FEBRUARY 1978 LRP 137/78

ASYMPTOTIC MATCHING

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Abstract

The matching of two asymptotic expansions, valid in different regions of parameter space, is examined. This type of matching is necessary when treating boundary layer, or certain other multiple scale phenomena. The matching region is represented as a double asymptotic series, and matching is achieved when two such double series are equivalent, except for rearrangement of terms. It is shown that this matching principle is valid, useful, pictorial, and intuitive.

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I. Introduction

The general ideas of singular perturbation theory have considerable antiquity, yet the detailed theory of their validity and application is still a subject of discussion. This is particularly true of the theory of matched asymptotic expansions, whose domain of utility arises when an asymptotic expansion in one parameter is not uniformly valid over the range of the other parameters. That is, one might wish to consider the behavior of $f(x,\epsilon)$ for small ϵ, and find separate asymptotic expansions valid for $x = O(1)$, and $x = O(\epsilon)$. The problem comes in identifying these two asymptotic expansions as a representation of a single function. This is particularly important when the expansions are known only as the approximate solutions of a differential equation. Then information relating to the boundary conditions must be passed from one expansion to another.

The important point of the theory is that the domain of validity of these two expansions can overlap in some sense. That is, it is useful to know that, for every value of x, either or both asymptotic expansions are valid, and that at the extremes of x for which each expansion is valid, both are in fact valid. When two expansions can be matched together this way one can safely infer that they are different approximations for a single function.

This overlap of the two expansions must occur when x, in the example above, is somewhat smaller than $O(1)$ and somewhat larger than $O(\epsilon)$. Thus intermediate limits $x = O(x^{\eta})$, $0 < \eta < 1$, play a role in the understanding.

This paper is an approach to these intermediate limits alternate to that developed by Kaplun and his coworkers. Excellent reviews of that approach have been given in Kaplun (1967) and Lagerstrom and Casten (1972). The regime of intermediate limits is treated here as a double series that encompasses all the intermediate limits. This concept of a double series replaces Kaplun's concept of a domain of overlap, where the difference between the two expansions vanishes to some order in ϵ.

Viewing the intermediate expansions as a double series has the advantage of being clear and pictorial. It bypasses the problem of finding the maximal domain of overlap. It is not expected that the present approach will yield different results than that of other valid approaches. In particular, it is often stated that Fraenkel (1969) has demonstrated counterexamples for which double series methods do not work. His objections do not apply to the methods given in this paper.

The basic principles of the viewpoint of this paper were established by Kruskal (1962). The aim here is only to expand and disseminate Kruskal's ideas.

A trivial model of overlap is developed in Section II to clearly illustrate the basic methods. Natural modifications that occur when treating more general cases are discussed in Section III. A non trivial example is given in Section IV, and a concise and sometimes convenient method of handling the double series is used.

II. Introductory Example

We first consider a simple equation, as an example. It is almost trivial, but serves to establish principles and notation. The chosen equation is

$$\epsilon \frac{dI}{dt} + I = \exp(-t), \tag{1}$$

with boundary condition

$$I(0) = 0. \tag{2}$$

For small ϵ there is an initial fast transient, followed, for larger times t, by a slower response to the driving inhomogeneous term. Thus it has the form of a typical boundary layer problem.

We first consider the outer region, the limit of small ϵ with t finite. For this simple problem it is safe to assume that I can be expanded in powers of ϵ:

$$I = \sum_{n=0}^{\infty} \epsilon^n I_n(t). \tag{3}$$

That is, we assume that I has a Taylor series, considered as a function of ϵ, around $\epsilon = 0$. Inserting this in Eq. (1), and equating coefficients of powers of ϵ to zero since the equations are assumed to be valid for a range of ϵ, we find,

$$\begin{aligned} I_0(t) &= \exp(-t), \\ I_n(t) &= -\frac{d}{dt} I_{n-1}(t), \qquad n \geq 1. \end{aligned} \tag{4}$$

Hence

$$I_n(t) = \exp(-t), \qquad n \geq 0. \tag{5}$$

This cannot be the whole solution because we are unable to apply the boundary condition at t = 0. Indeed, the prescribed boundary condition, Eq. (2), is not satisfied. There is an initial transient whose time scale depends on ϵ, and vanishes as it vanishes. To treat this we consider a new variable,

$$\tau = t/\epsilon. \tag{6}$$

We then consider the limit ϵ going to zero with τ held fixed. Substituting Eq. (6) into Eq. (1), we find

$$\frac{dI^*}{d\tau} + I^* = \exp(-\epsilon\tau), \tag{7}$$

where

$$I^*(\tau) = I(t). \tag{8}$$

Again, we can assume that I^* has a regular Taylor series in ϵ,

$$I^* = \sum_{n=0}^{\infty} \epsilon^n I_n^*(\tau). \tag{9}$$

We insert this into Eq. (7), and equate coefficients of powers of ϵ to zero. The exponential must also be expanded so that the coefficients are independent of ϵ. This yields

$$\frac{d}{d\tau} I_n^* + I_n^* = \frac{1}{n!}(-\tau)^n \quad , \quad n \geq 0. \tag{10}$$

The solutions of these equations, with the prescribed boundary conditions, are

$$\begin{aligned} I_0^* &= 1 - \exp(-\tau) \\ I_n^* &= \frac{1}{n!}(-\tau)^n + I_{n-1}^* \\ &= \sum_{p=0}^{n} \frac{1}{p!}(-\tau)^p - \exp(-\tau), \end{aligned} \tag{11}$$

by induction and summation.

We now have two separate solutions, one describing the initial transient response, the other describing the effect of a more slowly changing inhomogeneous term. The theory of asymptotic matching says that these two solutions can agree somewhere, so that together they can describe the entire solution. A more precise statement will be left to the next section.

In the present example the region of overlap, where the two solutions can agree, is not far after the end of the transient response. Thus we wish to compare the two solutions in the region where t is small and τ is large. When t is small, it is natural to expand each term of Eq. (3) in an asymptotic series in t, with the aid of Eq. (5), yielding an expression for I as a double power series in ϵ and t. This is shown in Fig. 1.

Similarly, we wish to expand each term in Eq. (9) in an asymptotic series in $1/\tau$. Note that the exponential terms in Eq. (11) are smaller, when τ is large, than any power of $1/\tau$, and thus do not enter. This operation also yields a double series, shown in Fig. 2.

These two double series, Figures 1 and 2, match term by term to every order in ϵ, t, and $1/\tau$. In particular, the top line of Fig. 1 is equal to the upper, left trending diagonal of Fig. 2 term by term, upon use of Eq. (6). Similarly, the second line of Fig. 1 is equal to the second diagonal of Fig. 2, etc.

Using the results of this section as a concrete example, the definition and meaning of asymptotic matching are discussed in the next section.

III. General Considerations

This section explores the concept of asymptotic matching.

Asymptotic matching will be defined as the term by term matching of two doubly infinite series, to all orders, in both expansion parameters as in the example of Sec. II. An asymptotic double series is formed from two successive asymptotic expansions. That is, in the previous example, it was formed from an asymptotic expansion in ϵ followed by asymptotic expansions of each term in powers of t or $1/\tau$. Of course, to demonstrate matching the two double series must be expressed in terms of a common variable. When they are matched, they then differ only in the ordering of the terms.

The example of the previous section was a special case in a number of respects. In the next few paragraphs we will consider some straightforward generalizations of that example, and their consequences for asymptotic matching. The most obvious simplification, the fact that Eq. (1) has an exact analytic solution, is irrelevant, since the methods have a more general applicability.

The most important specialization inherent in Eq. (1) is that the expansion of the I_n^* in powers of $1/\tau$ is truncated, according to Eq. (11). In more typical examples, such as those given in Coppi et al. (1966), this expansion is an infinite nonconverging series. Then matching demands that there be coresponding terms proportional to $(\epsilon/t)^n$ in the series analogous to that in Fig. 1. Thus in general the two double series will be nonvanishing in the hatched region of Fig. 3.

The borders of this region are defined by the condition that the leading terms can be normalized to be constant, $0(\epsilon^o)$, in the limit of small ϵ, in both the inner and outer regions.

Another simplification inherent in Eq. (1) is that it can be expanded as a regular Taylor series in ϵ in both regions. In fact, there is no difficulty in generalizing this expansion to include any sequence of gauge functions, as discussed in all the standard texts. It should be noted that finite subsequences of gauge functions, often powers of logarithms, can often most conveniently be smuggled along as constants, as discussed by Van Dyke (1964).

To see more clearly why asymptotic matching must succeed, consider an alternate form of Eq. (1),

$$I - 1 = \left[\exp(-t) - 1\right] - \epsilon \frac{dI}{dt} \; . \tag{12}$$

The two terms on the right are each small when ϵ and t are small. Thus, in this form, the equation can be solved recursively. The right hand side can be evaluated accurately to a given order in t and ϵ, using a lower order approximation for I , and thus it determines a more accurate value for I on the left. The statement that the two series, Figs. 1 and 2, are equivalent asymptotically follows from the fact that the recursively generated solution is independent of the relative magnitudes of the two terms on the right. Thus any two series that can be generated from a common recursive form should be equivalent.

Furthermore, the terms in this series can also be generated by a sequence of intermediate limits applied to Eq. (12). That is, any limit

$$\epsilon \to 0, \; \tau_g \equiv t/\epsilon^{g} \text{ fixed}, \; 0 < g < 1, \tag{13}$$

will also generate a series equivalent to Figs. 1 or 2, but each such series will be differently ordered. Each will be an asymptotic sequence, since the double series is generated by a sequence of asymptotic procedures. It follows that the double series contains all intermediate limits.

While the series generated by the limits $\epsilon \to 0$, τ_q fixed, for each q, $0 < q < 1$, are all asymptotic, these series are not asymptotic as $q \to 0$ or $q \to 1$. Here successive terms are all the same order of magnitude, along the top line of one of the double series. These limits thus define the range of validity of the matching region.

It is important to note that, while each sequence generated by an intermediate limit, $0 < q < 1$, is asymptotic, there is no partial sequence whose remainder vanishes uniformly for all q. Thus a sequence accurate to a given order over a range on q must contain an increasing number of terms to maintain the same accuracy when the range of intermediate limits is increased. An exception occurs when there is truncation of the series arising from the outer limit of the inner region, as in Fig. 2. Then it is possible to obtain finite sequences with higher order remainders for a range of q including $q = 1$. This has been done in Cole (1968) Sec. 2.2 and Lagerstrom and Casten (1972) Sec. 4.15.

This point constitutes the chief difference between the point of view of these authors and that of this paper. Superficially there is a good deal of similarity between finding sequences that are uniformly asymptotic over a range of q, and finding sequences that are asymptotic for each q. However, these points of view differ significantly.

It is occasionally stated that Fraenkel (1969) has shown that the present method does not work in general. However, this scheme differs from Fraenkel's in that he considers matching only the coefficients of the double series, rather than the series themselves. This simple difference becomes important when treating logarithms. Then the double series match, but not term by term since $\ln^n \epsilon t = (\ln \epsilon + \ln t)^n$ can contribute to several powers of $\ln t$. Then it is important that the two series should be expressed in a common variable before comparison. When this is done Fraenkel's objections do not apply.

The present method of matching is, of course, essentially equivalent to that proposed by Van Dyke (1964).

In the rest of this section we make a few more comments for completeness.

In most cases of interest, it is not possible to match the two double series without fixing a sequence of otherwise undetermined coefficients. This fixing of coefficients near the end of the range of validity of an asymptotic expansion has the role of determining boundary conditions in regular problems. This is by far the most important use of the theory. It is useful to remember that the theory is more general than this. In fact it can often be useful when exploring a complex, multidimensional parameter space as a means of insuring that all the nooks and crannies have been explored.

An important part of the theory concerns the case where matching is not successful. In that case it must be that the two recursions, that generate the inner and outer double series, are not equivalent. There has been some significant reversal of large and small terms between the two recursions. In that case, there must be some interesting intermediate limit, for example, corresponding to some particular q of Eq. (13). The relative sizes of various terms varies continuously with q. If these relative sizes make a difference, then the region where they are equal must be interesting. If this new region does not match asymptotically to the two previous regions, on either side, the process can be repeated. Such a case has been encountered by Glasser et al. (1975).

The object of introducing new regions should be clarity. The introduction of many trivial regions may reduce clarity, but there is no reason to avoid otherwise trivial regions that increase the clarity of the presentation. In many ways this theory has analogies with analytic extension in complex analysis. There one may be concerned at one level with maximal extension, but practically, one's first interest is in making the algebra come out. This should be one's primary interest in asymptotic matching also.

IV. A Model Equation

In this section another, more interesting example will be considered, taken from Lagerstrom and Casten (1972).

$$\frac{d^2f}{dr^2} + \left(\frac{1}{r} + f + \frac{df}{dr}\right)\frac{df}{dr} = 0, \tag{14}$$

$$f(\epsilon) = 0, \quad f(\infty) = 1. \tag{15}$$

The problem is to find the behavior of f for small ϵ. The solution of this problem is known. It is given here to demonstrate the power and versatility of the methods discussed in the previous sections. It is convenient to treat because we can borrow heavily from Lagerstrom and Casten (1972) for performing the algebraic manipulations.

Before undertaking a solution, it is a good idea to see the kind of things this equation can do. First, an arbitrary constant is an exact solution. Next, consider the behavior for very large r. A perturbation solution for large r around the constant solution shows that there is a one-parameter family that converges to each positive constant solution. Thus the second condition of Eq. (15) can be satisfied with one free parameter remaining, and this can be used to satisfy the first condition. Finally, as will be seen below, a one-parameter family of solutions diverges at $r = 0$. The only nondiverging solutions are those that are constant everywhere. It follows that the desired solution must be on its way to diverging when it vanishes at $r = \epsilon$. According to the boundary conditions, it is diverging slowly, so the solution must be close to unity over most of the range of r.

First consider the outer region with f and r finite. Equation (14) can be written in recursive form,

$$\frac{d^2f}{d\eta^2} + \left(\frac{1}{\eta} + f_o + 2\frac{df_o}{d\eta}\right)\frac{df}{d\eta} + \frac{df_o}{d\eta}f$$
$$-\left(f_o + \frac{df_o}{d\eta}\right)\frac{df_o}{d\eta} =$$
$$= -\left(f + \frac{df}{d\eta} - f_o - \frac{df_o}{d\eta}\right)\left(\frac{df}{d\eta} - \frac{df_o}{d\eta}\right), \qquad (16)$$

where f_o is chosen so that $f = f_o$ is a solution of the left side of Eq. (16). In particular, we will choose $f_o = 1$. Then, assuming an expansion of the form

$$f = f_o + \sum_{n=1}^{\infty} \eta^n f_n , \qquad (17)$$

where η is a parameter of smallness that will be related to ϵ, equating coefficients of powers of η to zero, and solving order by order,

$$f_1 = A_1 E(\eta) + B_1$$
$$f_2 = A_1^2 \left[-E(\eta)\exp(-\eta) + 2E(2\eta) - \frac{1}{2}E^2(\eta)\right]$$
$$-A_1 B_1 \exp(-\eta) + A_2 E(\eta) + B_2 , \qquad (18)$$

where

$$E(\eta) \equiv \int_{\eta}^{\infty} \frac{\exp(-\eta)}{\eta} \, d\eta . \tag{19}$$

The constants B_n must be set equal to zero to satisfy the boundary conditions at $r = \infty$. The arbitrary constants A_n confirm the earlier statement that a one-parameter family of solutions converges at infinity to the constant solution. That is, there is one arbitrary function of η, characterized by the set of constants A_n, that can be adjusted to match the boundary condition at $r = \epsilon$.

For small r, the double series expansion of f, corresponding to Fig. 1, is given by Fig. 4. Here the expansion for small r

$$E(\eta) \simeq -\ln(\eta) - \gamma + \eta + \cdots \tag{20}$$

has been used.

This double series contains arbitrarily high powers of $\ln r$ in the leading terms. It is thus necessary to treat it as an expansion in powers of η and $\ln r$. Then terms proportional to r are much smaller than any power of $\ln r$ and can be dropped in the matching procedure. The resulting double series is then truncated in the same way as that in Fig. 2.

It is possible with this simplification to obtain a more concise representation of the reduced double series. First, consider the recursion from which the series is generated,

$$\frac{d^2 f}{dr^2} + \left(\frac{1}{r} + 2\frac{df_0}{dr}\right)\frac{df}{dr} - \left(\frac{df_0}{dr}\right)^2 = -\left(\frac{df}{dr} - \frac{df_0}{dr}\right)^2 - f\frac{df}{dr} . \tag{21}$$

Again, f_0 is chosen so that $f = f_0$ satisfies the left hand side of Eq. (21). The right hand side is then small when r is small, derivatives are large, and f is close to f_0.

Fortunately, the general solution for f_0 can be found for this equation :

$$f_0 = \ln\left(C + D \ln r\right), \tag{22}$$

where C and D are the two constants of integration. It can be shown that in the recursion of Eq. (21), the first term on the right-hand side generates terms that can also be obtained by expanding the constants C and D. That is, these terms arise from the difference between the

the chosen values of C and D in f_o, and those desired for the true solution. The second term on the right hand side of Eq. (21) only generates terms proportional to r and smaller, that is, terms that were neglected in the double series Fig. 4.

It follows that the desired terms in the double series can be represented in the form of Eq. (22). Specifically, these terms are generated when

$$C = e\left\{1 - \eta\gamma A_1 - \eta^2\left[(\gamma + 2\ln 2)A_1^2 + \gamma A_2\right] + \cdots\right\}$$

$$D = -e\left\{\eta A_1 + \eta^2\left(A_1^2 + A_2\right) + \cdots\right\}. \tag{23}$$

Note that the outer solution of Eq. (16) has been used to evaluate the constants arising from the iteration of Eq. (21).

This outer solution is not valid near the boundary of the double series, $\eta \ln r \sim 1$. It has been assumed that the deviation from $f_o = 1$ is small, whereas, according to the boundary condition, Eq. (15), this deviation is unity. Thus it is necessary to find a more accurate approximation when r is small.

In the inner region that is required, Eq. (21) is still a valid recursion. The solution f_o^* that satisfies the boundary condition, Eq. (15), is

$$f_o^* = \ln\left[1 + F_o \ln(\eta/\epsilon)\right], \tag{24}$$

where F_o is arbitrary, and solutions valid in the inner region are distinguished by a star. The first correction in powers of η, assuming that f^* is also a Taylor series in η, is given by

$$f_1^* = \frac{F_1 \ln(\eta/\epsilon)}{1 + F_o \ln(\eta/\epsilon)} \; . \tag{25}$$

When $r \gg \epsilon$, away from the boundary, there is a region where $\ln r \ll \ln \epsilon$. Here each of the f_n^* can be expanded in powers of ln r. This procedure produces a double series in ln r and η that also contains powers of $\ln \epsilon$. These are essentially the same parameters as those of the double series of Fig. 4. Matching these series provides a relation between η and $\ln \epsilon$, and can also be used to evaluate all the unknown constants.

This matching can be done straightforwardly, but it is more convenient to express the inner double series in the concise form of Eq. (22), and then match the concise forms. The first two terms of f^*

can be expressed

$$f^* = \ln\left[1 + F \ln(\eta/\epsilon)\right], \tag{26}$$

with

$$F = F_0 + \eta F_1 + \cdots . \tag{27}$$

The condition that the two concise forms, Eqs. (22) and (26), generate identical double series is

$$C = 1 - F \ln \epsilon ,$$

$$D = F . \tag{28}$$

Eliminating F, substituting Eq. (23), and solving order by order yields

$$\eta = - \frac{1}{\ln \epsilon} ,$$

$$A_1 = - \frac{e-1}{e} ,$$

$$A_2 = \gamma A_1 - A_1^2 . \tag{29}$$

The essence of the matching in this example has been the summation of the important terms of the double series in compact form. This greatly simplified the matching. Such a summation is not necessary, because the compact form contains the same information as the double series. Note that the properties of Eqs. (22) and (23) as asymptotic approximations are most conveniently understood from the expanded form. Further, the double series works when the compact form is not available.

The choice of f_o^* in Eq. (24) is essential for obtaining the correct asymptotic form. This function is valid to order η both near $r = \epsilon$ and the region where f_o^* is of order unity. Perturbation around either $f_o^* = 0$ or $f_o^* = 1$ each break down, in that corrections become as large as the leading term. This has been discussed in Lagerstrom and Casten (1972). Note that the leading order of F vanishes as η goes to zero, according to Eqs. (28) and (23). This does not affect any of the results.

Acknowledgment

I am deeply indebted to Dr. Martin Kruskal for guiding me through this subject. I would like to acknowledge valuable discussions with Drs. P. A. Lagerstrom, M. J. Ablowitz, and R. E. O'Malley.

References

{1} J.D. Cole (1968), Perturbation Methods in Applied Mathematics, Blaisdell, Waltham, Mass.

{2} B. Coppi, J.M. Greene, and J.L. Johnson (1966), Resistive Instabilities in a Diffuse Linear Pinch, Nucl. Fusion 6, 101-117

{3} L.E. Fraenkel (1969),On the Method of Matched Asymptotic Expansions Proc. Cambridge Philos. Soc. 65, 209-283

{4} A.H. Glasser, J.M. Greene, and J.L. Johnson (1975), Resistive Instabilities in General Toroidal Plasma Configurations, Phys. Fluids 18, 875-888

{5} S. Kaplun (1967), Fluid Mechanics and Singular Perturbations, P.A. Lagerstrom, L.N. Howard, and C.S. Liu eds., Academic Press, New York

{6} M.D. Kruskal (1963), Mathematical Models in Physical Science, S. Drobot ed. Prentice Hall, Englewood Cliffs

{7} P.A. Lagerstrom and R.G. Casten (1972), Basic Concepts Underlying Singular Perturbation Techniques, SIAM Review 14, 63-120

{8} M. Van Dyke (1964), Perturbation Methods in Fluid Mechanics, Academic Press, New York

→ Powers of t

↓ Powers of ε

$$
\begin{array}{lllll}
I = \; 1 & - t & + \frac{1}{2} t^2 & - \frac{1}{6} t^3 & + \cdots \\
+ \varepsilon & - \varepsilon t & + \frac{1}{2} \varepsilon t^2 & + \cdots & \\
+ \varepsilon^2 & - \varepsilon^2 t & + \cdots & & \\
+ \varepsilon^3 & + \cdots & & & \\
+ \cdots & & & &
\end{array}
$$

Figure 1

$\longrightarrow$ Powers of $1/\tau$

$\downarrow$ Powers of ε

$$
I^{*} = \begin{array}{lllll}
 & & 1 & +\ 0 & +\ \cdots \\
 & -\ \varepsilon\tau & +\ \varepsilon & +\ \cdots & \\
+\ \frac{1}{2}\varepsilon^{2}\tau^{2} & +\ \varepsilon^{2}\tau & +\ \varepsilon^{2} & +\ \cdots & \\
+\ \cdots & & & &
\end{array}
$$

Figure 2

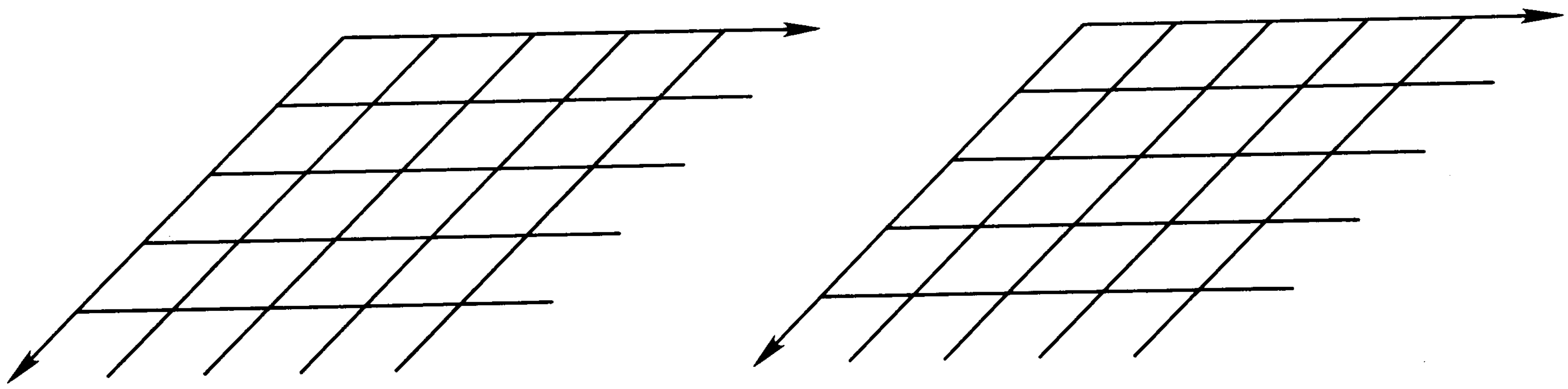

Figure 3

$$
\begin{aligned}
f = \quad & \qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + 0 \qquad + \ldots \\
& \qquad\qquad\qquad - \eta A_1 \ \ell n r \qquad\qquad - \eta A_1 \gamma \qquad\qquad\qquad\qquad\qquad\qquad\quad + \eta A_1 r \quad + \ldots \\
& - \frac{1}{2}\, \eta^2 A_1^2 \ell n^2 r \quad - \eta^2 [A_1^2 (1+\gamma) + A_2] \ell n r \quad - \eta^2 \left(\frac{1}{2}\, (\gamma^2 + \gamma + 2\ \ell n 2) A_1^2 + A_2 \gamma \right) \quad + \ldots \\
& + \ldots
\end{aligned}
$$

Figure 4