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Abstract

Saturation of the current-driven ion-acoustic instability in a weakly-ionized plasma is investigated within the context of a quasilinear model that includes the effects of collisions. The turbulent wave energy is calculated under the assumption that the waves are excited within a cone of a small angle. The results obtained compare favorably with the recent experimental observations of Ilić.

I. Introduction

In plasma physics, the current-driven ion-acoustic instability is one of the instabilities most studied. Experimentally, interest in the instability has been stimulated by its connection with anomalous resistivity, a useful mechanism for heating plasmas in controlled fusion devices. Different theories of plasma turbulence have been invoked to describe the saturation mechanism of the instability and to determine its spectrum. However, as yet there seems to be no satisfactory agreement between the theoretical results and experimental observations[1]. Apparently, there are two reasons for this situation. On the one hand, the theoretical models do not take into account real physical conditions in a particular device, while on the other hand, the results of experimental observations have not been sufficiently conclusive.

Recently, new detailed measurements of the ion-acoustic turbulent spectrum in a positive column have been reported[2]. It was argued that there are two nonlinear mechanisms which could be responsible for the saturation of the instability in this experiment. The first is electron trapping modified by collisions[3], the second is ion resonance broadening[4].

An objective of the present paper is to show that the experimental results obtained by Ilić can be interpreted by means of a simple quasilinear theory modified by collisions. The plan of the paper is now outlined. Firstly a brief formulation of the theoretical model is given. The resulting equations are then solved analytically, assuming that the system is in a stationary state. The spectral distribution of the turbulent oscillations is obtained in a purely one-dimensional case. In a pseudo-three-dimensional case, where the oscillations are excited within a cone of a small angle, we derive an approximate formula which relates the total wave energy to relevant physical parameters (Sec. II). In order to obtain a more accurate quantitative description of the turbulent system, the basic equations - somewhat generalized - are solved numerically in Sec. III. Finally, a discussion of the results obtained and their comparison with the experimental observations of Ilić are presented in Sec. IV.

II. Theory

1. Basic Equations

The plasma under consideration is assumed to be weakly ionized, uniform and unmagnetized. The electrons are hot ($T_e \gg T_i$) and drift with a velocity $\vec{v}_d = v_d \vec{e}_z$ ($v_d > 0$) relative to a cold ion background. The electron current is sustained by a weak, external dc electric field $\vec{E}_o = E_o \vec{e}_z$ which results in the generation of an ion-acoustic instability in the system. Since the dynamics of charged particle motions are dominated by collisions with neutrals, before the instability is excited the electron drift has to be several times greater than the ion sound speed $c_s \left[= (T_e/m_i)^{\frac{1}{2}} \right]$, in order to offset the damping by ion-neutral collisions.

The turbulence model adopted to describe the system consists of the quasilinear kinetic equations generalized to include the effects of the collisions between charged particles and neutrals. As will be verified a posteriori (Sec. III), the modification of the ion distribution function due to the turbulent oscillations is negligible for the range of the relevant physical parameters considered. Therefore, the ion distribution function may be taken as a Maxwellian with a temperature T_i. With this stipulation, the basic equations describing the problem can be given in the form[5,6]

$$\frac{\partial f}{\partial t} + \frac{eE_0}{m_e}\frac{\partial f}{\partial v_z} = \frac{\partial}{\partial \vec{v}}\cdot\overleftrightarrow{D}\cdot\frac{\partial f}{\partial \vec{v}} - \nu_{en}(f - f_0) \,, \tag{1}$$

$$\overleftrightarrow{D} = \pi\left(\frac{e}{m_e}\right)^2 \int \frac{d\vec{k}}{(2\pi)^3}\,\frac{\vec{k}\vec{k}}{k^2}\, I_{\vec{k}}\, \delta(\omega - \vec{k}\cdot\vec{v}) \,, \tag{2}$$

$$\frac{dI_{\vec{k}}}{dt} = 2\gamma_{\vec{k}} I_{\vec{k}} \,, \tag{3}$$

$$\gamma_{\vec{k}} = \frac{\pi}{2}\frac{\omega^3}{k^2}\frac{m_i}{m_e}\left[\int d\vec{v}\,\vec{k}\cdot\frac{\partial f}{\partial \vec{v}}\,\delta(\omega - \vec{k}\cdot\vec{v}) - \frac{\pi^{1/2}\omega^4}{k^3 v_i^3}\exp\left[-\left(\frac{\omega}{k v_i}\right)^2\right]\right] - \frac{\nu_{in}}{2} \,, \tag{4}$$

$$\omega = \frac{k C_s}{(1 + k^2\lambda_D^2)^{1/2}} \,. \tag{5}$$

Here f is the electron distribution function, $I_{\vec{k}}$ is the spectral distribution of the electric field associated with the oscillations, and e and m_e are the electron charge and mass, respectively; λ_D is the electron Debye length, m_i and $v_i\left[= \left(\frac{2T_i}{m_i}\right)^{\frac{1}{2}}\right]$ are the ion mass and thermal velocity, and ν_{en} and ν_{in} are the electron-neutral and ion-neutral

collision

frequencies, respectively. The last term in Eq. (1) is the simplest version of the collision term of the Bhatnagar-Gross-Krook[7] model, where f_o is a nondrifting Maxwellian with a temperature T_e. We have dispensed with the integral part of the collision term since for a uniform plasma $\int f dv = \int f_o dv = 1$. In the equation for the growth-rate (4), the first term describes the Čerenkov excitation of the ion-acoustic oscillations by the drifting electrons whereas the last two terms represent the ion Landau damping and the damping due to ion-neutral collisions. In deriving Eqs. (4) and (5) we have assumed that $v_d/v_e \ll 1$, $\nu_{in}/\omega \ll 1$, and $\nu_{en}/kv_e \ll 1$, where $v_e = (2T_e/m_e)^{\frac{1}{2}}$.

2. One-dimensional Model

Since the experimental results cited indicate that the turbulent spectrum is stationary, and essentially one-dimensional, it is tempting, initially, to seek one-dimensional stationary solutions to Eqs. (1) - (5). Thus, assuming that $I_{\vec{k}}(t) = (2\pi)^2 I_k \delta(k_x)\delta(k_y)$ and $f(\vec{v},t) = f(v)\delta(v_x)\delta(v_y)$, where $k \equiv k_z$ and $v \equiv v_z$, the integration over $d\vec{k}$ and $d\vec{v}$ in Eqs. (2) and (4) can be carried out explicitly. On integrating Eqs. (1) and (3) over $dv_x dv_y$ and $dk_x dk_y$, respectively, and discarding the terms with time derivatives the set of Eqs. (1) - (5) is reduced to

$$v_d \nu_{en} \frac{df}{dv} = \frac{d}{dv}\left(\frac{e}{m_e}\right)^2 \frac{I_v}{v\left(1-(v/c_s)^2\right)} \frac{df}{dv} - \nu_{en}(f - f_o) , \quad (6)$$

$$2\gamma_v I_v \equiv \left\{\pi\omega_{pi} v^2\left(1-\left(\frac{v}{c_s}\right)^2\right)^{1/2}\left(\frac{m_i}{m_e}\frac{df}{dv} - \frac{2v}{\pi^{1/2} v_i^3}\exp\left[-\left(\frac{v}{v_i}\right)^2\right]\right) - \nu_{in}\right\} I_v = 0 , \quad (7)$$

where $I_v = I_{k=\omega/v}$ and ω_{pi} is the ion plasma frequency. In Eq. (6) we have eliminated the external electric field by means of the relation $eE_o/m_e = v_d \nu_{en}$, anticipating that the contribution of the turbulent oscillations to the resistivity is negligible.

Let the oscillations be excited within an internal $\langle v_L, c_s \rangle$, where $v_L > 0$ is a lower boundary of the spectrum which will be specified later. Then, according to Eq. (7) the electron distribution function will be determined within this internal by the equation $\gamma_v = 0$. A simple integration yields

$$f = f(v_L) + \frac{m_e}{m_i}\left[\frac{\nu_{in}}{\pi\omega_{pi}}\left\{\frac{(1-(v_L/c_s)^2)^{1/2}}{v_L} - \frac{(1-(v/c_s)^2)^{1/2}}{v}\right\} + \frac{1}{\pi^{1/2} v_i}\left\{\exp\left[-\left(\frac{v_L}{v_i}\right)^2\right] - \exp\left[-\left(\frac{v}{v_i}\right)^2\right]\right\}\right], \quad v \in \langle v_L, c_s \rangle, \qquad (8)$$

where $f(v_L)$ is an arbitrary constant. Outside the interval $\langle v_L, c_s \rangle$, where $I_v = 0$, the electron distribution function satisfies the equation

$$v_d \frac{df}{dv} = f_o - f, \qquad (9)$$

which follows from Eq. (6). The solution to Eq. (9) is easily shown to be

$$f = \frac{1}{2v_d} \exp\left[\left(\frac{v_e}{2v_d}\right)^2 - \frac{v}{v_d}\right]\left(1 - \Phi\left(\frac{v_e}{2v_d} - \frac{v}{v_e}\right)\right), \quad v < v_L, \qquad (10)$$

$$f = f(c_s)\exp\left(\frac{c_s - v}{v_d}\right) + \frac{1}{2v_d}\exp\left[\left(\frac{v_e}{2v_d}\right)^2 - \frac{v}{v_d}\right]$$

$$\times \left\{ \Phi\left(\frac{v_e}{2v_d} - \frac{c_s}{v_e}\right) - \Phi\left(\frac{v_e}{2v_d} - \frac{v}{v_e}\right)\right\}, \quad v > c_s , \tag{11}$$

where $f(c_s)$ is an arbitrary constant and Φ is the probability integral[8].

Imposing the requirement that f be continuous at v_L and c_s, the constants $f(v_L)$ and $f(c_s)$ are fixed by Eqs. (10) and (8), respectively. In order to determine the quantity v_L we make use of the condition that the total number of particles is conserved : $\int f dv = 1$. Evaluating the integral by means of Eqs. (8), (10) and (11), and taking into account that $v_L < c_s \ll v_d \ll v_e$, after some algebra we find that the quantity v_L must satisfy the following transcendental equation

$$2\frac{m_i}{m_e}\frac{(v_L - c_s)v_d}{v_e^3} + \frac{\nu_{in}}{\pi^{1/2}\omega_{pi}}\frac{\left(1-(v_L/c_s)^2\right)^{1/2}}{v_L} + \frac{1}{v_i}\exp\left[-\left(\frac{v_L}{v_i}\right)^2\right] = 0. \tag{12}$$

An example of a numerical solution to Eq. (12) is given in Sec. III.

Having found the electron distribution function we can now determine the spectrum. On substituting f from Eq. (8) into Eq. (6) we obtain a first order differential equation for I_v which can easily be solved. Taking into account the condition that $I_v = 0$ at $v = v_L$, and neglecting the terms of the order c_s/v_d we finally obtain

$$\frac{I_v}{4\pi n T_e \lambda_D} = \frac{T_i}{T_e}\left(\frac{m_e}{m_i}\right)^{3/2}\frac{\nu_{en}}{\omega_{pe}}\frac{v_d}{c_s}v^3\left(1-\left(\frac{v}{c_s}\right)^2\right)^{3/2}$$

$$\times\frac{\exp\left[-\left(\frac{v_L}{v_i}\right)^2\right]-\exp\left[-\left(\frac{v}{v_i}\right)^2\right]+\frac{v_i\,\nu_{im}}{\pi^{1/2}\omega_{pi}}\left\{\frac{(1-(v_L/c_s)^2)^{1/2}}{v_L}-\frac{(1-(v/c_s)^2)^{1/2}}{v}\right\}}{\frac{v_i^3}{2\pi^{1/2}}\frac{\nu_{im}}{\omega_{pi}}+v^3\left(1-\left(\frac{v}{c_s}\right)^2\right)^{1/2}\exp\left[-\left(\frac{v}{v_i}\right)^2\right]}, \tag{13}$$

where n is the charged-particle density and ω_{pe} is the electron plasma frequency. If we discard the terms corresponding to the ion Landau damping and approximate v_L by v_i, formula (13) reduces to that obtained earlier by Vedenov, Velikhov and Sagdeev[5]. In order to find the total electrostatic wave energy density Eq. (13) must be integrated numerically. An example is given in Sec. III. Concluding this subsection, we can now show by means of Eqs. (8) and (13) that the contribution of the turbulent oscillations to the resistivity is indeed negligible. On defining the turbulent collision fequency as

$$\nu_{turb} = \frac{1}{v_d}\left(\frac{e}{m_e}\right)^2\int\frac{I_v}{v(1-(v/c_s)^2)}\frac{df}{dv}\,dv\,, \tag{14}$$

we estimate that

$$\frac{\nu_{turb}}{\nu_{en}} \sim \frac{m_e}{m_i}\left(\frac{T_e}{T_i}\right)^{1/2}\frac{\nu_{in}}{\omega_{pi}} \ll 1\,. \tag{15}$$

3. Pseudo-three-dimensional Model

Let us now consider a situation where the turbulent oscillations are excited within a cone of a small angle θ defined by $\cos\theta = \vec{k}\cdot\vec{e}_z/k$. A question arises as to how small the angle θ would have to be in order that the one-dimensional model developed in the preceding subsection would still apply. We put forward the following argument. In the one-dimensional model the number of electrons which are in resonance with the oscillations is approximately c_s/v_e. In the case of a finite angle we can estimate the number of resonant electrons to be of the order θ. Hence, we conclude that the one-dimensional model will apply for angles $\theta \lesssim c_s/v_e \sim (m_e/m_i)^{\frac{1}{2}}$. Consequently, for the spectra with an angular width greater than $(m_e/m_i)^{\frac{1}{2}}$ the problem becomes three-dimensional.

In what follows, we confine ourselves to the case where $(m_e/m_i)^{\frac{1}{2}} < \theta \ll 1$. It is then physically plausible to approximate the electron distribution function perpendicular to $\vec{e}_z$ by a Maxwellian with temperature T_e. Owing to this approximation we can integrate Eq. (1) over $dv_x dv_y$ to obtain an equation for the electron disbribution function in $v \equiv v_z$. We find

$$\frac{\partial f}{\partial t} + \frac{eE_o}{m_e}\frac{\partial f}{\partial v} = \frac{\partial}{\partial v} D \frac{\partial f}{\partial v} - \nu_{en}(f - f_o), \tag{16}$$

where

$$D = \pi\left(\frac{e}{m_e}\right)^2 \int \frac{d\vec{k}}{(2\pi)^3} \frac{I_{\vec{k}}}{\pi^{1/2} v_e k \theta} \exp\left[-\left(\frac{\omega - kv}{v_e k \theta}\right)^2\right]. \tag{17}$$

The expression for the diffusion coefficient D may be further simplified if we assume that the turbulent oscillations are uniformly distributed over an angular internal $<0,\theta_o>$. On performing the integration over $d\theta$ in Eq. (17), and introducing the spectral distribution per unit of k by means of the relation $\int I_{\vec{k}} d\vec{k}/(2\pi)^3 = \int_0^\infty I_k dk/\pi$, we obtain

$$D = \left(\frac{e}{m_e}\right)^2 \int_0^\infty \frac{I_k}{k} R_k(v)\, dk \quad , \tag{18}$$

where the resonance function $R_k(v)$ is given by

$$R_k(v) = \frac{|\omega - kv|}{\pi^{1/2} v_e^2 \theta_o^2 k} \Gamma\left(-1/2, \left(\frac{\omega - kv}{v_e \theta_o k}\right)^2\right), \tag{19}$$

and Γ is the incomplete gamma function[9]. As $\theta_o \to 0$ we find that $R_k \to \delta(v - \omega/k)$, as one would expect.

The same procedure as that used above can now be applied to Eqs. (3) and (4). We obtain

$$\frac{dI_k}{dt} = 2\gamma_k I_k \quad , \tag{20}$$

$$\gamma_k = \frac{\pi}{2} \frac{\omega^3}{k^2} \frac{m_i}{m_e} \int dv \frac{\partial f}{\partial v} R_k(v) - \frac{\nu_{in}}{2} \quad , \tag{21}$$

where we have neglected the term corresponding to the ion Landau damping. The reason for this approximation is discussed in Sec. III.

We are interested in stationary solutions to Eqs. (16) and (20). It is easy to see, however, that the mathematical structure of the equations, even after discarding the terms with the time derivative, is rather formidable. We have been unable to obtain an exact analytical solution to these equations. Therefore, to obtain at least a notion of how the total energy of the turbulence scales, with respect to relevant physical parameters, we resort to the following simplified model. Firstly, we replace the actual resonance function {Eq. (19)} by a model resonance function defined as

$$R_k(v) = \begin{cases} \dfrac{1}{v_e \Theta_o}\left(1 - \dfrac{|\omega - kv|}{v_e k \Theta_o}\right) & |\omega - kv| < v_e k \Theta_o , \\ 0 & |\omega - kv| > v_e k \Theta_o . \end{cases} \tag{22}$$

This function has the same area and approximately the same height, width, and shape as the actual R function. Secondly, we assume the spectrum to be sufficiently narrow so that we can put $I_k = I_o \delta(k-k_o)$. Here k_o is a characteristic wavenumber of the spectrum which cannot be determined within our rough model. Its value might be inferred either from the linear dispersion relation or from experimental observations.

With the assumptions listed above Eq. (16) becomes

$$v_d \nu_{en} \frac{df}{dv} = \left(\frac{e}{m_e}\right)^2 \frac{I_o}{v_e k_o \Theta_o} \frac{d}{dv}\left(1 - \frac{|\omega - k_o v|}{v_e k_o \Theta_o}\right)\frac{df}{dv} \tag{23}$$

$$- \nu_{en}(f - f_o) \quad , \quad |\omega/k_o - v| < v_e \Theta_o .$$

Introducing a new variable $x = (v-\omega/k_0)/(v_e\theta_o)$ Eq. (23) is reduced to a simple form

$$\alpha \frac{d}{dx}(1-|x|)\frac{df}{dx} - \beta\frac{df}{dx} + f_o - f = 0 \;, \quad |x| < 1 \;, \tag{24}$$

where

$$\alpha = \left(\frac{e}{m_e}\right)^2 \frac{I_o}{k_o} \frac{1}{(v_e\Theta_o)^3 \nu_{en}} \;, \quad \beta = \frac{v_d}{v_e\Theta_o} \tag{25}$$

The general solution to Eq. (24) can be expressed in terms of the Bessel functions $J_\nu(z)$ and $\underline{J}_\nu(z)$, where $\nu = \beta/\alpha$ and $z = 4(|x|-1)/\alpha$ [10]. For our purpose, however, it is sufficient to take a few significant terms of the series expansion of the Bessel functions since it turns out that $\alpha \gg 1$ for the range of physical parameters considered. Thus, approximately, we have

$$f = C_1\left(1 + \frac{1-x}{\alpha+\beta}\right) + \frac{1}{\pi^{1/2} v_e} \;, \qquad 0 < x < 1 \;, \tag{26}$$

$$f = C_2\left(1 + \frac{1+x}{\alpha-\beta}\right) + C_3(1+x)^{\beta/\alpha} + \frac{1}{\pi^{1/2} v_e} \;, \; -1 < x < 0 \;, \tag{27}$$

where C_1, C_2 and C_3 are arbitrary constants. In deriving Eq. (26) we have discarded the singular complementary function as being unphysical.

Evidently, for x < -1 the function f is given by Eq. (10). Imposing the condition that f be continuous at x = -1 we determine the constant C_2 as

$$C_2 = -\frac{2v_d^2}{\pi^{1/2}v_e^3} \quad . \tag{28}$$

The constants C_1 and C_3 are fixed by the condition that f and df/dv must be continuous accross x = 0. Thus, we obtain

$$C_1 = C_2\frac{\beta-1}{\beta+1} \quad , \quad C_3 = -C_2\frac{2\alpha^2+\beta(\alpha+\beta)}{(\beta+1)(\alpha^2-\beta^2)} \quad . \tag{29}$$

In order to determine the quantity I_o we need to calculate the integral $\int df/dvR_k(v)dv$. On making use of Eqs. (26) - (29) and (22) we find

$$\int\frac{df}{dv}R_k(v)\,dv = \frac{4}{v_e\Theta_o\alpha}\,\frac{v_d^2}{\pi^{1/2}v_e^3} \quad , \tag{30}$$

where we have assumed that

$$1 \ll \beta \ll \alpha . \tag{31}$$

We now combine (30) and (21), and put $\gamma_{k_o} = 0$. On substituting α from Eq. (25) into the resulting equation we finally obtain the following approximate expression for the total electrostatic wave energy density

$$\frac{W_{elst}}{nT_e} = \left(\frac{2^3}{\pi}\right)^{1/2}\left(\frac{m_e}{m_i}\right)^{1/2}\frac{\nu_{en}}{\nu_{in}}\left(\frac{v_d}{v_e}\right)^2\frac{\Theta_o^2(k_o\lambda_D)^2}{(1+k_o^2\lambda_D^2)^{3/2}} \quad , \tag{32}$$

where $W_{elst} = I_0/(8\pi^2)$. The condition (31) which has to be satisfied for formula (32) to be valid can now be written explicitly as

$$1 \ll \frac{v_d}{\Theta_0 v_e} \ll \frac{\omega_{pi}}{\nu_{in}} \left(\frac{v_d}{v_e}\right)^2 \frac{k_0 \lambda_D}{\Theta_0 (1 + k_0^2 \lambda_D^2)^{3/2}} \quad . \tag{33}$$

Although the principal result of this section - Eq. (32) - was derived with the aid of a number of simplifications, surprisingly good agreement is found between the values obtained from this equation and those determined by numerically integrating Eqs. (16) and (20).

III. Numerical Calculations

In order to provide a test of the predictions of the theory developed in the preceding section we shall now obtain numerical solutions to the basic equations of our model. Before doing this, however, we shall introduce a number of modifications into the equations. First of all, we allow for the quasilinear evolution of the ion distribution function F. Since the number of ions which can be in resonance with the turbulent oscillations is very small, regardless of whether the spectrum is one-dimensional or not, we shall ignore the finite angle effects when treating the ion distribution function. Thus, instead of Eq. (21) we consider the following expression for the growth-rate

$$\gamma_k = \frac{\pi}{2} \frac{\omega^3}{k^2} \left\{ \frac{m_i}{m_e} \int dv \, \frac{\partial f}{\partial v} R_k(v) + \left.\frac{\partial F}{\partial v}\right|_{v = \omega/k} \right\} - \frac{\nu_{in}}{2} \quad . \tag{34}$$

Next, in the particle dynamics we would like to take into account Coulomb collisions, which have been tacitly assumed to be negligible in the foregoing calculations. Inasmuch as the number of resonant particles is small, it is appropriate to make use of the collision integrals in Fokker-Planck form. They can be obtained from the linearized Landau collision integrals by integrating over the velocities perpendicular to $\vec{e}_z$ [11]. Consequently, the equation describing the dynamics of the ion distribution function can be written as

$$\frac{\partial F}{\partial t} = \left(\frac{e}{m_i}\right)^2 \frac{\partial}{\partial v} \frac{I_v}{v(1-(v/c_s)^2)} \frac{\partial F}{\partial v} - \nu_{in}(F - F_o)$$
$$+ \nu_{ii} \frac{\partial}{\partial v} \frac{v_i^3}{(v_i^2 + v^2)^{3/2}} \left(vF + \frac{T_i}{m_i} \frac{\partial F}{\partial v} \right) , \qquad (35)$$

where F_o is a Maxwellian with temperature T_i, and ν_{ii} is the ion-ion collision frequency. The electron distribution function is governed by Eq. (16), with the following collision term added onto the right-hand side

$$Sf = \nu_{ee} \frac{\partial}{\partial v} \frac{v_e^3}{(v_e^2 + (v - v_d)^2)^{3/2}} \left[(v - v_d) f + \frac{T_e}{m_e} \frac{\partial f}{\partial v} \right]$$
$$+ \nu_{ei} \frac{\partial}{\partial v} \frac{v_e^3}{(v_e^2 + v^2)^{3/2}} \left[vf + \frac{T_e}{m_e} \frac{\partial f}{\partial v} \right] , \qquad (36)$$

where ν_{ee} and ν_{ei} are the electron-electron and electron-ion collision frequencies, respectively.

Equations (16), (20) and (35) together with Eqs. (18), (19), (34) and (36) form a complete set of equations describing the dynamics of the turbulent system. This set, supplemented by the initial conditions : $f(t=0) = f_o(v-v_d)$, $F(t=0) = F_o$, $I_k(t=0) = I_k^{eq} = T_e k^2/\{1+(k\lambda_D)^{-2}\}$, is solved numerically using the finite-element method. The description of the algorithm is not given here since it has been published in detail elsewhere[12].

In order to proceed we first establish which independent parameters must be specified in the numerical calculations. From the structure of the equations it is easily seen that we may choose the following dimensionless parameters : θ_o, $v_d^* = v_d(m_e/T_e)^{\frac{1}{2}}$, $\mu = m_e/m_i$, $\eta = T_e/T_i$, $\sigma_{en}^* = \sigma_{en}/\lambda_D^2$, $\sigma_{in}^* = \sigma_{in}/\lambda_D^2$, $g = (n\lambda_D^3)^{-1}$, $\delta = n/(n+N)$, where σ_{en} and σ_{in} are the effective cross sections for the electron-neutral and ion-neutral collisions, respectively, and N is the neutral-particle density. It should be pointed out that during the computations we keep the electron drift velocity fixed rather than the dc electric field, which corresponds to the situation in the experiment.

We have performed two series of computations : one where the Coulomb collisions were "switched off", and one where they were included. Let us examine the former. In the first study we considered the one-dimensional situation, viz. $\theta_o = 0$. Figure 1 shows the spectral distribution at the saturation time for a typical set of plasma parameters : $v_d^* = 2.10 \times 10^{-1}$, $\mu = 1.36 \times 10^{-4}$, $\eta = 50$, $\sigma_{en}^* = 9.50 \times 10^{-12}$, $\sigma_{in}^* = 1.31 \times 10^{-10}$, $g = 4.76 \times 10^{-5}$, $\delta = 1.31 \times 10^{-5}$. The broken line represents the analytical solution given by Eq. (13) with $v_L = 2.16\ v_i$ which was determined numerically from Eq. (12). One can see that except for the largest values of $k\lambda_D$ the analytical spectrum agrees with that obtained from the computations. A better agreement is found if one compares quantities which are more relevant, viz. the total electrostatic wave energy density, the average wavenumber and the width of the spectrum. They are defined as

$$W_{elst} = \frac{1}{16\pi^2} I \quad , \quad I = \int I_k \, dk \, ,$$
$$\bar{k} = \frac{1}{I} \int k I_k \, dk \, , \quad (\Delta k)^2 = \frac{1}{I} \int (k - \bar{k})^2 I_k \, dk \, . \tag{37}$$

From the computations we obtain $W_{elst}/nT_e = 2.94 \times 10^{-8}$, $\bar{k}\lambda_D = 9.69 \times 10^{-1}$, $\Delta k\lambda_D = 3.16 \times 10^{-1}$, whereas the numerical evaluation by means of Eq. (13) yields $W_{elst}/nT_e = 2.92 \times 10^{-8}$, $\bar{k}\lambda_D = 9.47 \times 10^{-1}$, $\Delta k\lambda_D = 3.45 \times 10^{-1}$. Last but not least, we should mention that in this study we did not observe any significant modification of the ion distribution function due to the turbulent oscillations. Thus, the assumption made in deriving Eq. (13) is fully justified.

Next, we investigate the effect of a finite angle, keeping the other parameters fixed. The total electrostatic wave energy density as a function of the angle θ_o is plotted in Fig. 2. The broken line represents the value of W_{elst} determined by means of Eq. (13) whereas the solid line shows the behaviour of W_{elst} according to formula (32) multiplied by a factor 1/3 to give the best fit {hereafter called the modified formula (32)}. We observe that except for very small angles the modified formula (32) describes the behaviour of W_{elst} fairly well. Figures 3, 4 and 5 show the dependence of total electrostatic wave energy density upon the electron drift velocity, the electron-neutral effective cross section and the ion neutral effective cross section, respectively, for a fixed finite angle. Once again the solid lines correspond to the modified formula (32). The reasonably good agreement between the numerical and analytical results indicates that the best fit factor is to a good approximation independent of the plasma parameters.

We now turn our attention to the ion Landau damping. In the computations with relatively small values of T_i it appears that the spectrum is located in that portion of k-space where the ion Landau damping is negligible. For larger values of T_i the same is true at the time of saturation. However, in earlier stages of evolution the spectrum extends to larger wavenumbers, and consequently a part of the wave energy is absorbed by the ions via Landau damping before saturation is reached. Thus, the final wave energy obtained from the computations is somewhat smaller than that predicted by the analytical model which, of course, does not take the effect last mentioned into account. This discrepancy is clearly demonstrated in Fig. 6.

In the last study of this series we have investigated the dependence of the turbulent wave energy upon the ion mass. An inspection of Eq. (32) reveals that there should be no dependence. The computations for a variety of gases have shown a variation in wave energy of less than 30%, for the two extremes of hydrogen and argon.

Let us now turn to the case where Coulomb collisions are included in the numerical model. Evidently, the degree of ionization δ is the essential parameter which will control the competition between the collisions of charged particles with neutrals and the Coulomb collisions. Initially we assigned the same value to this parameter as in the foregoing studies, viz. $\delta = 1.31 \times 10^{-5}$, and simply repeated all the computations with the Coulomb collision terms "switched on". It turns out that for such a low degree of ionization the effect of the Coulomb collisions is negligible except in the case where the electron-neutral effective cross section is very small. As can be seen from Fig. 7, the Coulomb collisions become an important factor in determining the wave energy when $\sigma^*_{en} \lessdot 10^{-12}$.

The state of the turbulent system changes appreciably as we increase the degree of ionization. As an example, Fig. 8 shows the angular dependence of the wave energy for $\delta = 1.31 \times 10^{-3}$. We notice that the wave energy is roughly two orders of magnitude higher than that for $\delta = 1.31 \times 10^{-5}$. Moreover, the variation with angle no longer follows the parabolic scaling given by Eq. (32). The reason for the energy increase can easily be understood if one realises that the Coulomb collision term, due to its differential structure, is much more efficient in restoring the electron distribution function than the BGK term. (We can estimate that these terms are comparable if $\nu_{en}/\nu_{ee} \sim (v_e/\Delta v)^2 \gg 1$, where Δv is the width of the wave-particle interaction region in velocity space.) Consequently, the wave energy must reach a higher value in order to achieve a balance between the quasilinear and the collisional modifications of the electron distribution function. The wave energy as a function of the degree of ionization is plotted in Fig. 9. We observe that for $\delta \gtrsim 10^{-4}$ the plot is a linear function of δ. This is the part corresponding to plasmas which are dominated by Coulomb collisions.

IV. Discussion

Let us now apply the results obtained in the previous sections to the ion-acoustic turbulence which has been observed in the positive column of a helium discharge[2]. In order to calculate the theoretical values of the turbulent wave energy corresponding to the observations we need the experimental values of the steady-state parameters of the discharge. Moreover, we must know the angular width of the observed spectrum. Following Ilić[2], we take : T_e = 5 eV, T_i = 0.1 eV, the neutral-gas pressure p in the range 50 to 200 mTorr, the discharge current I in the range 0.5 to 3A, and the angular width of the spectrum $\theta_o = 5 \times 10^{-2}$. For the calculations, however, we need the values of the electron drift velocity and the charged-particle density rather than the discharge current. The values of these quantities for different pressures and currents were inferred from the earlier works of Ilić[13], and Ilić et al.[14], and are summarized in tables 1 and 2. Consistent with the temperature data we take the following values of the electron-neutral[15] and the ion-neutral[16] effective cross section, respectively : $\sigma_{en} = 5.6 \times 10^{-16} cm^2$, $\sigma_{in} = 7.7 \times 10^{-15} cm^2$.

TABLE 1

I = 3A		
p(mTorr)	v_d^*	$10^{-10}n(cm^{-3})$
50	0.29	3.4
100	0.21	4.7
150	0.18	5.6
200	0.15	6.4

TABLE 2

p = 100 m Torr		
I(A)	v_d^*	$10^{-10}n(cm^{-3})$
0.5	0.13	1.3
1	0.15	2.2
2	0.19	3.6

It is now easy to recognize that the numerical data used in Fig. 1 are typical of the experiment cited. The values of the electrostatic wave energy corresponding to this data, and to the observed angular width of the spectrum, are indicated in Figs. 2 - 9 by a circle. As can be seen in Fig. 2, the wave energy is about two orders of magnitude higher than that calculated from the one-dimensional model. Thus, even for such a narrow spectrum as that found in the experiment the angular width plays a crucial role in determining the saturation state. On the other hand, the effect of the Coulomb collisions appears to be negligible for these experimental conditions as can be seen in Figs. 7 and 9.

In order to compare the theoretical results with the experimental observations we have calculated the total wave energy for different neutral-gas pressures and discharge currents. The results are shown in Fig. 10. We see that the wave energy increases with the discharge current, and decreases with the neutral-gas pressure. If we consult Fig. 8 of Ref. 2 we find that the measured wave energy shows the same tendency. Moreover, the calculated values of the normalized wave energy compare favorably with those found in the experiment. They are typically 10^{-5} to 10^{-4}. As far as the average wavenumber of the spectrum is concerned we find in all calculations that $\bar{k}\lambda_D \approx 0.73$, which is the same value as in the observations.

In conclusion, we have shown that the experimental results obtained by Ilić[2] may be satisfactorily interpreted by means of a quasilinear theory generalized to include the effects of the collisions between the charged particles and the neutrals. We do not claim that the other interpretations advanced do not apply. However, ours seems to be the simplest one.

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Figure Captions

Fig. 1 Spectral distribution of the electrostatic wave energy for $\theta_o = 0$. The broken line represents the analytical expression (13). The parameters used are : $v_d^* = 2.10 \times 10^{-1}$, $\mu = 1.36 \times 10^{-4}$, $\eta = 50$, $\sigma_{en}^* = 9.50 \times 10^{-12}$, $\sigma_{in}^* = 1.31 \times 10^{-10}$, $g = 4.76 \times 10^{-5}$, $\delta = 1.31 \times 10^{-5}$.

Fig. 2 Electrostatic wave energy as a function of the angle θ_o. The solid line represents the modified formula (32) with $k_o^2 \lambda_D^2 = 0.5$ and the broken line the value of W_{elst} obtained from Eq. (13). The parameters used are the same as in Fig. 1. The circle indicates the value corresponding to the experiment (see Sec. IV).

Fig. 3 Electrostatic wave energy as a function of the electron drift velocity for $\theta_o = 0.05$. The other parameters used are the same as in Fig. 1. The solid line is the analytical expression.

Fig. 4 Electrostatic wave energy versus the electron-neutral effective cross section for $\theta_o = 0.05$. The other parameters used are the same as in Fig. 1. The solid line is the analytical expression.

Fig. 5 Electrostatic wave energy versus the ion-neutral effective cross section for $\theta_o = 0.05$. The other parameters used are the same as in Fig. 1. The solid line is the analytical expression.

Fig. 6 Electrostatic wave energy as a function of the ion temperature for $\theta_o = 0.05$. The other parameters used are the same as in Fig. 1. The solid line is the analytical expression.

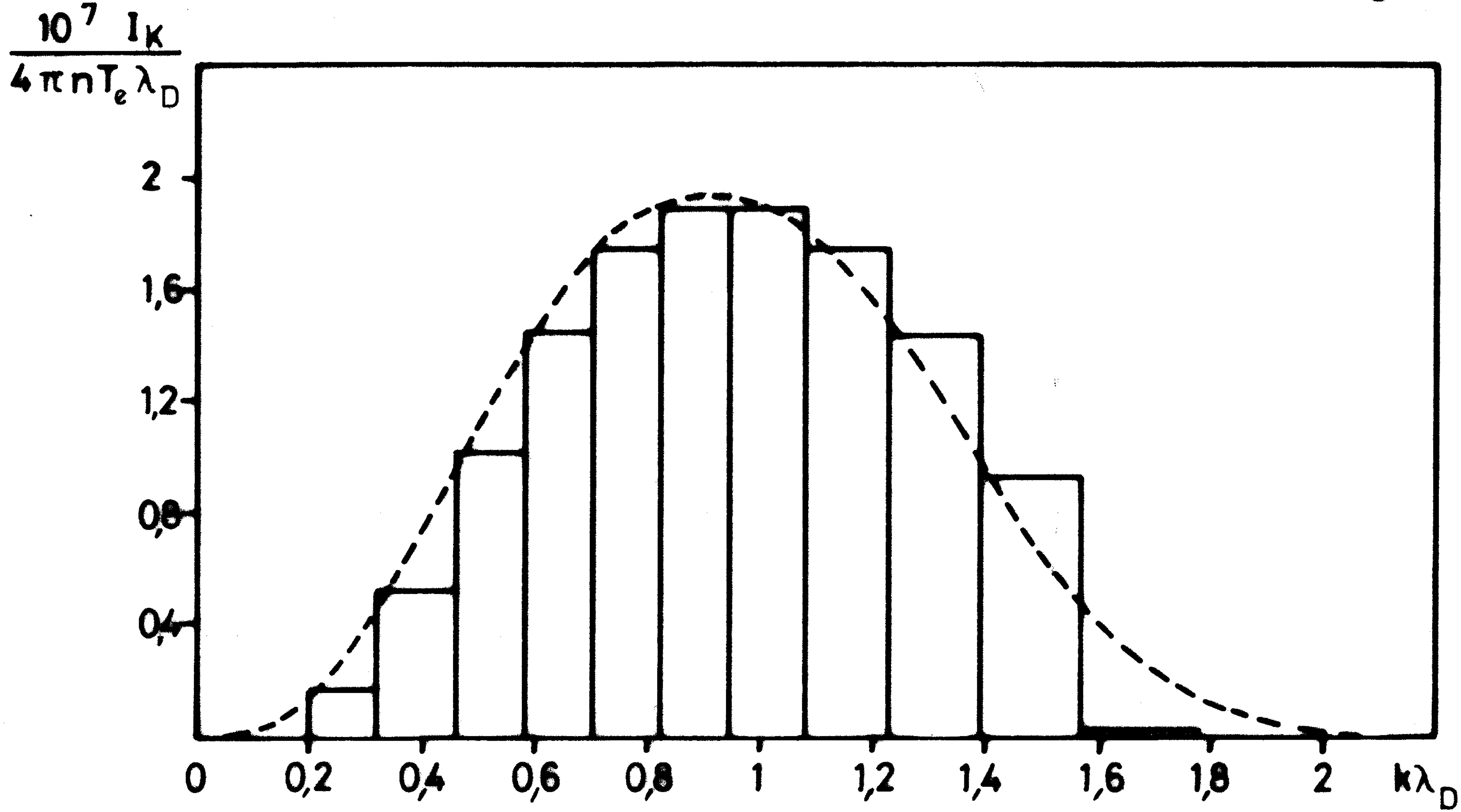
Fig. 7 Electrostatic wave energy versus the electron-neutral effective cross section in the case where the Coulomb collisions are taken into account (the triangles). The dots refer to the case without the Coulomb collisions. $\theta_o = 0.05$, and the other parameters used are the same as in Fig. 1.

Fig. 8 Electrostatic wave energy as a function of the angle θ_o (with Coulomb collisions included). The triangles and the dots correspond to $\delta = 1.31 \times 10^{-3}$ and $\delta = 1.31 \times 10^{-5}$, respectively. $\theta_o = 0.05$, and the other parameters used are the same as in Fig. 1.

Fig. 9 Electrostatic wave energy as a function of the degree of ionization (with Coulomb collisions included). $\theta_o = 0.05$, and the other parameters used are the same as in Fig. 1.

Fig. 10 Total wave energy versus the neutral-gas pressure and the discharge current.

Fig. 1



$\frac{10^7 I_K}{4\pi n T_e \lambda_D}$
2
1,6
1,2
0,8
0,4
0
0,2
0,4
0,6
0,8
1
1,2
1,4
1,6
1,8
2
$k\lambda_D$

Fig. 2

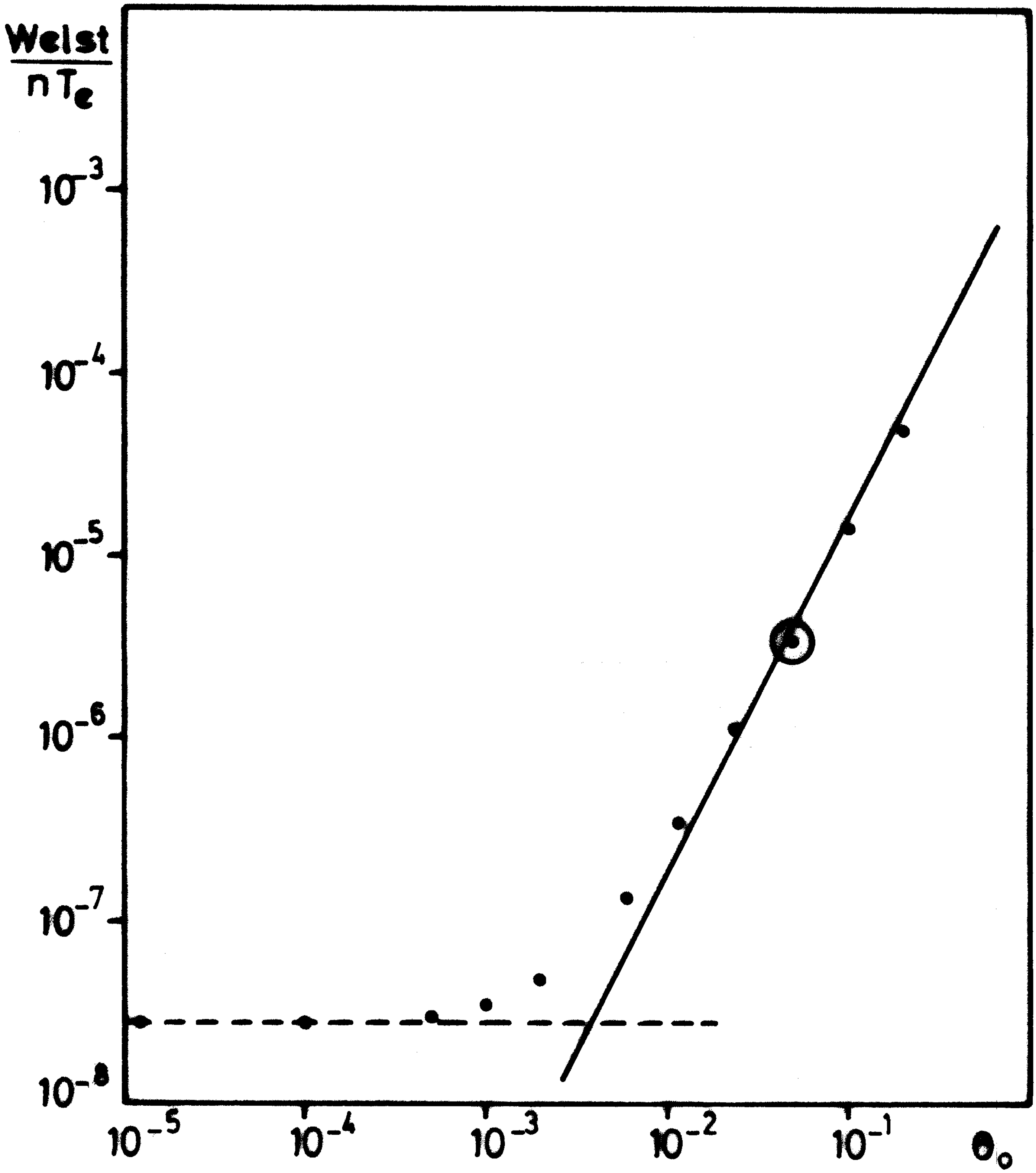


Fig. 3

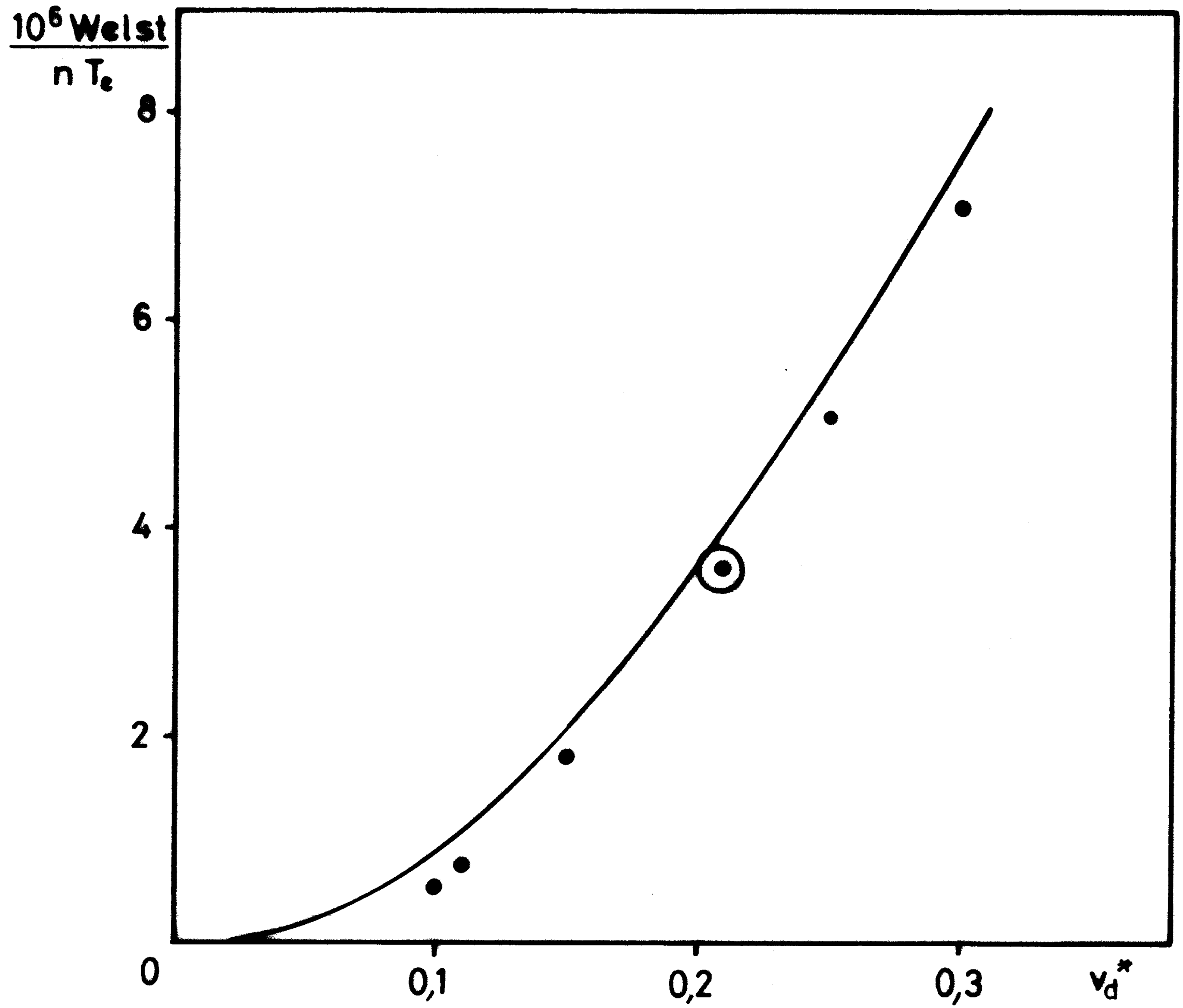


Fig. 4

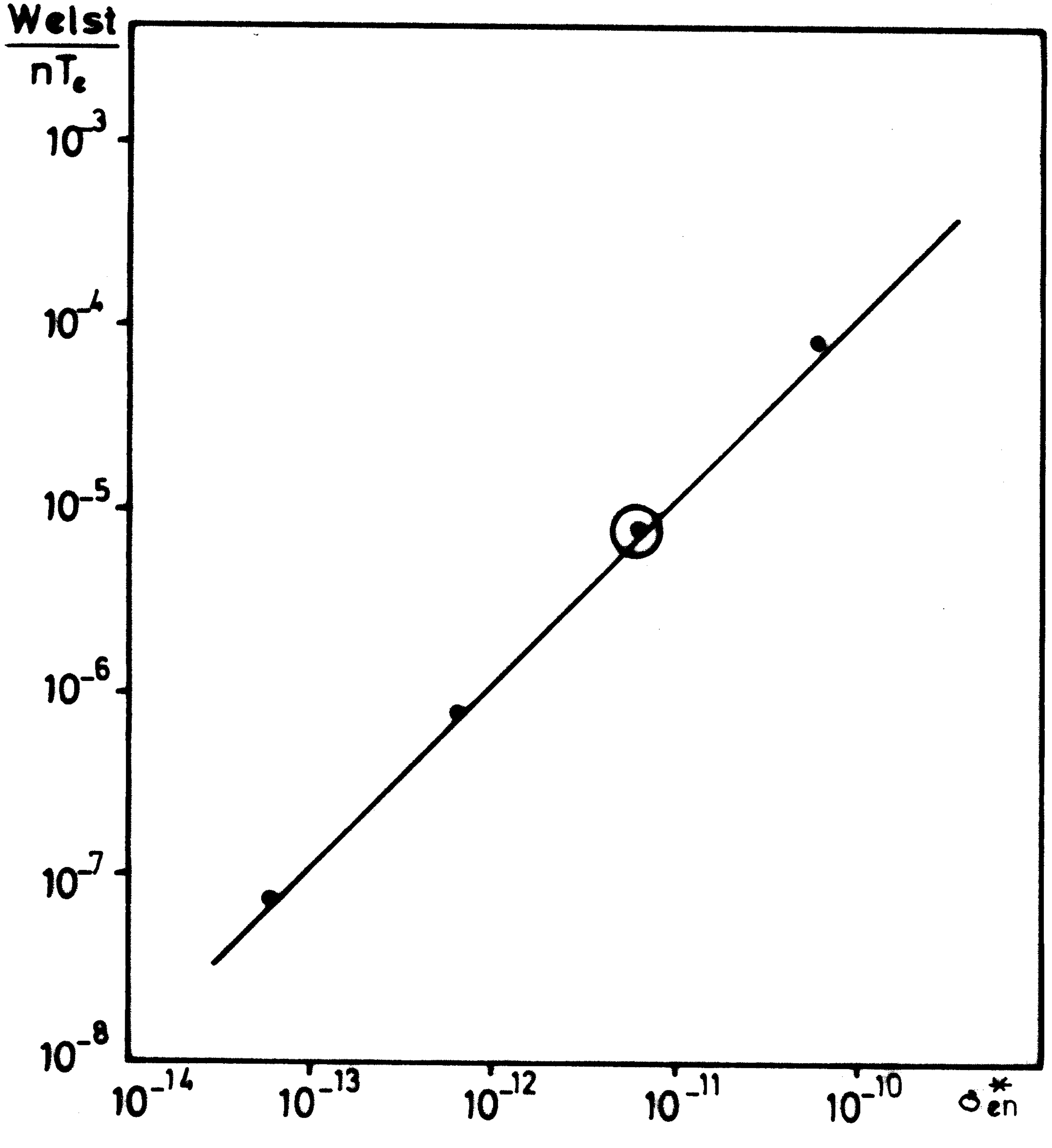


Fig. 5

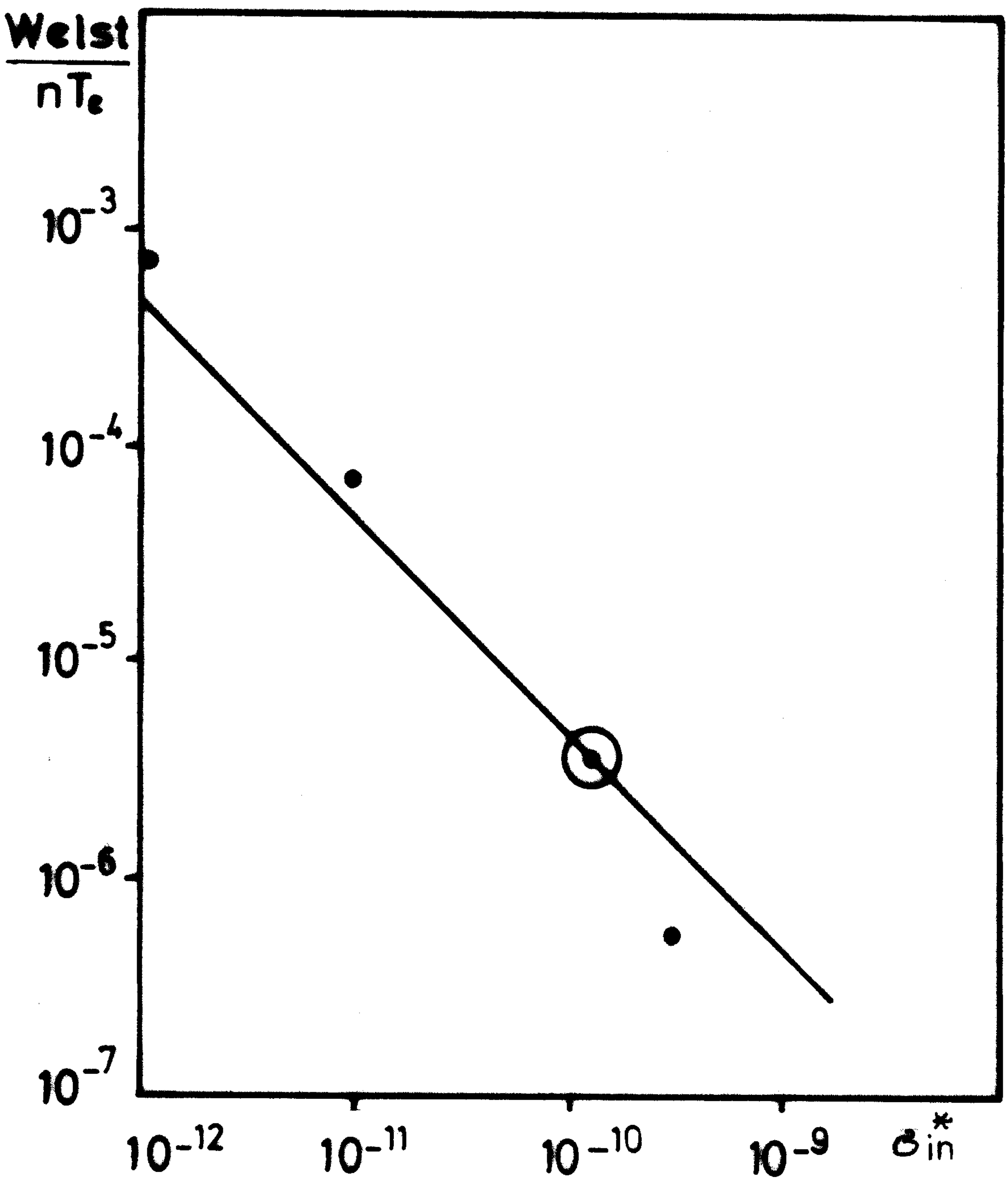


Fig. 6

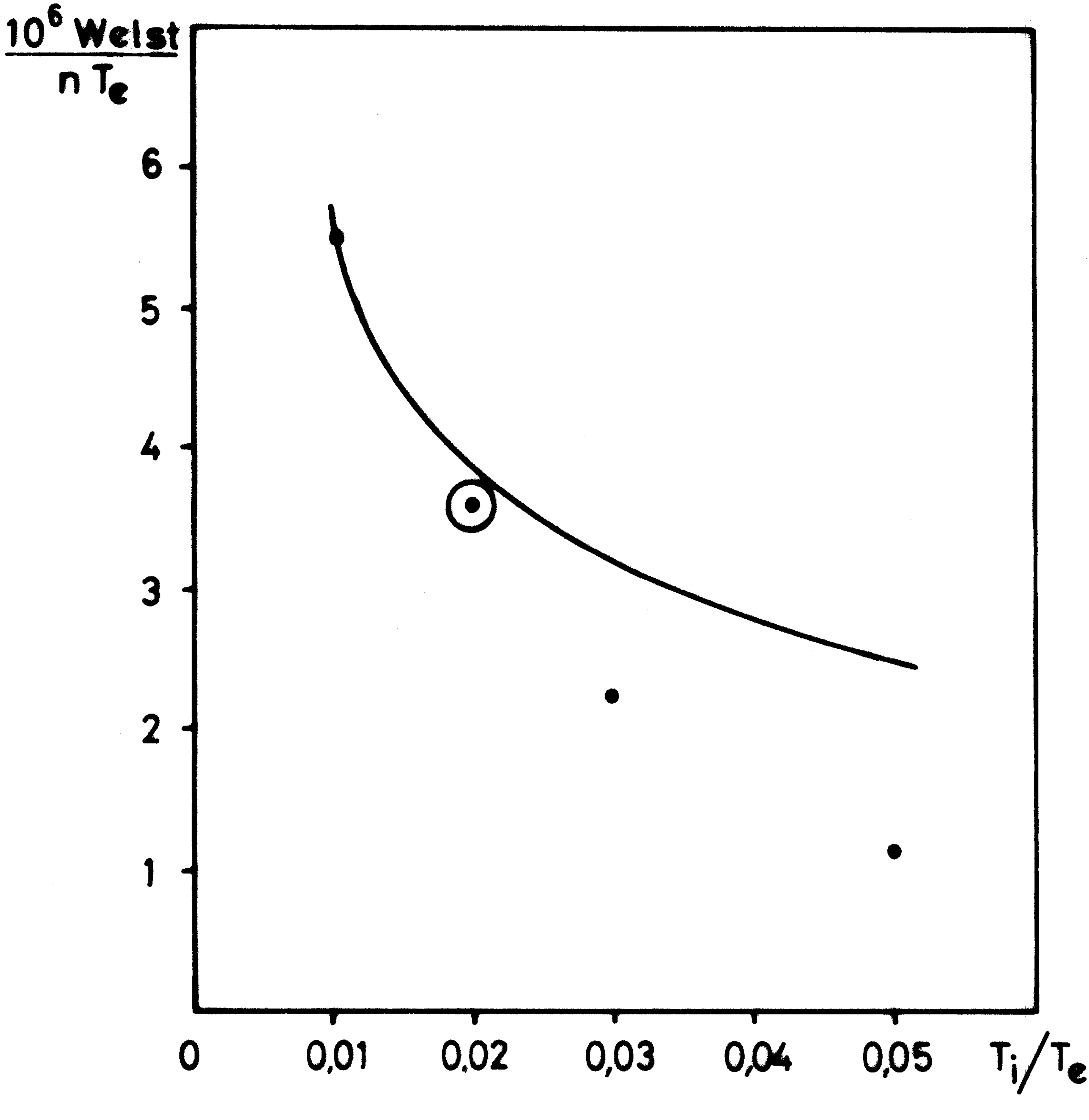


Fig. 7

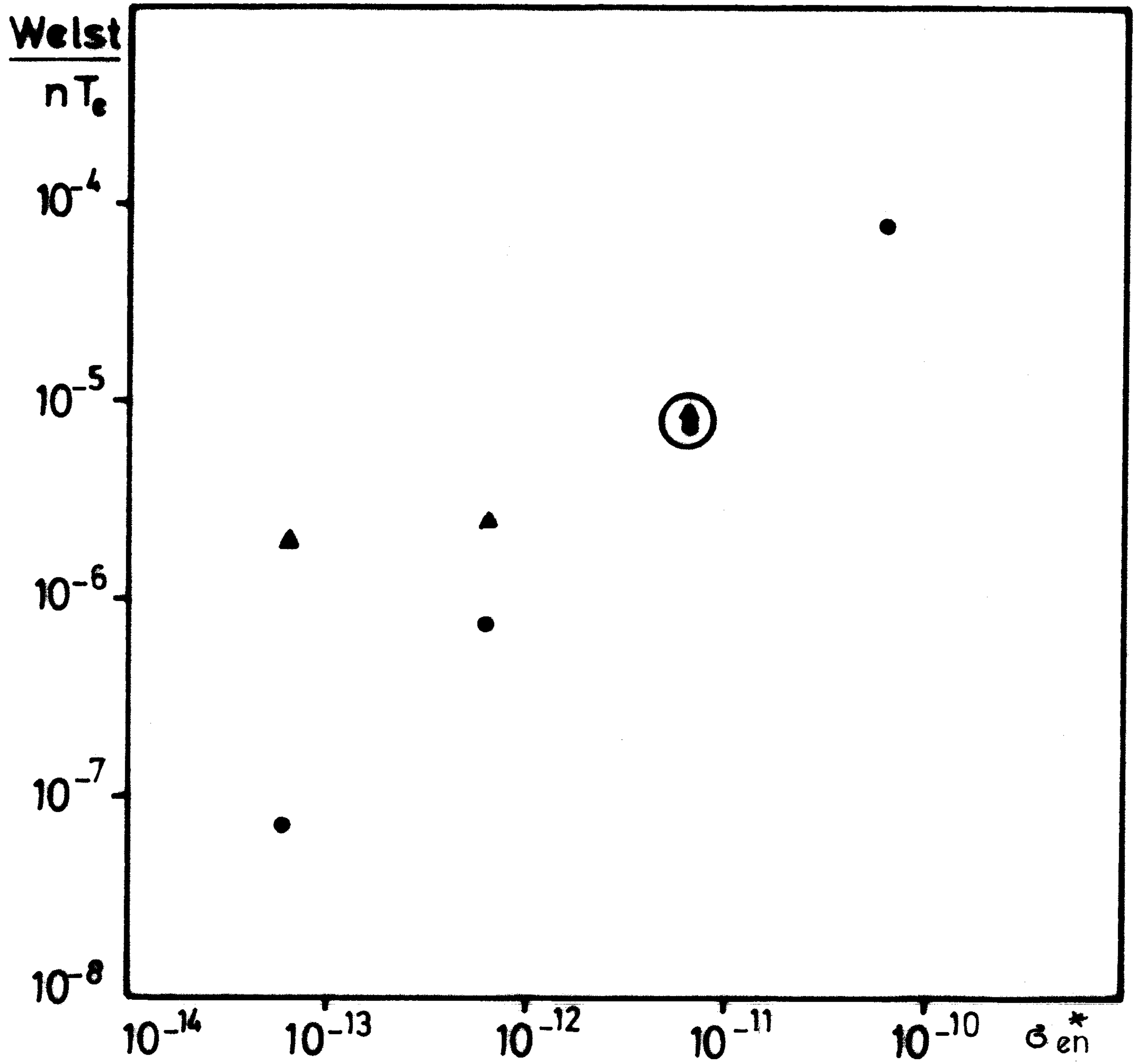


Fig. 8

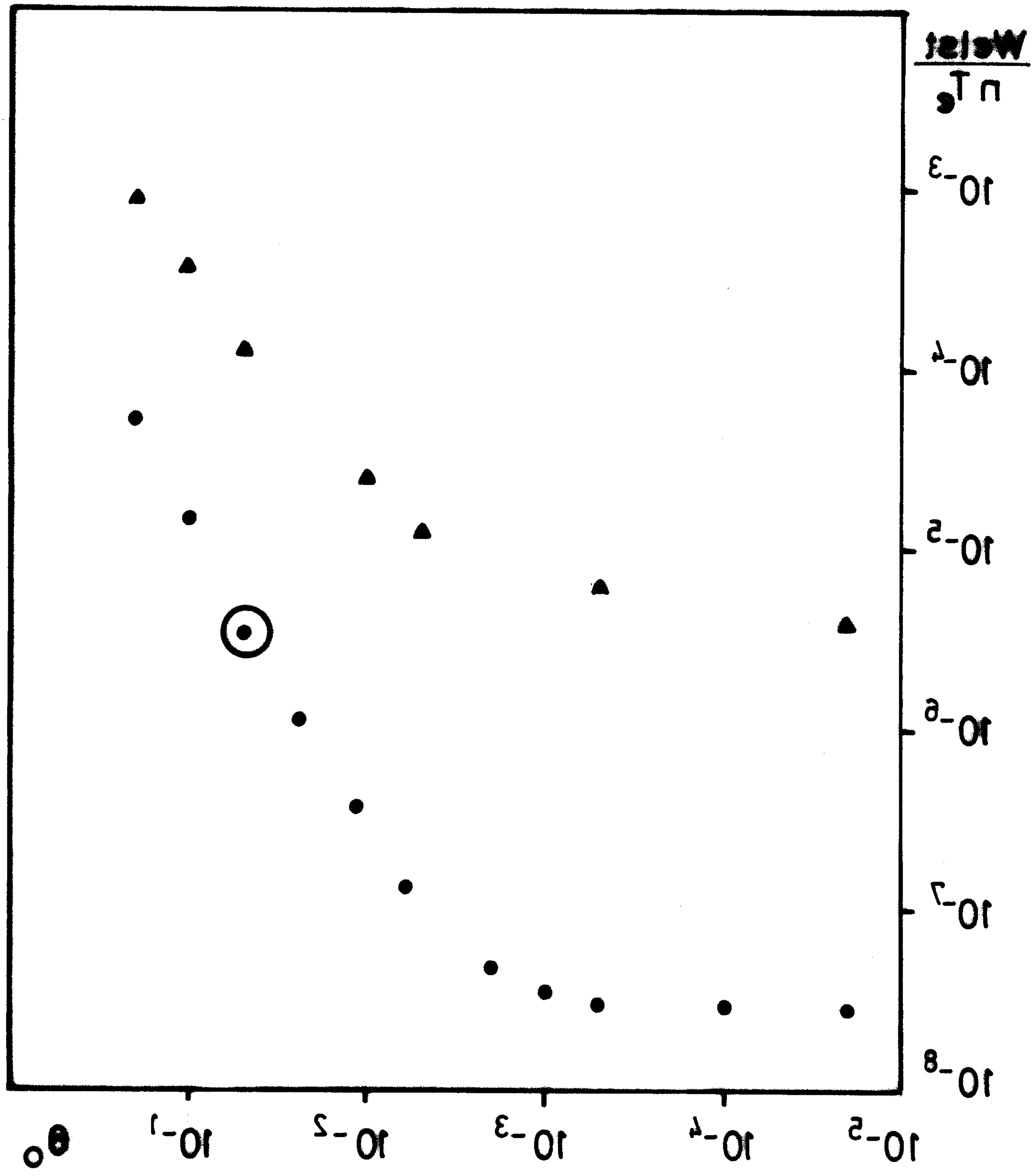


Fig. 9

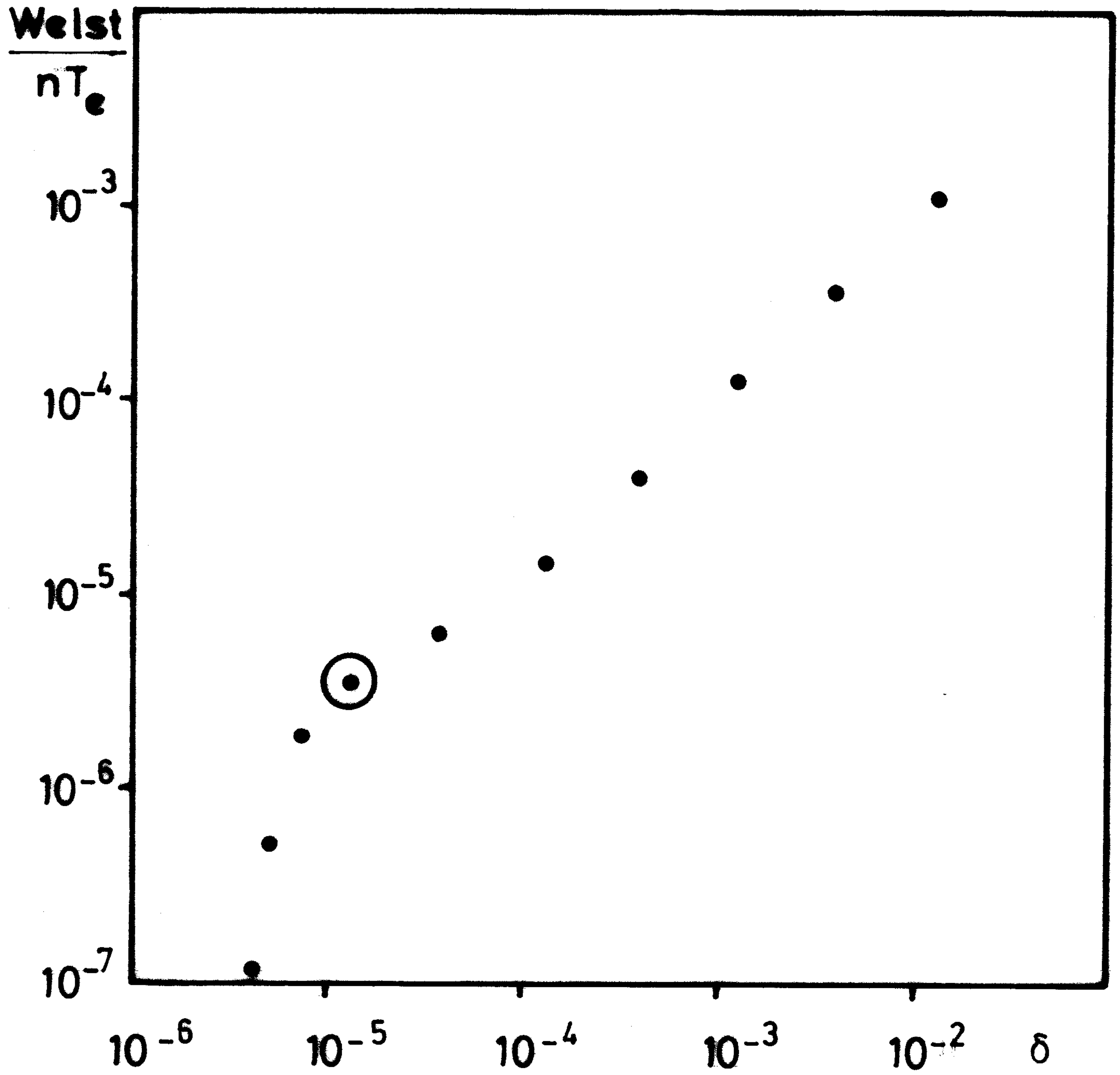


Fig. 10