20}} \exp \left[ \frac{E_0^2 - E^2}{4\gamma(1-\gamma)} \right] . \quad (25)$$

the maximum attainable ratio therefore is

$$\frac{n_1}{n_2} = \frac{n_{10}}{n_{20}} \exp \left[ \frac{E_c^2}{4\gamma(1-\gamma)} \right] . \quad (26)$$

## 6. NUMERICAL INTEGRATION

Numerical solutions have been obtained by direct integration of Eq. (18) using the method of Runge-Kutta. Four typical cases are shown on Fig. 2, 3, 4 and 5. In the cases A1 and A2 the values of  $\alpha$  and  $\beta_i$  apply to the Uranium isotopes  $U^{235}$  and  $U^{238}$ , while the cases B1 and B2 represent the Neon isotopes  $Ne^{20}$  and  $Ne^{22}$ . The dimensionless parameters chosen for these solutions are listed in tabel I, where  $\varphi$  is the maximum concentration of the minority element in percent.

To obtain an idea of the actual physical values of the fields and the wavelengths we have to compute the units of field and of distance as given by (12) and (13). In the international system of units, MKSA, they become



$$\Sigma = \left( \frac{\alpha k T_i}{m} \right)^{1/2} B_0 ,$$

$$\mathcal{L} = \frac{\alpha^{1/2}}{1 + \alpha - 2\alpha\gamma} \left( \frac{m}{\mu_0 e^2 n_{e0}} \right)^{1/2} ,$$

while

$$\eta = \frac{\Sigma_0 \alpha B_0^2}{m n_{e0}}$$

As an example we choose  $B_0 = 0.1$  Tesla,  $T_i = 10^5$  °K,  $n_{e0} = 10^{15} \text{ m}^{-3}$ ,  $\gamma =$  and we consider the two isotopes mixtures  $\text{Ne}^{22}$ ,  $\text{Ne}^{20}$  and  $\text{U}^{235}$ ,  $\text{U}^{238}$ .

Thus we obtain the values listed in table II.

## 7. CONCLUSIONS

We have demonstrated the existence of large amplitude non-linear solutions of the wave equation for a three fluid plasma. It represents a stationary left circularly polarized wave which creates spatially periodic variations of the relative concentration of the two ion species given by

$$\text{case A: } \frac{n_2}{n_1} = \frac{n_{20}}{n_{10}} \exp \left[ \frac{m E_{\max}^2}{4\gamma(1-\gamma) \alpha B_0^2 k T_i} \right]$$

or

case B: 
$$\frac{n_1}{n_2} = \frac{n_{10}}{n_{20}} \exp \left[ \frac{m(E_0^2 - E_{min}^2)}{4\gamma(1-\gamma) \propto B_0^2 kT_i} \right] .$$

Collisions, which we have neglected, tend to broaden the resonances. It will be important to keep the collision frequency, due to all effects, small compared to the separation of the two isotope resonances.

$$v_{coll} \ll \propto \Omega .$$

This is possible<sup>1,2</sup>. If one wishes to enhance the separation by choosing  $\omega$  very near to  $\Omega_1$ , or  $\Omega_2$  that is  $\gamma \ll 1$  or  $1-\gamma \ll 1$  then the condition on the collisions becomes more stringent

$$v_{coll} \ll \min \left\{ \gamma \propto \Omega, (1-\gamma) \propto \Omega \right\} .$$

Finally, any device for isotope separation based on the ponderomotive force must be large compared to the Larmor radius of the ions which is

$$R = \frac{E}{\propto B_0 \omega} .$$

Within the framework of the ideal three fluid theory the solutions obtained are exact to any amplitude. This does not mean that they represent physically realisable configurations since they may be unstable. This will have to be investigated. Furthermore, the solutions which we present are plane waves which do not fit into any finite device. It is therefore not yet evident how one should use the ponderomotive force in a practical method for isotope separation.

Nevertheless, these idealized solutions show the existence of the effect of separation which is large and therefore promising. It should be possible to find configurations in a finite volume which have the same property of segregating isotopes in space. This will be the subject of a forthcoming report.

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FIGURE CAPTIONS

- Fig. 1: Qualitative behaviour of the function  $F(E^2)$   
for two cases  $\eta < \eta_0$  and  $\eta > \eta_0$
- Fig. 2: Normalized electric field  $E$ , densities  $\psi_1$ , and  
 $\psi_2$  versus distance  $z$ . Case A1
- Fig. 3: Normalized electric field  $E$ , densities  $\psi_1$ , and  
 $\psi_2$  versus distance  $z$ . Case A2
- Fig. 4: Normalized electric field  $E$ , densities  $\psi_1$ , and  
 $\psi_2$  versus distance  $z$ . Case B1
- Fig. 5: Normalized electric field  $E$ , densities  $\psi_1$ , and  
 $\psi_2$  versus distance  $z$ . Case B2

TABLE I

$\gamma = 0.5$	Uranium		Neon	
$\alpha$	$6.35 \cdot 10^{-3}$		$4.76 \cdot 10^{-2}$	
$\beta_1$	.9928		0.0882	
$\beta_2$	0.0027		0.9118	
$\eta_0$	0.9856		-0.8236	
	Case A1	Case A2	Case B1	Case B2
$\eta$	0	1	0	0
$E_{\max}$	3.14	1.43	1	2.5
$\lambda/4$	4.75	3.77	1.80	0.78
$\vartheta (\%)$	99.3	5.41	20.8	91.1

TABLE II

$B_0 = 0.1$  Tesla,  $T_i = 10^5$  K,  $n_{e0} = 10^{15} \text{ m}^{-3}$ ,

$\gamma = 0.5$	Uranium	Neon
$\alpha$	$6.35 \cdot 10^3$	$4.76 \cdot 10^{-2}$
$\beta_1$	0.9928	0.0882
$\beta_2$	0.0072	0.9118
$\eta_0$	0.246	-0.206
$m$	$3.96 \cdot 10^{-25} \text{ kg}$	$3.50 \cdot 10^{-26} \text{ kg}$
$\Sigma$	14.9 V/m	137 V/m
$\mathcal{L}$	8.84 m	7.20 m
$\eta$	$1.42 \cdot 10^{-6}$	$1.205 \cdot 10^{-4}$













