February 1980 LRP 163/80

NUMERICAL TREATMENT OF THE QUASI-LINEAR FAKE DIFFUSION

K. Appert and J. Vaclavik

Numerical Treatment of the Quasilinear Fake Diffusion

K. Appert and J. Vaclavik

Centre de Recherches en Physique des Plasmas

Association Euratom - Confédération Suisse

Ecole Polytechnique Fédérale de Lausanne

CH-1007 Lausanne / Switzerland

ABSTRACT

A formulation of the quasilinear equations is proposed which allows for the nonresonant interaction of particles with damped waves without leading to a "negative diffusion problem".
In an application to current-driven ion-acoustic turbulence the proposed equations are shown to be amenable to a numerical treatment.

1. INTRODUCTION

Ten years ago Vahala and Montgomery[1] have pointed out that the standard quasilinear theory of weakly turbulent plasmas may lead to an ill-posed problem whenever initially unstable waves become damped in the later evolution of a turbulent system. The basic equation of quasilinear theory is a diffusion equation for the particle distribution function $f(v,t)$, in velocity space whose coefficient $D(v,t)$, is not positive everywhere whenever damped waves are involved.

In subsequent publications[2,3], however, it became clear that the problem may be well posed, when the resonant and the nonresonant wave-particle interactions are treated separately. This separation has been achieved in two slightly different ways. On the one hand Kaufman[2] seperates the particle distribution function into two distribution functions, f_o and f_2, which are defined everywhere in velocity space. The evolution of f_o is proportional to the turbulent energy and is governed by a diffusion equation with a positive coefficient D_R, stemming from the resonant interaction. f_2 in turn can be expressed directly in terms of the turbulent energy and is in size proportional to it. On the other hand, Davidson[3] separates the velocity space into a resonant and a nonresonant region and shows that a negative diffusion coefficient D_{NR}, may only arise in the nonresonant region, where local features of the distribution function go beyond the approximations made in the derivation of the quasilinear equations. Both methods[2,3] permit to demonstrate conservation of particles, momentum and energy. However, as they stand they do not permit an exchange of resonant and nonresonant particles in course of time, a piece of physics which has been shown to be important for the creation of high energy ions in the problem of ion-acoustic turbulence.[4,5] Davidson's method does permit this exchange under the condition that the distribution function is approximated by an evolving global function (e.g. a Maxwellian).[6] The price to pay is a poor description of the evolving distribution function in the resonant region.[5] We therefore seek a formulation of the quasilinear equation which has both the right conservation properties and no negative diffusion problem, and which permits exchange of particles between the resonant and the nonresonant regions without spoiling the detailed response of the distribution function.

2. BASIC IDEA

The formulation we would like to put forward is neither sophisticated nor really original but convenient for numerical calculations. The basic idea is in fact older than the dispute about the fake diffusion. Kadomtsev[7] mentions in his book the possibility of replacing $f(v,t)$ in the nonresonant diffusion term by $f(v,t=o)$ arguing that this replacement results in a change of higher order than to which the equation itself has been derived. In his formulation no negative diffusion problem arises because the nonresonant term is a simple source term and yet all quantities are conserved and particle exchange is permitted. However, the difference between f(v,t) and f(v,t=o) in the nonresonant region is only small in a closed system. Whenever the system is driven by an external force such as an electric field, this difference may become substantial. In the ion-acoustic problem for instance, temperature increases of one order may be observed.[4,5] In such cases one may use the following formulation of the quasilinear equation :

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial \underline{v}} \cdot \underline{\underline{D}}_R \cdot \frac{\partial f}{\partial \underline{v}} + \frac{\partial}{\partial \underline{v}} \cdot \underline{\underline{D}}_{NR} \cdot \frac{\partial f_M}{\partial \underline{v}} \tag{1}$$

where f_M is a shifted Maxwellian with the same drift velocity and temperature as f:

$$\int d^3v \begin{pmatrix} \underline{v} \\ v^2 \end{pmatrix} f_M(\underline{v},t) = \int d^3v \begin{pmatrix} \underline{v} \\ v^2 \end{pmatrix} f(\underline{v},t) \tag{2}$$

Hence the nonresonant term in Eq.(1) depends nonlinearly on integral quantities defined by f and acts as a source term to the diffusion equation. Eq.(1) possesses all the properties we have asked for, in particular energy and momentum are conserved if the wave dispersion is consistently evaluated with the function $f_M(\underline{v},t)$. Eq.(1) is only correct under the restriction $(\partial\omega/\partial t)/\omega \ll \gamma$ where ω and γ denote the waves' frequencies and growth rates respectively. In general, the quasilinear equation comprises on the right hand side terms proportional to $\partial\omega/\partial t$ and $\partial f/\partial t$.[7,8] Obviously, Eq.(1) is only tractable by numerical means due to its integro-differential character.

3. APPLICATION TO ION-ACOUSTIC TURBULENCE

We describe now an application of the quasilinear equations in the form of Eqs.(1) and (2) to the problem of ion-acoustic turbulence. We shall compare the results with the ones obtained with what we call the "inconsistent method". The latter consists in treating the nonresonant interaction as a diffusion process, $f_M = f$ in Eq.(1), as long as $\underline{\underline{D}}_{NR}$ is positive definite, in the opposite case the nonresonant interaction is simply neglected. This method is inconsistent in so far as the total energy is not conserved when $\underline{\underline{D}}_{NR}$ is not positive definite. For positive definite $\underline{\underline{D}}_{NR}$, however, both methods should yield roughly the same result, because they are equivalent within the accuracy to which the quasilinear equations are derived.

The plasma under consideration is assumed to be collisionless, uniform and nonmagnetized. The electrons are hot ($T_e >> T_i$) and drift with a constant velocity $\underline{v}_d = v_d \underline{e}_x$ relative to a cold ion background which results in the generation of ion-acoustic turbulence in the system. Since the problem exhibits axial symmetry with respect to the current axis we confine ourselves, for convenience, to a two-dimensional model. Even so the structure of the resonant interaction terms is rather formidable. As for the electrons one can show that the nonresonant interaction is negligible compared to the resonant interaction as long as $v_d^2 \ll T_e/m_e$. In order to make the ion nonresonant interaction term amenable to a numerical procedure we take the motion of ions as one-dimensional. However, in their resonant terms, where 2-D behaviour is important, we replace the 1-D δ-function by the function obtained by averaging the 2-D δ-function over the angles assuming that the ion distribution is isotropic. With this stipulation, the quasilinear equations describing the problem can be given in the form

$$\frac{\partial f^{(e)}}{\partial t} + E\,\frac{\partial f^{(e)}}{\partial v_x} = \frac{\partial}{\partial \underline{v}} \cdot \int \frac{d^2k}{4\pi}\,\frac{\underline{k}\,\underline{k}}{k^2}\, I_{\underline{k}}\, \delta(\omega_{\underline{k}} - \underline{k}\cdot\underline{v}) \cdot \frac{\partial f^{(e)}}{\partial \underline{v}} , \tag{3}$$

$$\frac{\partial f^{(i)}}{\partial t} - \mu E\,\frac{\partial f^{(i)}}{\partial v_x} = \mu^2 \frac{\partial}{\partial v_x} \int \frac{d^2k}{4\pi^2}\,\frac{k_x^2}{k^2}\, I_{\underline{k}}\, \frac{2\,\omega_{\underline{k}}}{k_x v_x \left(k_x^2 v_x^2 - \omega_{\underline{k}}^2\right)^{1/2}}\,\frac{\partial f^{(i)}}{\partial v_x} + \frac{\partial}{\partial v_x} D_{NR}\,\frac{\partial f_M^{(i)}}{\partial v_x} , \tag{4}$$

$$\frac{1}{2 I_{\underline{k}}}\,\frac{\partial I_{\underline{k}}}{\partial t} = \gamma_{\underline{k}} \equiv \frac{1}{k^2\,\partial\epsilon/\partial\omega_{\underline{k}}} \left\{ \pi \int d^2v\; \underline{k}\cdot\frac{\partial f^{(e)}}{\partial \underline{v}}\; \delta(\omega_{\underline{k}} - \underline{k}\cdot\underline{v}) \right.$$

$$\left. + \, 2\mu\,\omega_{\underline{k}} \int dv_x\, \frac{\partial f^{(i)}}{\partial v_x}\; \frac{1}{v_x \left(k_x^2 v_x^2 - \omega_{\underline{k}}^2\right)^{1/2}} \right\} , \tag{5}$$

$$D_{NR} = \mu^2 \mathcal{P} \int \frac{d^2k}{4\pi^2} \frac{k_x^2}{k^2} \frac{\gamma_{\underline{k}} I_{\underline{k}}}{2(\omega_{\underline{k}} - k_x u_x)^2} , \tag{6}$$

$$k^2 \frac{\partial \epsilon}{\partial \omega_{\underline{k}}} = 2\left(\frac{\mu k_x^2}{\omega_{\underline{k}}^3} + \frac{k_x v_d}{k^2 T_e}\right) , \quad \omega_{\underline{k}} = \frac{(\mu T_e)^{1/2} k_x}{[1 + k^2 T_e - k_x^2 v_d^2/(k^2 T_e)]^{1/2}} \tag{7}$$

---

where $f^{(e)}$ and $f^{(i)}$ are the electron and ion distribution functions, respectively, $I_{\underline{k}}$ is the spectral distribution of the fluctuating electrostatic field, E is the electric field associated with the current, and $\mu = m_e/m_i$ is the electron-to-ion mass ratio. Equations (1)-(4) are in dimensionless units; the units of time, space, distribution function, electric field, temperature and spectral distribution are, respectively, ω_{pe}^{-1}, λ_D, $m_e n/T_{eo}$, $(4\pi n\, T_{eo})^{\frac{1}{2}}$, T_{eo} and $4\pi n\, T_{eo}\lambda_D^2$. Here n and T_{eo} are the electron density and initial temperature, respectively, and λ_D is the Debye length. Equations (1)-(4) are solved numerically using the finite-element method[9]. The initial conditions are

$$f^{(e)}(t=0) = \frac{1}{2\pi} \exp\left\{-\frac{1}{2}\left[(v_x - v_d)^2 + v_y^2\right]\right\} ,$$

$$f^{(i)}(t=0) = \frac{1}{(2\pi\mu T_{io})^{1/2}} \exp\left(-\frac{v_x^2}{2\mu T_{io}}\right) ,$$

$$I_{\underline{k}}(t=0) = \text{const} .$$

It turns out that the numerical procedure is not stable in the nonresonant region where the changes of f_i are merely due to the source term. A slight change of Eq.(4), however, removes the difficulty: We replace the nonresonant term by

$$\frac{\partial}{\partial v_x}\left(D_{NR} - \sigma|D_{NR}|\right)\frac{\partial f_M^{(i)}}{\partial v_x} + \frac{\partial}{\partial v_x}\,\sigma|D_{NR}|\,\frac{\partial f^{(i)}}{\partial v_x}$$

where σ is a small positive number, typically 0.05 to 0.1. The term proportional to $\sigma|D_{NR}|$ provides us with a small positive diffusion in the nonresonant region which is big enough to make the procedure stable. The new equation is equivalent to Eq.(4) within the accuracy of the general quasilinear equation.

In Fig. 1 we show the temporal evolution of the total electrostatic wave energy W, for the initial conditions $v_d = 0.12$, $T_{io} = 0.02$, $W/n\,T_{eo} = 4 \times 10^{-10}$ and $\mu = 1/1836$. Both methods used yield the same overall behaviour. As long as the wave energy is increasing both methods are consistent. During this phase the relative difference between the two energies is of the order of 10%. This difference provides one with a feeling of how correct either one of the methods is. The effect of the inconsistency is, at least in the problem at hand, of the same order as the above-mentioned difference. This effect should be most pronounced at the quasi-saturation time where damped and growing waves are equally important. At later times no further deviation is observed, because the nonresonant interaction becomes negligible compared to the resonant interaction.

../9

In the framework of turbulence calculations a difference of 10% is immaterial, because experimental uncertainties are usually much greater. Seen from this standpoint therefore, the two methods are equally good. In general we would nevertheless opt for the consistent treatment of the nonresonant interaction, because it does not essentially increase the numerical effort over the one needed for the inconsistent treatment. Moreover, it could well be that in a problem other than the ion-acoustic one the inconsistency would affect the final result more pronouncedly.

4. CONCLUSION

We have proposed a formulation of the quasilinear equations which permits us to treat the nonresonant interaction of particles with damped waves in a consistent manner. In a numerical application to ion-acoustic turbulences we have shown how this formulation can be used. We have certainly not enriched the theory of weak turbulence from a formal point of view, but we have shown in a pragmatic manner how this theory can be used when damped waves are in the game.

../10

ACKNOWLEDGEMENTS

We are indebted to Dr. A. D. Cheetham, who carefully read and criticized the manuscript.

This work was partly supported by the Swiss National Science Foundation.

REFERENCES

1 G. Vahala and D. Montgomery, J. Plasma Phys. 4, 677 (1970)

2 A.N. Kaufman , J. Plasma Phys. 8, 1 (1972)

3 R.C. Davidson, Methods in Nonlinear Plasma Theory, (Academic, New York, 1972)

4 C.T. Dum, R. Chodura, and D. Biskamp, Phys. Rev. Lett. 32, 1231 (1974)

5 K. Appert, R. Bingham, J. Vaclavik, and E.S. Weibel in 9th European Conference on Controlled Fusion and Plasma Physics (Culham Lab., 1979) BP9

6 M.Z. Caponi and R.C. Davidson, Phys. Rev. Lett. 31, 86 (1973)

7 B.B. Kadomtsev, Plasma Turbulence, (Academic, New York, 1965)

8 W. Horton, Jr. and Duk-In Choi, Physics Reports 49, 273 (1979)

9 K. Appert, T.M. Tran, and J. Vaclavik, Comput. Phys. Commun. 12, 135 (1976)

FIGURE CAPTION

Fig. 1 Temporal evolution of the electrostatic wave energy W, according to our method (————) and according to the inconsistent method (--------).

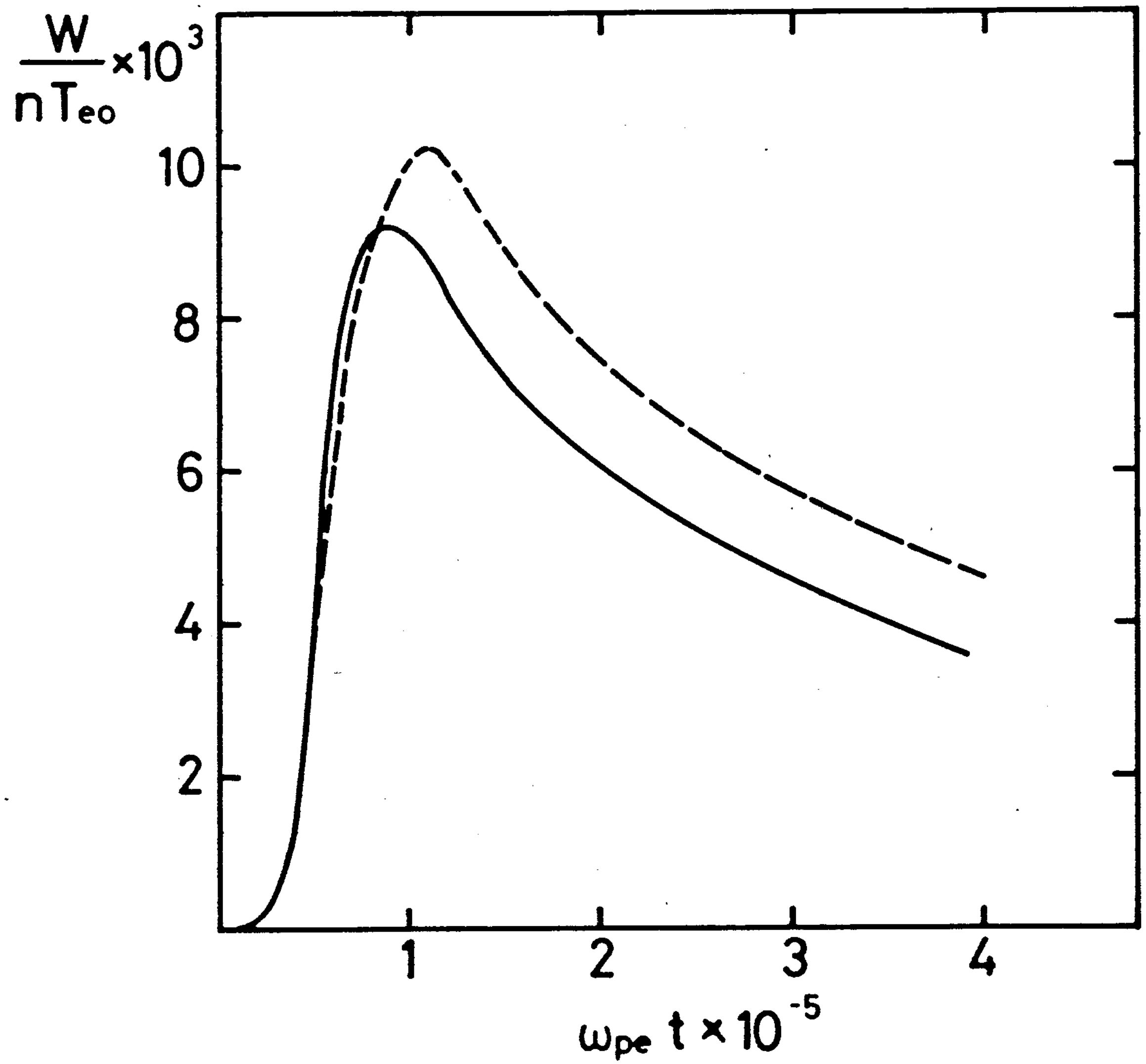

Fig.1