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THEORY OF DOUBLE RESONANCE PARAMETRIC
EXCITATION IN PLASMAS. II

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Abstract

Using a simpler formalism than in the original paper on this subject, we verify the earlier result that for a pump frequency separation Δ approximately equal to twice the ion acoustic frequency Ω , the use of two long wavelength pumps can reduce the threshold for parametric excitation of ion acoustic waves when, and only when, the Langmuir wave damping rate γ is much larger than Ω . The threshold is reduced by a factor of order Ω/γ , the optimum value of Δ being $2\Omega - \Gamma$ for equal

pump amplitudes, where Γ is the ion acoustic wave damping rate, $\Gamma \ll \Omega$. The analysis presented in a recent paper is shown to be valid only for $\gamma \ll \Omega$, where there is no threshold reduction, and for weak ion acoustic wave damping ($T_e/T_i \ll 1$). There it gives thresholds in agreement with the earlier work, but its results for T_e/T_i of order 1 are invalid.

I. Introduction

In an earlier publication¹ a complete and detailed account was given of the parametric excitation of ion acoustic waves and Langmuir waves resulting from two long wavelength pump waves, whose frequencies are near the electron plasma frequency, while their difference is of order of ion acoustic wave frequencies. Naturally, the use of two pumps is of interest only if the threshold for the resulting instability is smaller (or the growth rates above threshold larger) than for a single pump, and while it seems reasonable that the use of two such pumps might result in lower thresholds and larger growth rates, it is not clear a priori whether, or under what circumstances, this will indeed be the case. A principal result of Ref. 1 was that the threshold is reduced when, and only when, the damping rate, γ , of the Langmuir waves exceeds the ion acoustic wave frequency, Ω . Recent experiments by Akiyama et al.² appear to confirm this threshold reduction.

In a recent publication on this same subject³, results strongly at variance with those given in Ref. 1 have been reported. Although Ref. 3 uses two different methods of calculation to obtain results which are in

agreement with each other but, allegedly, quite different from those of Ref. 1, the authors do not identify any errors in the earlier work, offering only a vague speculation that the authors of Ref. 1 "might have made over-simplified assumptions about their resonance functions". Since an analysis of two pump excitation inevitably entails considerable algebraic complication and since, of the two methods of computation in Ref. 3, the one which is described in most detail is quite different from the approach of Ref. 1, some effort is required to ascertain which is correct.

We have carefully reviewed the theory of two pump excitation, and find that, notwithstanding the assertions of Ref. 3, the results of Ref. 1 for pump power thresholds are correct, although the value cited there for the optimum value of the pump frequency separation is very slightly in error. Ref. 3, on the other hand, is found to have serious limitations and also some errors. Where it is correct, it is essentially in agreement with the analysis of Ref. 1. Unfortunately, the only regime treated correctly in Ref. 3 is the inherently uninteresting "weak damping" case, $\gamma \ll \Omega$, where the threshold with two pumps is always greater than that for a single pump; the case of "strong damping", $\gamma \gg \Omega$, where the use of two pumps can indeed give a lower threshold, is not examined. Ref. 3 also contains statements concerning the importance of phase relations between waves of different frequencies which would be surprising, if true, but which seem, in fact, to be without foundation.

A further complication arises from the use, in Ref. 3, of kinetic theory expressions for the ion susceptibility, rather than resonant fluid approximations. While the former is, of course, correct for a collisionless plasma, the authors seem not to have realized that the truncation of the infinite set of coupled mode equations which underlies their work, as well as that of Ref. 1, is justified only when the ion acoustic modes involved are highly resonant, i.e., when the electron-ion temperature ratio, T_e/T_i , is large enough to make the Landau damping small. In that case, a resonant fluid approximation is valid and makes it possible to obtain closed form results from which general conclusions can be drawn, whereas use of the kinetic susceptibilities forces the authors of Ref. 3 to rely upon numerical calculations for specific parameter values, making it hard to see the forest for the trees. Thus, the thresholds given in Ref. 3 for small T_e/T_i values of 1, 2 and, probably, 4 are incorrect, since the ion acoustic resonance is then so broad that there is no justification for ignoring additional low frequency modes. The thresholds for larger T_e/T_i are in substantial agreement with Ref. 1.

In the present paper, we re-exam the theory of two pump excitation, simplifying and extending somewhat the work of Ref. 1 and formulating the analysis in a manner which facilitates comparison with Ref. 3 and which should serve to resolve the confusion engendered by that work. Comparison of the two treatments is complicated by the fact that Ref. 3 uses a heuristic approach, quite different from that of Ref. 1 (although a more conventional method is also briefly sketched). In addition, Ref. 3 deals with the high frequency sidebands, while Ref. 1 involves the low

frequency waves. We therefore present here both the high and low frequency approaches.

The treatment in Ref. 1 is quite formal, and while this provides a useful tool for examining all of the possible cases and for rigorously justifying the approximations used, it can, apparently, be difficult to follow.

In this paper we have therefore taken the opposite tack, presenting everything in the simplest possible way, using approximations whose careful substantiation can be found in Ref. 1, and emphasizing the physical picture of the instability. For example, since only electrostatic waves are involved, and since the lowest threshold or largest growth rate corresponds to a common polarization, with all electric fields parallel to the pump fields, it suffices to use a one dimensional analysis.

In Section II, we summarize the essential aspects of two pump excitation and discuss several approaches to the calculation of thresholds and growth rates. Some general properties of the coupled modes are discussed in Section III. In Section IV we analyze the weak damping and strong damping cases, avoiding some of the approximations used in Ref. 1 to obtain specific results. We obtain threshold values in general agreement with those of Ref. 1 but find that the minimum threshold is obtained when the separation of the pump frequencies differs slightly from twice the ion acoustic frequency, a shift overlooked in Ref. 1. In particular, we show that the threshold for two pumps always exceeds that for one pump in the weak damping case, whereas two pumps can be advantageous in the

strong damping case. In Section V we explain the relation between our approach and that of Ref. 3 and briefly discuss the errors and limitations of the latter work. Conclusions, relations to experiments and suggestions for future work are given in Section VI.

II. Parametric Excitation with Two Pumps

In this section we derive the basic equation (16) which expresses the coupling of high and low frequency components of the fluctuating electric field, E . This equation, or rather set of equations, contains all of the physics of the problem and there remains only the question of how to obtain thresholds and growth rates from it. We discuss three alternative approaches which, though formally different, must, of course, yield the same results. In the weak damping case, one of these, as will be shown explicitly in Section V, leads to the dispersion relation (7) of Ref. 3 when the damping is weak ($\gamma \ll \Omega$).

We assume a spatially uniform, high frequency pump electric field

$$E_0 = E_1 \exp(-i\omega_1 t) + E_2 \exp(-i\omega_2 t) + c.c. \quad (1)$$

in a homogeneous plasma, where ω_1 and ω_2 are of order of the electron plasma frequency, $\omega_p^2 = (4\pi n e^2 / m)$, with $\omega_1 > \omega_2$. This field causes the

electrons to oscillate with a displacement

$$X_o(t) = (e/m) \sum_s E_s \exp(-i\omega_s t) \omega_s^{-2} + c.c. \quad (2)$$

where, as in subsequent equations, s is to be summed over $s = 1, 2$. (The field also causes ion oscillations, but since these are smaller in amplitude by the electron-ion mass ratio, m/M , we shall neglect them.) Without loss of generality we can take E_1 and E_2 to be real, since the relative phase of two waves of different frequencies can have no physical significance in a linear theory; it only determines the absolute time at which the two waves are instantaneously in phase, and in absence of something which depends upon the absolute time, this cannot have any physical consequences.

Before proceeding with the formal analysis, we describe the essential aspects of the physics. Because of the oscillatory electron motion (2), a charge density fluctuation or perturbation at a high (or low) frequency ω will give rise to similar perturbations at the low (or high) frequencies $\omega \pm \omega_s$. (In the heuristic discussion of this paragraph and the following one we regard the frequencies as real, but thereafter ω , as a Laplace transform variable, is correctly treated as a complex quantity). The net result, under proper circumstances, is an amplification of the original perturbation, i.e., an instability. The relation of the various frequency components is illustrated in Fig. 1, where a low frequency fluctuation at ω is seen to give rise to high frequency sidebands at $\omega \pm \omega_s$.

It is important to note that two of these, (a and b in Fig. 1) lie between the pumps. We shall call these the inner sidebands, whereas the other two, which lie above the higher pump or below the lower one (c and d in Fig. 1) we shall call the outer sidebands. (Of course, since we are using complex exponentials, a negative frequency, such as $\omega - \omega_1$ is always accompanied by, and hence can be considered as being entirely equivalent to, the corresponding positive frequency, $\omega_1 - \omega$). All four of these high frequency sidebands generate low frequency perturbations at ω , just as in the case of a single pump, but also, as a consequence of having two pumps, at the frequencies $\omega \pm \Delta$ where

$$\Delta = \omega_1 - \omega_2 \quad (3)$$

These low frequency components, in turn, generate high frequency sidebands, some at the old frequencies, $\omega \pm \omega_s$, and others at new frequencies so that, as usual, an infinite set of both high and low frequency modes are coupled. Under certain circumstances, a finite subset of these can be, to good approximation, decoupled from the remainder and these may reinforce one another, giving rise to an instability. Although there are several possibilities for this situation, as shown in Ref. 1, we concentrate here on the simplest, which is also the subject of Ref. 3.

Suppose we choose the pump frequency separation to be approximately twice the ion acoustic frequency

$$\Delta \doteq 2 \Omega \equiv 2 k c_s (1 + k^2/k_D^2)^{-\frac{1}{2}} \quad (4)$$

for some wave number $k \ll k_D \equiv (4\pi n e^2 / T_e)^{1/2}$. Then if the ion acoustic resonance is sharp, i.e., if its damping rate Γ is small compared to Ω , there can be two resonant low frequency modes, namely at

$$\omega \doteq \Omega \quad \text{and} \quad \omega - \Delta \doteq -\Omega \quad (5)$$

In computing the threshold and the growth rate for pump amplitudes not too far above threshold it should be a good approximation to keep just these two low frequency modes and neglect all the others, i.e., those at frequencies $\omega + n\Delta$ with $n \neq -1, 0$.

In this way, the infinite set of coupled modes is reduced to just the six frequencies ω , $\omega - \Delta$, and $\omega \pm \omega_s$. The term "double resonance" is used because two low frequency perturbations, at ω and $\omega - \Delta$, are involved and both can be resonant, whereas for a single pump the low frequencies which are coupled to ω are only its harmonics, which are not, for given k , resonant when ω is. Of course, if $\omega = +\Delta/2$ so that $\omega - \Delta = -\omega$, then we will have only a single low frequency, but this is simply a limiting case of what are, in general, two distinct low frequency modes. Clearly, if the ion acoustic waves are strongly damped, due to either collisions or to Landau damping (as when T_e/T_i is of order 1), then the restriction to just two low frequency modes and the consequent truncation of the infinite set of coupled modes is completely unjustified and results obtained from the 6-dimensional subset have no validity. Finally, we mention that, of course, the threshold will be lowest if the high

frequency modes are also resonant, i.e., if the ω_s lie near the Bohm-Gross frequency,

$$\omega_k \equiv \omega_p \left(1 + 3k^2/k_D^2 \right)^{1/2} \quad (6)$$

We now proceed to the formal analysis. The basic equation describing the coupling of high and low frequency components can most easily be obtained using the Dawson transformation to a frame oscillating with the electrons,

$$\tilde{X} = X - X_o(t) \quad (7)$$

We shall use the tilde to denote quantities in the oscillating frame,

$$\tilde{E}(\tilde{X}, t) = E(x, t) + E(\tilde{X} + X_o(t), t), \text{etc.} \quad (8)$$

In this frame, the electrons do not feel the pump field and hence will execute their usual thermal motions, unperturbed by the pump. Consequently, in this frame the linear electron density response to an electric field with wave number k and frequency ω will be determined by the usual linear relation between the Fourier-Laplace transforms of the density and field,

$$\tilde{\rho}_e(k, \omega) = -(ik/4\pi) \chi(k, \omega) \tilde{E}(k, \omega) \quad (9)$$

where ρ_e is the electron density, χ is the usual electron susceptibility,

$$\chi(k, \omega) = -(\omega_p/k)^2 \int dv f'_{e0}(v) (v - \omega/k)^{-1} \quad (10)$$

and f_{e0} is the unperturbed electron velocity distribution. (This physically obvious result can, of course, be derived formally using the Vlasov equation¹).

To make use of (9) we need to know also the relation between the Fourier-Laplace transforms of variables such as E and ρ_e in the oscillating frame and the corresponding quantities in the laboratory frame. This is provided by a straightforward calculation of the transforms, in which we retain only terms of order kx_0 , which is assumed to be small, and neglect terms of order $(kx_0)^2$ or higher. (The neglect of quadratic terms, both here and in Eq. (12) below, is justified in Ref. 1, where terms of all orders in kx_0 are retained in the general analysis). From the definition of $E(k, \omega)$ we have

$$\begin{aligned} E(k, \omega) &\equiv \int_0^{\infty} dt \int_{-\infty}^{\infty} dx E(x, t) \exp[-i(kx - \omega t)] \\ &= \int_0^{\infty} dt \int_{-\infty}^{\infty} d\tilde{x} \tilde{E}(\tilde{x}, t) \exp[-ik(\tilde{x} - \omega t)] (1 - ikx_0) \\ &= \tilde{E}(k, \omega) - i \int_0^{\infty} dt \tilde{E}(k, t) \left[\sum_s \lambda_s e^{-i\omega_s t} + c.c. \right] \exp(i\omega t) \end{aligned}$$

with
$$\lambda_s \equiv k (e E_s / m \omega_s^2)$$

Thus,

$$E(k, \omega) = \tilde{E}(k, \omega) - i \sum_s \lambda_s \left[\tilde{E}(k, \omega - \omega_s) + \tilde{E}(k, \omega + \omega_s) \right] \quad (11)$$

and a similar relation holds for ρ and $\tilde{\rho}$. (Since k appears in (11) only as a parameter, unaffected by the transformation from one frame to the other, we shall generally omit it in the interests of notational simplicity). The inverse relation, expressing \tilde{E} in terms of E , differs from (11) only in the sign of the λ_s terms:

$$\tilde{E}(k, \omega) = E(k, \omega) + i \sum_s \lambda_s [E(k, \omega - \omega_s) + E(k, \omega + \omega_s)]$$

Using the transformation equations together with (9) we have (omitting the k dependence)

$$\begin{aligned} \rho(\omega) &= \tilde{\rho}(\omega) - i \sum_s \lambda_s [\tilde{\rho}(\omega - \omega_s) + \tilde{\rho}(\omega + \omega_s)] \\ &= -(ik/4\pi) \left\{ \chi(\omega) [E(\omega) + i \sum_s \lambda_s (E(\omega - \omega_s) + E(\omega + \omega_s))] \right. \\ &\quad \left. - i \sum_s \lambda_s [\chi(\omega + \omega_s) E(\omega + \omega_s) + \chi(\omega - \omega_s) E(\omega - \omega_s)] \right\} \quad (12) \end{aligned}$$

where we have again omitted terms quadratic in the λ_s . Finally, we make use of Poisson's equation,

$$ikE = 4\pi(\rho_e + \rho_i) \quad (13)$$

and the usual linear relation between the density of ions (which, as noted, are scarcely affected by the pump fields) and the electric field,

$$\rho_i = -(ik/4\pi) \chi_i E \quad (14)$$

where χ_i is the usual ion susceptibility, given by an expression analogous to (10). From (13) and (14) we have

$$E(\omega) = (4\pi/ik) \rho_e(\omega) (1 + \chi_i)^{-1} \quad (15)$$

and substituting this into (12) to eliminate ρ_e gives us the desired equation relating the high and low frequency components of E induced by the pump:

$$\epsilon(\omega) E(\omega) + i \sum_s \lambda_s \left\{ \left[\chi(\omega + \omega_s) - \chi(\omega) \right] E(\omega + \omega_s) + \left[\chi(\omega - \omega_s) - \chi(\omega) \right] E(\omega - \omega_s) \right\} = 0 \quad (16)$$

where

$$\epsilon(\omega) = 1 + \chi_i(\omega) + \chi(\omega) \quad (17)$$

is the usual linear dielectric function. (To simplify the notation in subsequent equations, we have omitted the subscript e which would normally be carried by the electron susceptibility, and have written it

simply as χ . The dielectric function, ϵ , of course involves both the ionic and electronic contributions).

The infinite set of equations (16) contains all of the relevant physics (correct to lowest order in the pump fields) and there remains only the question of finding circumstances under which truncation of the set can be justified. As the analysis of Ref. 1 shows, there are two interesting possibilities for the choice of the pump frequency separation Δ :

- 1) $\Delta \doteq 2\Omega$, in which case we have what might be described as a generalization of the usual decay instability;
- 2) $\Delta \doteq \Omega$, in which case there is, roughly speaking, a combination of the decay and oscillating two stream instabilities. It is only the first of these which is considered in Ref. 3 and since it has the lower threshold, we shall only examine that case.

Of the infinite number of components $E(\omega)$ in (16), the important ones will be those which are resonant, i.e., for which the coefficient of E on the left hand side, $\epsilon(\omega)$, is very small. This occurs only when $|\omega|$ is near either the Bohm-Gross frequency ω_k or the ion acoustic frequency Ω . (For consistency we should perhaps write Ω_k , but since k is a fixed parameter in all of the equations, we omit it whenever possible). With the choice $\Delta \doteq 2\Omega$ there will then be, as explained above, altogether 6 such resonant frequencies:

$$\omega \doteq \Omega ; \quad \omega - \Delta \doteq -\Omega ; \quad \omega \pm \omega_s \doteq \pm \omega_k \quad (18)$$

It is trivially straightforward, in principle, to obtain a dispersion relation for the truncated system of equations. We simply select from (16) the six equations corresponding to the 6 frequencies (18) ; compute their determinant; and equate it to zero. An alternative, and in some cases less onerous and hence more common, procedure, which may provide more insight into the physics, is to work directly with the 6 equations, eliminating dependent variables until we are left with only one or two components of E. Moreover, it is only in this way that we can make contact with the somewhat curious approach used in Ref. 3 and thus understand fully its limitations.

Before commencing this elimination of dependent variables, we interpolate a remark concerning the role of initial conditions and also introduce an assumption concerning the symmetry of the velocity distribution functions which, while not essential, has the advantage of simplifying the algebraic formalism. As regards the first point, in order to establish the connection between the Fourier-Laplace components considered here and the physical space-time fields E (x,t) we should, of course, include on the right hand side of (14) an inhomogeneous term representing the initial perturbation whose long time growth is described by the unstable or least damped roots of the dispersion equation. If, for example, the initial perturbation is in the ion distribution functions,

$$f_i(x, v, t=0) = \pi^{-1} F(v) \cos(k_0 x) \quad (19)$$

giving $f_i(k, v, t=0) = F(v) [\delta(k-k_0) + \delta(k+k_0)] \quad (20)$

then (14) would be replaced by

$$\rho_i = -(i k / 4\pi) (\chi_i E + r) \quad (21)$$

where

$$r = (4\pi n e k^{-2}) [\delta(k+k_0) + \delta(k-k_0)] \int dv F(v) (v - \omega/k)^{-1} \quad (22)$$

and n is the unperturbed density. The net result is that an inhomogeneous term, namely r , should be added to the left hand side of (16). We shall assume that the perturbation, $F(v)$, is symmetric, $F(v) = F(-v)$, and that this is also true of the unperturbed electron and ion velocity distribution functions, i.e., that there is no streaming motion either initially or in the perturbation. Then

$$r(k, \omega) = r(k, -\omega^*)^* \quad (23)$$

and similarly

$$\epsilon(k, \omega) = \epsilon(k, -\omega^*)^*, \quad \chi(k, \omega) = \chi(k, -\omega^*)^*, \quad \text{etc.} \quad (24)$$

It follows that when the inhomogeneous version of (16) is solved for the $E(k, \omega)$, these will have a similar property,

$$E(k, \omega) = E(k, -\omega^*)^* \quad (25)$$

This reflection symmetry proves useful since (16) inevitably mixes frequencies having positive real parts with those having negative real parts and (25) allows us to consider only the former, provided we admit as dependent variables both $E(\omega)$ and $E^*(\omega)$. (In absence of this assumed symmetry in the velocity distributions we would have, instead, the usual relations

$$E(k, \omega) = E(-k, -\omega^*)^*, \quad \text{etc.} \quad (26)$$

We restrict the discussion to symmetric distributions since we can then work with fixed $k > 0$ and need not bring in the $-k$ components. This simplifies the formalism but does not, of course, affect the results for threshold or growth rates).

We now return to an examination of the subset of (16) corresponding to the 6 frequencies (18). In dealing with such mode coupling equations it is customary either to eliminate the high frequency ($\omega \doteq \omega_k$) components, leaving a smaller set of equations involving only low frequency ($\omega \doteq \Omega$) components, or to do the converse. As will become apparent, it is significantly more convenient to work with the low frequency components, as was done in Ref. 1, but since Ref. 3 formulates the problem in terms of high frequency components we shall consider these as well.

We note that if ω_h and ω_l denote typical high and low frequencies, respectively, then we have

$$\chi(\omega_h) \doteq -1 \quad (27)$$

$$\chi(\omega_\ell) \doteq (k_0/k)^2 \gg 1 \quad (28)$$

If the unperturbed distribution functions are Maxwellian, then (27) and (28) can be obtained formally from the expression of χ in terms of the plasma dispersion function⁴, Z ,

$$\chi(k, \omega) = -(\omega_p/ka)^2 Z'(\omega/ka); \quad a = (2Te/m)^{1/2} \quad (29)$$

and from the small argument and asymptotic expressions

$$Z'(s) \doteq \begin{cases} -2 & |s| \ll 1 \\ s^{-2} & |s| \gg 1 \end{cases} \quad (30)$$

More generally, (28) is simply a manifestation of Debye shielding while (27) follows from the fact that the high frequencies are near the roots of ϵ so that

$$\epsilon = 1 + \chi_i + \chi \doteq 1 + (m/M) + \chi \doteq 0 \quad (31)$$

Since the right hand side of each of the equations (16) involves the difference of a $\chi(\omega_h)$ and a $\chi(\omega_\ell)$, and since we assume $k \ll k_D$, we can always neglect $\chi(\omega_h)$ compared to $\chi(\omega_\ell)$. That is, of the two terms $\chi(\omega)$ and $\chi(\omega \pm \omega_s)$ in any of the equations (16), only one need be retained (the former, if ω is a low frequency, the latter if ω is a high frequency)

and the one which is kept can be set equal to $(k_D/k)^2$. Consequently, in the following it suffices to simply write χ , sans argument, and to regard it as an abbreviation for $(k_D/k)^2$.

To eliminate the high frequency components from (16), let ω be a low frequency, with $\text{Re } \omega = \Omega > 0$. Then from (16) we have

$$\epsilon(\omega) E(\omega) = -i \chi \sum \lambda_s [E(\omega - \omega_s) + E(\omega + \omega_s)] \quad (32)$$

Using (16) again, with ω replaced by $\omega \pm \omega_s$ gives

$$\begin{aligned} \epsilon E(\omega \pm \omega_1) &= i \chi [\lambda_1 E(\omega) + \lambda_2 E(\omega \pm \Delta)] \\ \epsilon E(\omega \pm \omega_2) &= i \chi [\lambda_2 E(\omega) + \lambda_1 E(\omega \mp \Delta)] \end{aligned} \quad (33)$$

where $\epsilon E(\omega)$ is an abbreviation for $\epsilon(\omega) E(\omega)$, etc. Note that these equations involve both the inner sidebands, $\omega - \omega_1, \omega + \omega_2$, and also the outer sidebands, $\omega + \omega_1, \omega - \omega_2$. Substituting (33) into (32) we immediately obtain

$$\epsilon_L E(\omega) = M_-(\omega) E(\omega - \Delta) + M_+(\omega) E(\omega + \Delta) \quad (34)$$

where

$$\epsilon_L(\omega) = \epsilon(\omega) - \chi^2 \sum_s \lambda_s^2 [E^{-1}(\omega - \omega_s) + E^{-1}(\omega + \omega_s)] \quad (35)$$

plays the role of a nonlinear dielectric function and

$$M_{\pm}(\omega) = \chi^2 \lambda_1 \lambda_2 \left[\epsilon^{-1}(\omega \pm \omega_1) + \epsilon^{-1}(\omega \mp \omega_2) \right] \quad (36)$$

are mode coupling coefficients. Writing (34) with ω replaced by $\omega - \Delta$ and neglecting the non-resonant components $E(\omega + n\Delta)$, $n \neq -1, 0$, we obtain a pair of equations for the two coupled, resonant, low frequency components $E(\omega)$ and $E(\omega - \Delta)$:

$$\begin{aligned} \epsilon_L E(\omega) - M_-(\omega) E(\omega - \Delta) &= 0 \\ -M_+(\omega - \Delta) E(\omega) + \epsilon_L E(\omega - \Delta) &= 0 \end{aligned} \quad (37)$$

The vanishing of the determinant of these two equations then gives the dispersion equation for this system,

$$D_L(\omega) = \epsilon_L(\omega) \epsilon_L(\omega - \Delta) - M_-(\omega) M_+(\omega - \Delta) = 0 \quad (38)$$

Aside from differences of notation and a factor of $(1 + \chi)^2$ in each term, this equation is identical with Eq. (35) of Ref. 1. The terms $\epsilon^{-1}(\omega + \omega_1)$, $\epsilon^{-1}(\omega - \omega_2)$ arise, of course, from the outer sidebands, and while they will be small compared to those arising from the inner sidebands if the damping is weak, $\gamma \ll \Omega$, they will all be of comparable magnitude when $\gamma \geq \Omega$. This is an essential point, to which we shall return in Section V.

To obtain specific results from (38) requires explicit expressions for the

susceptibilities and either algebraic or numerical analysis of the resulting equation. This was done in Ref. 1 using simple fluid expressions with phenomenological damping terms for the susceptibilities and we shall present additional calculations of this character in Section IV. One can, of course, use kinetic expressions, as is done in Ref. 3 for χ_i , but that will affect the results of (38) only if the ion damping is significant, i.e., only if the ion acoustic resonance is appreciably broadened, in which case the essential approximation of neglecting low frequency components other than ω and $\omega - \Delta$, on which (38) relies, is no longer valid. Thus, the use of the kinetic expression for χ , while correct, is at best a useless and unwarranted complication. Even worse, it can, as in Ref. 3, lead to erroneous results: having gone to the trouble of introducing the kinetic χ_i , one may be tempted to explore the case $T_e/T_i \doteq 1$, where the argument of the Z' function is not large and a simple resonant approximation to χ_i is inadequate. However, since the basic truncation (neglect of components at $\omega + n\Delta$, $n \neq -1, 0$) which leads to (38) is unjustified, the results are of no use.

We now turn to the, unfortunately more complicated, analysis of (16) based on elimination of low frequency components. Assuming, then, that ω is a high frequency, $\omega \doteq \omega_k > 0$, we have from (16)

$$\epsilon E(\omega) = i \chi \sum \lambda_s E(\omega - \omega_s) \quad (39)$$

where we neglect non-resonant components having frequencies of order $2\omega_k$.

Applying (16) again, with ω replaced by $\omega - \omega_s$ we obtain

$$\epsilon E(\omega - \omega_1) = -i\chi \left\{ \lambda_1 [E(\omega) + E(\bar{\omega} - \Delta)] + \lambda_2 [E(\bar{\omega}) + E(\omega - \Delta)] \right\} \quad (40)$$

where $\bar{\omega} \equiv \omega - \omega_1 - \omega_2 \doteq -\omega_k$ (41)

and

$$\epsilon E(\omega - \omega_2) = -i\chi \left\{ \lambda_2 [E(\omega) + E(\bar{\omega} + \Delta)] + \lambda_1 [E(\bar{\omega}) + E(\omega + \Delta)] \right\} \quad (42)$$

Note that the frequency $\bar{\omega}$ has a simple physical significance: if ω is one of the inner sidebands, say at a in Fig. 1, then ω corresponds to the other sideband, at b. To eliminate the several new high frequency components, at $\bar{\omega}$, $\omega \pm \Delta$, $\bar{\omega} \pm \Delta$, which have entered, we must again use (16), evaluated at each of these new frequencies. If we assume that ω corresponds to an inner sideband, so that $|\omega - \omega_2| \doteq |\omega - \omega_1| \doteq \Omega$ and if we neglect non-resonant components near $2\omega_k$ and at $\omega - \omega_2 + \Delta \doteq \Omega + \Delta$, etc. then no new frequencies enter. Leaving the algebraic details to the appendix, we simply state here the resulting coupled equations for the two inner sideband components, $E(\omega)$ and $E(\bar{\omega})$:

$$\begin{aligned} [\epsilon(\omega) - F_1(\omega)] E(\omega) - F_2(\omega) E(\bar{\omega}) &= 0 \\ -F_2(\omega) E(\bar{\omega}) + [\epsilon(\omega) - F_3(\omega)] E(\bar{\omega}) &= 0 \end{aligned} \quad (43)$$

with

$$F_1(\omega) = \chi^2 \sum \lambda_s^2 \epsilon_s^{-1}(\omega)$$

$$F_2(\omega) = \lambda_1 \lambda_2 \chi^2 \sum \epsilon_s^{-1}(\omega)$$

$$F_3(\omega) = \chi^2 \sum \bar{\lambda}_s^2 \epsilon_s^{-1}(\omega)$$

$$\epsilon_1(\omega) = \epsilon(\omega - \omega_1) - \chi^2 \left[\lambda_1^2 \epsilon^{-1}(\bar{\omega} - \Delta) + \lambda_2^2 \epsilon^{-1}(\omega - \Delta) \right] \quad (44)$$

$$\epsilon_2(\omega) = \epsilon(\omega - \omega_2) - \chi^2 \left[\lambda_1^2 \epsilon^{-1}(\omega + \Delta) + \lambda_2^2 \epsilon^{-1}(\bar{\omega} + \Delta) \right]$$

$$\bar{\lambda}_1 = \lambda_2, \quad \bar{\lambda}_2 = \lambda_1$$

The greater simplicity of both the derivation and the results when we work with low frequencies is impressive. In particular, we note that the quantities $\epsilon_s(\omega)$ which appear in the denominators of the coefficients F_1 , F_2 and F_3 involve terms quadratic in the λ_s , with coefficients which will be large if the outer sidebands are resonant.

The vanishing of the determinant of the equations (43) provides a dispersion equation

$$\mathcal{D}_H(\omega) = \left[\epsilon(\omega) - F_1(\omega) \right] \left[\epsilon(\bar{\omega}) - F_3(\omega) \right] - F_2^2(\omega) = 0 \quad (45)$$

which must have the same roots as (38) since both (38) and (43) are obtained from the same initial set of six coupled equations. However, the actual dispersion relations, i.e., (38) with (35) and (36) substituted

into it, and (45) with the quantities (44) substituted into it, are so different in form that this coincidence of roots is by no means obvious.

III. Properties of the Coupled Modes

In this section we briefly consider some of the general properties of the two coupled modes described by (37) or (43). Of course, there are, in general a total of 6 coupled modes: two near the ion acoustic resonance and four (two inner sidebands and two outer sidebands) near the Bohm-Gross frequency. As we have seen, the simplest procedure is to eliminate the four high frequency modes, leaving us with the two coupled low frequency modes described by (37). Alternatively, motivated by a desire to understand the results of Ref. 3, we can eliminate the two outer sidebands and the two low frequency modes, leaving the two coupled inner sidebands, described by (43).

Although it may not be immediately evident, the roots of the low frequency dispersion equation (38) must be symmetrically located with respect to the frequency $\Delta/2$. This can be seen most clearly by defining

$$z = \omega - \Delta/2 \quad (46)$$

and

$$f(z) = D_L(\Delta/2 + z) \quad (47)$$

From the definitions (35) and (36) and from the reflection symmetry property (24) it follows that

$$f(z) = f^*(-z^*) \quad (48)$$

so that if z is a root of f , then so is $-z^*$. If $\text{Re}(z) \neq 0$, then there will be two distinct roots, with the same imaginary part but equal and opposite real parts, i.e., the complex roots must occur in pairs. In addition, there may be pure imaginary roots, since they satisfy $z = -z^*$. To a pair of complex roots correspond two low frequency waves, with frequencies lying symmetrically above and below $\Delta/2$; two inner sidebands, with frequencies lying symmetrically above and below the mean pump frequency,

$$\omega_0 = (\omega_1 + \omega_2)/2 \quad ; \quad (49)$$

and two outer sidebands. To an imaginary root in the z plane there corresponds a single low frequency wave, with the real part of its frequency at $\Delta/2$; a single high frequency wave, with the real part of its frequency at ω_0 ; and two outer sidebands.

We can expect that for $E_1 = E_2 = 0$, the roots of f , which must lie in the lower half plane if damping is included in ϵ , will in general have non-zero real parts and hence, as a consequence of (48), lie symmetrically about the imaginary z axis. As the pump amplitudes are increased, the trajectories of the roots can follow one of several patterns, depending

upon the values of the free parameters (pump frequencies, pump amplitudes, damping rates, etc.). One of the symmetrical pairs could move up to the real axis, in which case the spectrum at marginal stability would have two low frequency waves, two inner sidebands, and two outer sidebands. Alternatively, the two roots of a pair could move towards the negative imaginary axis; merge; and then separate again, with one moving down the imaginary axis and the other moving up until it reached the origin. In this case, the marginally stable spectrum would consist of a single low frequency wave, at $\Delta/2$; a single inner sideband, at ω_0 ; and two outer sidebands. Naturally, many other topological possibilities could occur, including the coalescent of two distinct roots at $z = 0$, i.e., a double root of f . We shall consider in detail only the second of the cases described above, where marginal stability occurs at $z = 0$, which we refer to as the case of coincident inner sidebands, since there is, in fact, only a single inner sideband frequency (and, correspondingly, a single low frequency) in this case. (The analysis of Ref. 3 and the specific results, albeit not the general analysis, of Ref. 1, are also limited to this case, although we are not aware of any formal proof that this necessarily corresponds to a lower threshold than the case of two distinct, marginally stable roots).

To obtain the physical fluctuation field $E(x,t)$ it is necessary to use the inhomogeneous form of (37) which results from including the initial value term r , defined by (19) in the mode coupling equations (16). These inhomogeneous equations have the form

$$\begin{aligned} \epsilon_L E(\omega) - M_-(\omega) E(\omega - \Delta) &= R(\omega) \\ - M_+(\omega - \Delta) E(\omega) + \epsilon_L E(\omega - \Delta) &= R(\omega - \Delta) \end{aligned} \quad (50)$$

where $R(\omega)$, whose exact form is of no concern here, is just a linear combination of r , evaluated at arguments $\omega, \omega \pm \omega_1$, and $\omega \pm \omega_2$, with coefficients involving the λ_s and $\epsilon(\omega \pm \omega_s)$. It is convenient to separate out explicitly the delta functions of k which occur in r , and hence in R , and to set

$$R = \pi \left[\delta(k - k_0) + \delta(k + k_0) \right] Q \quad (51)$$

We then have

$$E(k, \omega) = i\pi N(\omega) \left[\delta(k - k_0) + \delta(k + k_0) \right] / D_L(\omega) \quad (52)$$

where

$$iN = R(\omega) \epsilon_L(\omega - \Delta) + R(\omega - \Delta) M_-(\omega) \quad (53)$$

Assuming a marginally stable root of f at $z = 0$ and inverting the Fourier-Laplace transform gives a standing wave at $\Delta/2$:

$$\begin{aligned} E_L(x, t) &= \left[N(\Delta/2) / D'(\Delta/2) \right] \exp(-i\Delta t/2) \cos k_0 x + c.c. \\ &= A \cos(\Delta t/2 + \phi) \cos k_0 x \end{aligned} \quad (54)$$

where

$$2 N(\Delta/2) / D'(\Delta/2) = A e^{-i\phi} \quad (55)$$

The subscript L on E indicates that (54) describes only the low frequency portion of the field. An explicit expression for the high frequency part i.e., for the inner sideband at ω_0 and the outer sidebands at $\omega_0 + \Delta$ and $\omega_0 - \Delta$ can be found by using (52) and (33) to find the high frequency components of E (k, ω) and again inverting the Fourier-Laplace transform.

As is clear from (53) and (55), the phase ϕ of the standing wave depends, in an essential way, upon the initial conditions and can be anything from 0 to 2π . This is consistent with the fact that ϕ can have no physical significance, since it is the only wave in the system with frequency $\Delta/2$. Exactly the same remarks apply to the phase of the high frequency waves, for example the standing wave at ω_0 : its phase, which can likewise assume any value, depending on the initial conditions, can have no physical significance. Since there is only a single wave with frequency ω_0 .

IV. Threshold Calculations

In this section we obtain explicit results for thresholds from the general low frequency dispersion equation (38). As we have explained previously, our truncation of the infinite set (16) is justified only if the ion acoustic resonance is sharp, so it is sensible to use a resonance approximation. Thus, if ω is a low frequency, with $\text{Re } \omega \approx \Omega > 0$, we have

$$\begin{aligned} \epsilon(\omega) &= 1 + \chi - \omega_{pi}^2 (\omega + i\Gamma)^{-2} \\ &= \chi [1 - \Omega^2 (\omega + i\Gamma)^{-2}] \doteq (2\chi/\Omega)(\omega - \Omega + i\Gamma) \end{aligned} \quad (56)$$

If, on the other hand, ω is a high frequency, with $\omega \doteq \omega_k > 0$, then

$$\epsilon(\omega) = 1 - \omega_k^2 (\omega + i\gamma)^{-2} \doteq (2/\omega_p)(\omega - \omega_k + i\gamma) \quad (57)$$

We always assume the damping of the Langmuir waves to be small compared to the plasma frequency, $\gamma \ll \omega_p$, so that the resonance approximation (70) for high frequencies is always valid, whether γ is small or large compared to Ω .

Once we have introduced these resonance approximations, the dispersion equations (38) and (45) reduce to polynomials in ω . However, considerable algebraic complications remain because we have four parameters at our disposal, the amplitudes and the frequencies of the two pumps, and would like to choose these so as to minimize the threshold or maximize the growth rate of the instability.

To minimize the rather tedious algebra, we shall, as remarked in Section III, consider only the simplest case, where marginal stability occurs at the origin in the plane of $z = \omega - \Delta/2$. We shall further simplify the analysis by examining only the two particular cases of weak damping, $\gamma \ll \Omega$, and strong damping $\gamma \gg \Omega$.

A. Weak Damping $\gamma \ll \Omega$

Consider the high frequency form of the dispersion equation (45). As is clear from Fig. 1, not all of the high frequency sidebands can be resonant

in this case, but if the two inner sidebands are close (or, as we assume here, coincident), then we can choose both of them to lie within γ of the Bohm-Gross frequency, i.e., take

$$\delta \equiv \omega_0 - \omega_k \lesssim \gamma \quad (58)$$

(Of course, we could also choose the pump frequencies so that one of the outer sidebands was resonant, but the choice (58) is preferable since it results in two resonant sidebands). We can then neglect the outer sidebands, since their resonant denominators will be larger, by a factor of Ω/γ , than those for the inner sidebands.

In general, the pump parameters would be specified and (45) would be solved to find the roots of $D(\omega)$. Here we are specifying $\omega = \omega_0$ (i.e., postulating coincidence of the inner sidebands and marginal stability) in which case (45) gives a relation which must be satisfied by the ω_s and λ_s . Our goal is find what combination of these parameters, satisfying (53), minimizes the total pump power, or, equivalently, the dimensionless quantity

$$P = (E_1^2 + E_2^2)/4\pi nT = \Lambda_1^2 + \Lambda_2^2 = (f_1^2 + f_2^2) \Lambda^2 \quad (59)$$

Here, consistent with the notation in Ref. 1, we have defined

$$\Lambda_s^2 = \chi \lambda_s^2 = (k_D \lambda_s / k)^2 = E_s^2 / 4\pi nT \quad (60)$$

and also written

$$\Lambda_s = f_s \Lambda$$

where Λ is a convenient measure of the general pump power level and the f_s determine the ratio of the two pump amplitudes.

For the case of coincident inner sidebands, we have $\omega = -\bar{\omega}^*$ and it then follows from (44) that $F_1(\omega) = F_3^*(\omega)$ and that F_2 is real, while (24) gives $\epsilon(\bar{\omega}) = \epsilon^*(\omega)$. Thus, (45) can be written

$$|\epsilon(\omega) - F_1(\omega)|^2 - F_2^2(\omega) = 0 \quad (61)$$

where we have, so far, not made use of the resonance approximations (56) and (57) i.e., (61) is valid in general for the case of coincident inner sidebands. If we now introduce the resonance approximation only for the Langmuir waves, using (57) to set

$$\epsilon(\omega) = 2(\delta + i\gamma)/\omega_p \quad (62)$$

then (61) can be written as

$$(2\delta/\omega_p - \text{Re } F_1)^2 + (2\gamma/\omega_p - \text{Im } F_1)^2 = F_2^2 \quad (63)$$

It follows from the expressions (44) for the F_i that, independent of the precise form of χ_i (kinetic, fluid, or whatever) contained in ϵ , $F_1(\omega)$ and $F_3(\omega)$ depend only on the pump frequency separation, Δ , and not on the

displacement δ of the average pump frequency ω_0 from the Bohm-Gross frequency. The only dependence on δ is that which appears explicitly in the first term of (63).

To find the minimum threshold we should set $\omega = \omega_0$ in (63); solve for Λ^2 as a function of the parameters at our disposal, namely δ , Δ , and the f_s ; and then vary these parameters so as to minimize P . However, since the only dependence on δ is in the first term of (63), differentiation with respect to δ yields the result that the first term must vanish and hence that

$$2\gamma/\omega_p - \text{Im } F_1 = \pm F_2 \quad (64)$$

$$2\delta/\omega_p = \text{Re } F_1 \quad (65)$$

We now introduce the resonance approximation for the ion waves as well, i.e., (56), and, consistent with our assumption of weak damping, $\gamma \ll \Omega$, neglect the outer sideband contributions to F_1 , i.e., the terms involving $\epsilon^{-1}(\omega \pm \Delta)$ and $\epsilon^{-1}(\bar{\omega} \pm \Delta)$. (The argument for neglecting the outer sidebands is most easily made in terms of the low frequency form (38) of the dispersion equation: in the expression (35) for $\epsilon_L(\omega)$ the inner and outer sideband contributions appear additively and the latter will be smaller than the former by a factor of γ/Ω .) From (44) we then obtain

$$F_1(\omega) = \chi^2 \sum \lambda_s^2 \epsilon^{-1}(\omega - \omega_s) = (\Omega/2) [g(\Lambda_1^2 + \Lambda_2^2) + i\Gamma(\Lambda_1^2 - \Lambda_2^2)] (g^2 + \Gamma^2)^{-1}$$

$$F_2(\omega) = \chi^2 \lambda_1 \lambda_2 \sum \epsilon^{-1}(\omega - \omega_s) = \Omega \Lambda_1 \Lambda_2 g (g^2 + \Gamma^2)^{-1} \quad (66)$$

where we have, to simplify the notation, defined

$$g = (\omega_0 - \omega_2) - \Omega = (\omega_1 - \omega_0) - \Omega = \Delta/2 - \Omega \quad (67)$$

For $g = 0$, we have the special case where the pump separation Δ is exactly equal to twice the ion acoustic frequency, 2Ω , but in general we are free to choose Δ or g so as to minimize the threshold. Substituting (65) and (66) into (64), we obtain

$$2\gamma/\omega_p - \Omega \Gamma (f_1^2 - f_2^2) \Lambda^2 / 2(g^2 + \Gamma^2) = \pm f_1 f_2 g \Omega \Lambda^2 (g^2 + \Gamma^2)^{-1} \quad (68)$$

which can be immediately solved for Λ^2 :

$$\Lambda^2 = (4\gamma\Gamma/\omega_p\Omega)(u^2 + 1) [f_1^2 - f_2^2 + 2f_1 f_2 u]^{-1} \quad (69)$$

where

$$u = g/\Gamma = (\Delta/2 - \Omega)/\Gamma \quad (70)$$

We have chosen here the + sign on the right side of (68) since for $f_1 > f_2$ this gives a smaller value of Λ^2 than does the - sign, while for $f_1 < f_2$ this is the only choice consistent with $\Lambda^2 > 0$.

From (69) we can now read off all of the results for the weak damping, coincident sideband case. For example, setting $f_2 = 0$ we obtain the single pump result,

$$\Lambda^2 = (4\gamma\Gamma/\omega_p\Omega)(u^2 + 1) \quad (71)$$

which, for $u = 0$, takes on its minimum value,

$$\Lambda^2 = \Lambda_0^2 \equiv (4\gamma\Gamma/\omega_p\Omega) \quad (72)$$

the well known single pump weak damping result⁵. For equal pump amplitudes, $f_1 = f_2$, we have

$$\Lambda^2 = \Lambda_0^2(u^2 + 1)/2u \quad (73)$$

which is a minimum for $u = 1$, i.e., when the pump separation Δ is shifted away from $\Omega/2$ by $\Gamma/2$. The minimum value is again Λ_0^2 , but the total pump power is now $2\Lambda_0^2$. so there is no advantage in having two pumps. In general for any choice of the f_s , the total power, P , achieves its minimum value,

$$P_{MIN} = \Lambda_0^2 \left(1 + f_2^2/f_1^2 \right) \quad (74)$$

for $u = f_2/f_1$, and hence in the weak damping, degenerate mode case the two pump threshold always exceeds that for the single pump ($f_2 = 0$).

Of course, having thus determined the optimum u and Λ^2 values, we can use (65) to find the shift δ of the average pump frequency from the Bohm-Gross frequency:

$$2\delta/\omega_p = \text{Re } F_1 = \Omega u (f_1^2 + f_2^2) \Lambda^2 / 2\Gamma(u^2 + 1) \quad (75)$$

or $\delta = \gamma f_2 / f_1$

B. Strong Damping, $\gamma \gg \Omega$

In this case it is easiest to use the low frequency dispersion equation (38). Remembering that ω is now a low frequency, with $\text{Re } \omega \doteq \Omega > 0$, we have from (44)

$$\begin{aligned} \epsilon(\omega \pm \omega_s) &= 1 - \omega_k^2 (\omega \pm \omega_s + i\gamma)^2 \\ &= (2/\omega_p) [\delta_s \pm (\omega + i\gamma)] \end{aligned} \quad (76)$$

where the quantities

$$\delta_s \equiv \omega_s - \omega_k \quad (77)$$

which give the displacements of the respective pumps from the Bohm-Gross frequency are simply linear combinations of δ and Δ :

$$\Delta = \delta_1 - \delta_2, \quad \delta = (\delta_1 + \delta_2)/2 \quad (78)$$

From (76) and (56) we then obtain (for $\text{Re } \omega > 0$)

$$\epsilon_L(\omega) = \chi \left\{ (2/\Omega)(\omega - \Omega + i\Gamma) - \omega_p \sum \Lambda_s^2 \delta_s [\delta_s^2 - (\omega + i\gamma)^2]^{-1} \right\} \quad (79)$$

and

$$M_-(\omega) = M_+(\omega - \Delta) = \omega_p \chi \Lambda_1 \Lambda_2 \delta [\delta^2 - (z + i\gamma)^2]^{-1} \quad (80)$$

where z is defined by (56). If, as in the weak damping case, we consider marginal stability ($\text{Im } \omega = 0$) and degenerate roots, $\omega = \Delta - \omega = \Delta/2$, or $z = 0$, then $\epsilon_L(\omega - \Delta) = \epsilon_L(-\omega) = \epsilon_L^*(\omega)$, $M_-(\omega)$ is real, and (38) becomes

$$|\epsilon_L(\omega)|^2 = [M_-(\omega)]^2 \quad (81)$$

In the denominator of the second term on the right side of (79) we can set

$$[\delta_s^2 - (\omega + i\gamma)^2]^{-1} \doteq (\delta_s^2 + \gamma^2)^{-1} [1 + 2i\gamma\omega(\delta_s^2 + \gamma^2)^{-1}] \quad (82)$$

since $\omega = \Delta/2 \doteq \Omega \ll \gamma$. Then (81) gives

$$\begin{aligned} & [2g/\Omega - \omega_p \sum \Lambda_s \delta_s (\delta_s^2 + \gamma^2)^{-1}]^2 + [2\Gamma/\Omega - 2\gamma\omega\omega_p \sum \Lambda_s^2 (\delta_s^2 + \gamma^2)^{-2}]^2 \\ & = (\omega_k \Lambda_1 \Lambda_2 \delta)^2 (\delta^2 + \gamma^2)^{-2} \end{aligned} \quad (83)$$

From this we can easily recover the single pump threshold for strong damping. On setting $\Lambda_2 = 0$ it follows that each of the positive definite terms on the left side of (82) must be zero. The vanishing of the second

term gives

$$\Lambda_1^2 = \Gamma (\delta^2 + \gamma^2)^2 / \gamma \omega_p \Omega^2 \delta, \gamma \quad (84)$$

which, for $\delta = \gamma/3^{1/2}$, achieves its minimum value,

$$\Lambda_{MIN}^2 = (4/3^{3/2}) \Lambda_0^2 (\gamma/\Omega) = 0.77 (\gamma/\Omega) \Lambda_0^2 \quad (85)$$

Thus, as originally shown by Nishikawa⁵, this threshold exceeds the weak damping one by a factor of order $\gamma/\Omega \gg 1$. From the vanishing of the first term in (83) we find the value of the pump separation Δ corresponding to this minimum threshold:

$$g/\Omega = (\Delta/2 - \Omega)/\Omega = 2\gamma\Gamma/3\Omega^2 \quad (86)$$

The fractional shift of $\omega = \Delta/2$ away from Ω may be small or large, depending on the ratio of the two small quantities Ω/γ and Γ/Ω . However, the shift is always large compared to Γ :

$$(\omega - \Omega)/\Gamma = 2\gamma/3\Omega \gg 1 \quad (87)$$

To see whether two pumps can give a lower threshold, we consider the case of equal pump amplitudes, $f_1 = f_2$. We shall find that the optimum δ_1 and δ_2 are of order γ and hence large compared to $\delta_1 - \delta_2 = \Delta$, so we can approximate $\delta_1 \doteq \delta_2 \doteq \delta$, except in places where the difference $\delta_1 - \delta_2$ appears. Then (83), which is in any case simply a biquadratic in Λ , can be

written as

$$3y^2 - 8uy + 4(u^2 + 1) = 0 \quad (88)$$

where u is given by (70)

and

$$y = (\Lambda/\Lambda_0)^2 4\gamma\delta(\gamma^2 + \delta^2)^{-1} \quad (89)$$

Solving (88) for y and minimizing the result with respect to u , for fixed δ , gives a minimum value, $y_{\min} = 2$ for $u = 2$, or, from (89)

$$(\Lambda/\Lambda_0)^2 = (\gamma^2 + \delta^2)/2\gamma\delta \quad (90)$$

Minimizing (90) with respect to δ gives

$$\Lambda_{\min} = \Lambda_0 \quad (91)$$

for $\delta = \gamma$. Thus, with a pump separation Δ which is nearly equal to 2Ω ,

$$\Delta - 2\Omega = 2\Gamma \ll 2\Omega$$

we obtain a threshold for the strong damping case with two equal amplitude pumps which is the same as for a single pump in the weak damping case, namely $\Lambda^2 = \Lambda_0^2$, whereas the single pump threshold with strong damping, (85), is larger by the factor $0.77 \gamma/\Omega \gg 1$.

In summary, the results found in this section are in agreement with those of Ref. 1 regarding the pump power thresholds. As before, we find that in the weak damping case the threshold for two pumps is no less than that for a single pump. In fact, we have shown here that, at least in the case of coincident inner sidebands the threshold with two pumps is always higher than with a single pump. In the strong damping case we find, again in agreement with Ref. 1, that the two pump threshold can be lower, by a factor of order Ω/γ , than that for a single pump. By carrying out the algebraic analysis of the dispersion equation (81) somewhat more carefully than was done in Ref. 1, we find that the minimum threshold with two pumps does not occur for $\Delta = 2 \Omega$, as stated there, but for Δ shifted slightly away from 2Ω . Specifically, in the weak damping case

$$\Delta - 2 \Omega = 2 \Gamma f_2 / f_1 \quad (92)$$

while in the strong damping case with equal pump amplitudes

$$\Delta - 2 \Omega = \Gamma \quad (93)$$

In both cases, the fractional displacement of Δ from 2Ω is small, of order $\Gamma/\Omega \ll 1$, but it is not strictly zero.

V. Relation to the Calculations of Fejer, et al.

In Ref. 3, two different methods of calculating the threshold are presented. The most straightforward, which is only briefly sketched (in their Section 3), involves two coupled, low frequency equations which, aside from notational differences, appear to be equivalent to eqs. (37) of the present paper. The authors state that the vanishing of the determinant of their coupled equations yields eq. (35) of Ref. 1, which is, in turn, identical, save for a factor of $(1 + \chi)^2$, to eq. (38) of the present paper. Moreover, since they proceed to use resonance approximations for the dielectric functions which are equivalent to (56) and (57) of the present paper and consider the case of coincident, marginally stable low frequency perturbations, $\omega = \Delta - \omega = \Delta/2$ one would expect the results of that method of approach to coincide with those of Section IV and of Ref. 1.

Unfortunately, the authors of Ref. 3 drop from their coupled equations the terms corresponding to the outer sideband resonances, i.e., they implicitly, albeit never explicitly, limit themselves to the uninteresting case of weak damping. For that case, their results actually, as should be expected, disagree with those of Ref. 1 only as regards the optimum choice of the pump frequency separation Δ . As discussed at the end of Section IV, the minimum threshold corresponds to a value of Δ whose fractional difference from 2Ω is of order $\Gamma/\Omega \ll 1$, a shift which was neglected in the approximations used in Ref. 1. Of course, if one insists upon having Δ precisely equal to 2Ω , then one can indeed find nonsensical results, such as the

"infinite thresholds" which are emphasized in Ref. 3. (The origin of these becomes clear if we examine the weak damping result (69). As explained in Section IV, Δ or, equivalently, the quantity u defined in (70) must be considered as one of the free parameters, to be chosen in such a way as to minimize the total pump power P , and, as shown there, the optimum value of u is just f_2/f_1 . If one, instead, requires $u = 0$ for two equal amplitude pumps ($f_1 = f_2$), then (69) indeed gives an infinite value for Λ^2 . In fact, for any choice of f_1 and f_2 there is a value of u , namely $u = (f_2^2 - f_1^2) / 2 f_1 f_2$, which makes the right side of (69) infinite, but such considerations are clearly irrelevant to the problem of finding thresholds for the instability).

The other approach to the calculation of thresholds, described in considerable detail in Ref. 3, can, in the case of weak damping, with degenerate eigenvalues, lead to correct answers (although it is also applied in Ref. 3 to the case of strongly damped ion acoustic waves, $T_e/T_i \doteq 1$, where it does not give valid results). The derivation given in Ref. 3 is a heuristic one, in which a single high frequency sideband, precisely at $\omega_0 = (\omega_1 + \omega_2) / 2$, is assumed. (Actually, two travelling waves at this frequency are postulated, but since they turn out to have equal amplitudes, this is equivalent to a single standing wave). A physical argument involving ponderomotive forces and energy conservation is then used to obtain a dispersion equation which turns out to be equivalent to (78).

To relate that work to the analysis of the present paper, it is easiest to consider the coupled high frequency equations (43). If we specialize

to the case of coincident sidebands, $\omega = -\omega^*$, then the first of eqs. (43) becomes

$$[E(\omega) - F_1(\omega)] E(\omega) - F_2(\omega) E^*(\omega) = 0 \quad (94)$$

while the second equation is simply the complex conjugate of the first. It is then reasonable to expect that all the information about the system is contained in (94). In fact, if we set

$$E(\omega_0) = |E| e^{i\theta} \quad (95)$$

multiply (94) by E^* ; and take the real and imaginary parts of the resulting equation, we obtain, since F_2 is real in this case,

$$\text{Re } E = \text{Re } F_1 + F_2 \cos 2\theta \quad (96)$$

$$\text{Im } E = \text{Im } F_1 + F_2 \sin 2\theta \quad (97)$$

The angle θ which appears in these equations is unknown, but we can eliminate it. Since $\sin^2 2\theta + \cos^2 2\theta = 1$ we have

$$F_2^2 = (\text{Re } E - \text{Re } F_1)^2 + (\text{Im } E - \text{Im } F_1)^2 \quad (98)$$

an equation which is, not surprisingly, identical with the high frequency dispersion equation (45) in the special case of coincident inner sidebands.

In Ref. 3, however, (97) is used in a quite different way. In principle, we could solve (97) for Λ^2 and then minimize the total pump power P , defined by (67), but since we would have to minimize with respect to θ , as well as the free parameters δ , Δ , and the f_s , this approach appears to be no simpler than the conventional method of calculation based on the dispersion equation (98) in which θ does not appear. Of course, if we could, by inspection, choose θ so as to minimize P (or even to minimize Λ for fixed f_s), then this procedure might offer some computational advantages. In general, this does not seem possible, due to the somewhat complicated dependence of the F_i on the λ_s , as given by (44), with the λ_s appearing in the quantities ϵ_s . However, in the weak damping case the F_i are simply linear functions of the λ_s , since we can, as noted in Section IV, neglect the outer sideband terms, $\epsilon^{-1}(\omega \pm \Delta)$ and $\epsilon^{-1}(\omega \pm \Delta)$ which account for the more complicated dependence of the F_i on the λ_s . Then $\epsilon_s(\omega) = \epsilon(\omega - \omega_s)$ and with the resonant approximation (57) or (62) for the high frequency dielectric function (97) becomes

$$2\gamma/\omega_p - \text{Im} F_1 = F_2 \sin 2\theta \quad (99)$$

Save for notational differences, this is precisely the dispersion equation (7) of Ref. 3, on which the results cited in that paper are based. Since $\text{Im}F_1$ and F_2 are linear functions of Λ^2 for fixed f_s , one can argue, as in Ref. 3, that the minimum Λ^2 will be obtained if $\sin 2\theta$ is chosen to be either 1 or -1, depending on the signs of F_2 and $\text{Im}F_1$. Indeed, if we introduce the resonance approximation (56) for the low frequency dielectric function, then (99) just reduces to (68), with $\sin 2\theta$ replaced

by the \pm sign of the latter equation.

In summary, the heuristic approach of Ref. 3 does lead to a threshold equation, namely (99), which is correct in the special case of weak damping and coincident inner sidebands. Of course, since (99) is only a relation for determining the marginal stability threshold, it cannot be used to find growth rates, as one can do with the conventional dispersion equations (38) or (45). Moreover, this approach appears to be limited to the uninteresting case of weak damping: although (91) and, consequently, (99) are valid in general for the case of coincident inner sidebands, there is no simple way to determine the value of θ which will minimize Λ^2 for fixed f_s when the complete expressions (44) for the F_i are used in (99). The restriction to weak damping and coincident inner sidebands is, of course, built into the derivation in Ref. 3, which assumes at the outset only a single sideband frequency, ω_o , without even considering the possibility of distinct inner sidebands or the existence of outer sidebands.

The phase θ which appears in (95) through (99) is taken, in Ref. 3, to be the physical phase of the high frequency sideband field. If true, this would be quite surprising, since the fact that minimum threshold corresponds to $\sin 2\theta = 1$ would then imply that only those fluctuations having this phase would reach marginal stability. In fact, this point of view is adopted in Ref. 3 and the authors attach some importance to what they call a "coherent instability". However, as is well known from linear stability theory, the conditions for instability or marginal stability are entirely

determined by dispersion equations, like (38) or (45), which are completely independent of the phase of the fluctuations involved. If the dispersion equation has unstable roots (in the upper half ω plane), then essentially any initial fluctuation will grow. If it has real roots, then the fluctuations will oscillate with constant amplitude, again independent of phase. As the analysis of Section III clearly shows, the actual phase of the growing or marginally stable waves depends in an essential way upon the initial conditions for the perturbation, and can certainly not be determined by the homogeneous form of the coupled mode equations, such as (104). Moreover, as explained in the final paragraph of Section III, the phase ϕ of the low frequency wave, and, equally, the phase of the high frequency sideband, a) should not have any physical significance and b) will depend in an essential way upon the initial conditions. It thus does not seem that either can have any simple relation to the angle θ , which must, we believe, be regarded as simply a mathematical artifact of the special technique used in Ref. 3 to obtain (68) from (97), rather than from the dispersion equation (61) in the usual way, as described in Section IV. (Of course, these remarks concerning the importance of the phase of the excited wave do not apply to the oscillating two stream instability (OTSI), which is mentioned in Ref. 3 but actually has no direct pertinence to the case analyzed there and here. For the OTSI, the parametrically excited wave can have, at threshold, the same frequency as the pump, and hence their relative phase can be of physical significance).

Finally, we reiterate that even under circumstances where (99) is correct (weak damping, coincident inner sidebands) it can only be applied to the case where the ion acoustic resonance is fairly sharp, $\Gamma \ll \Omega$. In the collisionless regime, where Γ arises from Landau damping, this requires that T_e/T_i be large. To use (99) with the kinetic expression for the ion susceptibility and with T_e/T_i equal to 1 or 2, as done in Ref. 3, is clearly unjustified and will lead to invalid results: when $\Gamma \geq \Omega$ there is no reason to neglect the other low frequency components, e.g. at $\omega + \Delta \doteq 3\Omega$, etc., so we can not truncate the set (16) and would have to analyze a much larger subset of the coupled modes than the six considered here.

To summarize this section, we have shown that the work of Ref. 3 is limited or incorrect with regard to the following points:

1. Their equation for the threshold pump power which results from their heuristic derivation is correct only for the weak damping case, with coincident inner sidebands, and a consistent analysis of that equation shows that the two pump excitation is then of no interest, since it always leads to a larger threshold than single pump excitation.
2. Their alternate derivation, using the low frequency dispersion equation, is correct in principle, but since the terms corresponding to the outer sidebands are dropped, the results which they obtain from it are, again, valid only in the weak damping case.
3. Insofar as the weak damping, coincident inner sideband case is

concerned, it is clear that the two methods of Ref. 3 should not only yield the same results but that these should coincide with those of Ref. 1 and of Section IV of the present paper for this particular case. As shown in Section IV, the optimum choice for the pump separation Δ differs from 2Ω by a small term of order $\Gamma \ll \Omega$. The fact that this was neglected in Ref. 1 seems a rather minor point, considerably inflated in Ref. 3 by a discussion of "infinite thresholds", but, as shown above, these are simply a consequence of rigidly requiring $\Delta = 2\Omega$ rather than allowing Δ to be chosen so as to minimize the pump power.

4. The use of the threshold equation in Ref. 3 for $T_e/T_i \doteq 1$, where $\Gamma \geq \Omega$, is totally unjustified and the results obtained therefrom are not valid, since there is then no basis for the truncation of the infinite set of coupled modes.
5. The claim that there is a preferred phase for the marginally stable waves is both unphysical and unfounded.

VI. Conclusions and Discussion of Results

Using a simplified version of the analysis of two pump excitation presented in Ref. 1, we have confirmed the principal results of that paper concerning thresholds when the pump frequency separation $\Delta = \omega_1 - \omega_2$ is

approximately twice the ion acoustic frequency: for weak damping (Langmuir wave damping rate γ small compared to ion acoustic frequency Ω) the second pump is of no help, the threshold always being greater than with a single pump; for strong damping ($\gamma \gg \Omega$) the two pump threshold with equal pump amplitudes, is less than that for a single pump by a factor of order of Ω/γ . The optimum choice of Δ is found to be not exactly 2Ω , as stated in Ref. 1, but rather to be shifted from that by a small term of order of the ion acoustic wave damping Γ . Since the reduction of the infinite set of coupled mode equations to a subset of 6 modes is justified only when $\Gamma \ll \Omega$, this shift in Δ will be unimportant whenever the assumptions underlying the entire analysis are satisfied. As in Ref. 1, these specific results pertain to the case where marginal instability of the low frequency waves occurs at $\Delta/2$, the associated high frequency sidebands being at $\omega_0 = (\omega_1 + \omega_2)/2$ and $\omega_0 \pm \Delta$. We give below some physical arguments suggesting that this situation probably corresponds to at least as low a threshold as the one where marginal stability occurs with two distinct low frequency waves, located symmetrically about the frequency $\Delta/2$, but, the extensive exploration of parameter space needed to provide a formal proof has not been undertaken.

Using the known properties of the usual single pump excitation⁵ we can present a simple physical argument which indicates why two pumps are advantageous only in the strong damping case. For a single pump with weak damping, the minimum threshold is achieved when the pump frequency, ω_1 , is above the Bohm-Gross frequency ω_k by approximately Ω . Thus, when

the low frequency wave at ω is resonant, $\omega \doteq \Omega$, the lower sideband, at $\omega_1 - \omega \doteq \omega_k$, will also be resonant. The upper sideband, at $\omega_1 + \omega$, is separated from ω_k by $2\omega \doteq 2\Omega \gg \gamma$, so it will be far off resonance. Of course, all three waves, the low frequency one and both sidebands grow at the same rate (if we are slightly above threshold or oscillate, undamped, just at threshold) but the amplitude of the upper sideband, as given by linear theory, will be smaller than that of the lower sideband by a factor of order γ/Ω . (This can be seen, for example, from (33) which shows that the amplitudes of the sidebands will be inversely proportional to the corresponding values of ϵ , which is of order γ for the lower, resonant sideband and of order Ω for the upper, non-resonant sideband). Thus, within the linear theory, the upper sideband is relatively insignificant. (Naturally, the actual saturation amplitudes will be determined by non-linear effects). If we now add a second pump, at $\omega_2 < \omega_1$, then its lower sideband will be far off resonance if we continue to keep the lower sideband of the first pump resonant. The upper sideband of the second pump can coincide with the lower sideband of the first pump, and hence also be resonant, but, as we have seen, it is unimportant in the decay of the second pump and hence not likely to have much effect. In fact, it probably does not matter very much whether or not the two inner sidebands coincide, so that the threshold we have found should be approximately correct also for the case of non-coincident inner sidebands.

If the damping is strong, then in single pump excitation the amplitudes of the two sidebands are comparable, since the optimum pump frequency is shifted away from ω_k by something of order γ and the same will therefore

be true of both sidebands, which are separated from the pump by a frequency of order $\Delta/2 \approx \Omega \ll \gamma$. It is then plausible that adding a second pump, at such a frequency that its upper sideband is near or even equal to the lower sideband of the first pump, could result in an enhancement of the instability, particularly if the inner sidebands have the same frequency.

An examination of the analysis given in Ref. 3 shows that of the two approaches used there, one is inherently restricted to the weak damping case, since the outer sidebands are ignored. The other is in principle equivalent to the dispersion equations used here and in Ref. 1, but in applying it the outer sideband terms are neglected in Ref. 3 so that, again, only the weak damping case is analyzed. Within the confines of the weak damping case, the results of Ref. 3 for large T_e/T_i are correct and differ from those of Ref. 1 only with respect to the exact value of the optimum Δ , which, as noted above, should be shifted from 2Ω by a term of order Γ , an effect not taken into account in Ref. 1. Ref. 3 also gives results for the case of strong Landau damping of the ion acoustic waves, i.e., for T_e/T_i of order 1, but these are invalid, since for $\Gamma \gtrsim \Omega$, the truncation of the infinite set of coupled modes on which the work of the present paper, as well as that of Refs. 1 and 3 is based, is no longer justified. Remarks in Ref. 3 concerning the determination of the phase of the marginally stable wave and its physical importance appear to be without foundation.

More detailed experimental tests of the predicted lowering of the threshold consequent on the use of two pumps appear quite feasible using large volume

laboratory plasmas, for example a large "Mackenzie bucket" (in which the ionizing electrons of a d.c. discharge are confined by rows of permanent magnet dipoles on the surface of the vacuum vessel). From the parameter values for a typical xenon discharge shown in Table 1, we see that the pump powers required are only of order of tens of watts, even for a plasma with dimensions of the order of 1m. Moreover, in the low frequency portion of the ion acoustic spectrum, say, $k/k_D \doteq .004$, (corresponding to a wavelength of 3 cm and an ion acoustic frequency of order 60 kHz) we have strong damping, $\gamma/\Omega \doteq 10$, while in the high frequency part of the spectrum, say $k/k_D \doteq 0.4$ ($\lambda \doteq 3$ mm, $\Omega/2\pi \doteq 6$ MHz) we have weak damping, $\gamma/\Omega \doteq 0.1$. Thus, one could pass from the weak damping case to the strong damping one by simply varying the pump frequency separation from 12 MHz to 120 kHz. The threshold should rise by a factor of 10 with a single pump but should remain approximately constant for two equal amplitude pumps. (A variation of this sort was not possible in the experiment of Akiyama, et al² since the use of a resonant cavity required that the separation of pump frequencies be less than the resonance width of the cavity.)

It may also be possible to observe double resonance excitation in the F layer of the ionosphere using existing high power antennae. In recent experiments⁶ two pumps with a frequency separation of order Ω were used and for $\Delta = 2\Omega$ much stronger excitation of sidebands was observed than with a single pump of comparable power.

Numerical studies, now in progress, will be required to settle two questions left unanswered here :

- 1) In the strong damping case, can values of $E_1/E_2 \neq 1$ give smaller thresholds

n_e	$4 \times 10^{11} \text{ cm}^{-3}$
P_0	10^{-6} Torr
T_e	2 ev.
T_i	0.2 ev.
ω_{pi}	10^8 sec^{-1}
γ/ω_{pe}	10^{-4}
Γ/Ω (Landau damping)	10^{-3}
k_D	500 cm^{-1}
γ/Ω	$0.04 k_D/k$
Λ_o^2	4×10^{-7}
$E_o = (4\pi n_e T_e)^{1/2} \Lambda_o^2$	0.6 v/cm
$P_o = cE_o^2/2\pi$	2 mw/cm^2

Table 1. Typical parameter values for a Xenon discharge plasma in a multidipole plasma configuration.

than the value, $\Lambda_{\min} = \Lambda_0$, which we have found for $E_1/E_2 = 1$?

2) Are there parameter values for which marginal stability with two distinct low frequencies occurs at a lower threshold than when the frequencies coincide ?

More valuable, but also more difficult, would be the inclusion of additional sidebands, as observed in the ionospheric experiments², and a calculation of the nonlinear saturation effects.

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Appendix

Detailed Derivation of the High Frequency Coupled Mode Equations

We give here the algebraic details leading to (43) and (44). To eliminate the high frequency components at $\omega \pm \Delta$ and $\bar{\omega} \pm \Delta$ which occur in (40) and (42) we use (16) evaluated at these frequencies, obtaining

$$\epsilon E(\omega - \Delta) = i\chi\lambda_2 E(\omega - \omega_1) \quad (A1)$$

$$\epsilon E(\bar{\omega} - \Delta) = i\chi\lambda_1 E(\bar{\omega} - \omega_1) \quad (A2)$$

$$\epsilon E(\omega + \Delta) = i\chi\lambda_1 E(\omega - \omega_2) \quad (A3)$$

$$\epsilon E(\bar{\omega} + \Delta) = i\chi\lambda_2 E(\bar{\omega} - \omega_2) \quad (A4)$$

where we have neglected non-resonant components near $2\omega_k$ and near $(\Omega + \Delta)$.

Substituting (A1) and (A2) into (40),

$$\epsilon_1(\omega) E(\omega - \omega_1) = -i\chi [\lambda_1 E(\omega) + \lambda_2 E(\bar{\omega})] \quad (A5)$$

where $\epsilon_1(\omega)$ is given by (44) and similarly, substituting (A3) and (A4) into (42), gives

$$\epsilon_2(\omega) E(\omega - \omega_2) = -i\chi [\lambda_2 E(\omega) + \lambda_1 E(\bar{\omega})] \quad (A6)$$

where $\epsilon_2(\omega)$ is given by (44). Finally, substituting (A5) and (A6) into (39) gives

$$\epsilon E(\omega) = F_1(\omega) E(\omega) + F_2(\omega) E(\bar{\omega}) \quad (\text{A7})$$

To obtain the corresponding equation for $E(\bar{\omega})$, we evaluate (16) at $\bar{\omega}$, obtaining

$$\epsilon E(\bar{\omega}) = i \chi \sum \bar{\lambda}_s E(\omega - \omega_s) \quad (\text{A8})$$

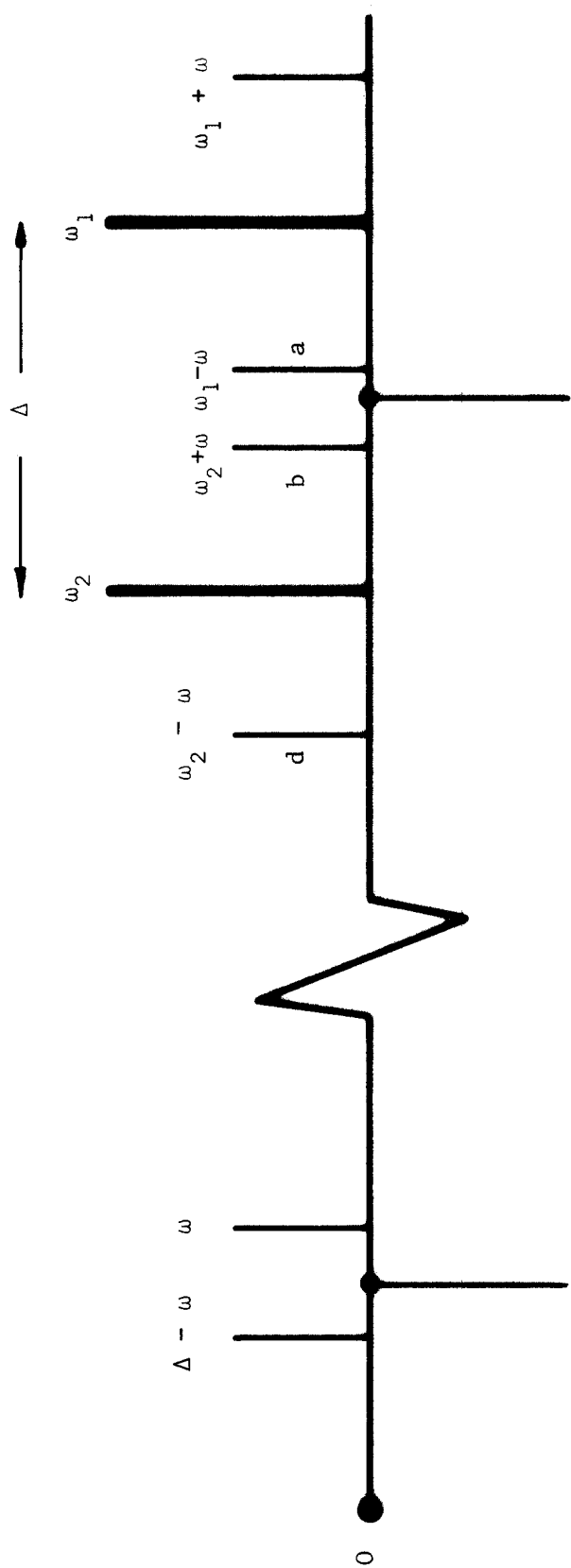
and then substitute into this (A5) and (A6) to give

$$\epsilon E(\bar{\omega}) = F_3(\omega) E(\bar{\omega}) + F_2(\omega) E(\omega) \quad (\text{A9})$$

where F_3 is given by (44). The set (A7) and (A9) give equations (43).

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$$\Omega = kC_s(1+k^2/k_D^2)^{-\frac{1}{2}}$$

$$\omega_k = \omega_p(1+3k^2/k_D^2)^{\frac{1}{2}}$$

Fig. 1. Schematic representation of the high and low frequency spectra associated with two high frequency pumps at the frequencies ω_1 and ω_2 with a separation $\Delta = \omega_1 - \omega_2 = 2\Omega$.