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NONLINEAR DEVELOPMENT OF THE LANGMUIR MODULATIONAL INSTABILITY AND LANGMUIR SOLITONS

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1. INTRODUCTION

In this paper we shall present an exact non-linear analysis of the development of a long wavelength finite amplitude Langmuir wave in a uniform plasma. The initial Langmuir wave is chosen to satisfy conditions ($k_o \lambda_{De} < \gamma_e^{-1}(m_e/m_i)^{\frac{1}{2}}$ where λ_{De} is the electron Debye length and γ_e is the ratio of specific heats) such that the decay instability is forbidden. Under these conditions a long wavelength Langmuir wave, whose amplitude exceeds a certain threshold, is subject to the modulational instability which was discovered by Vedenov and Rudakov (1964).

The work described in this paper complements that of Morales and Lee (1976) who recently gave a very interesting analysis of a similar problem. Morales and Lee carried out a numerical analysis of the fields generated by a travelling electrostatic wave of finite spatial extent. In this work the driving (or pump) wave was assumed to remain constant in time but its amplitude was assumed to vary in some prescribed way (it was actually chosen to model a recent observation of Wong and Quon (1975)). The non-linear interaction between the perturbations generated by this pump wave were then computed numerically but the reaction of the perturbations on the pump wave was ignored. In our work we assume the pump wave amplitude is initially uniform but we take full account of pump depletion. The variation in the pump amplitude we obtain is due to the reaction of the perturbations on the pump wave.

Another feature of the analysis we shall present is in the method of treating the high frequency fields. Many authors describe the high frequency fields by a single envelope model using the non-linear Schrodinger equation (e.g. Zhakharov (1972), Rudakov (1973), Nishikawa, Lee and Liu (1975), and Nicholson and Goldman (1976)). However, in many situations of practical interest the pump wave and the perturbations occur at very different wave numbers. In view of this and the fact that the basic instability is a four wave interaction we describe the high frequency fields by three separate wave envelopes - the pump wave and two sidebands. These three wave envelopes are treated as physically distinct throughout the interaction. This method has recently been applied by us to the problem of the non-linear development of the filamentation of an electromagnetic wave which is also a four wave interaction. (The four wave interaction consists of two pump waves and two excited waves.) The above points are discussed more fully later.

2. MODEL AND DERIVATION OF THE NON-LINEAR EQUATIONS

We shall consider a uniform, infinite plasma in which a small but finite amplitude Langmuir wave is propagating. We shall restrict the analysis to one spatial dimension and the initial Langmuir wave is described through its electric field

$$\underset{\sim}{E}_{LO}(x,t) = \hat{\imath}_x\, \mathcal{E}_{LO}\, e^{i(k_o x - \omega_o t)} \tag{1}$$

Associated with the electric field $E_{LO}(x,t)$ is a density perturbation $n_{LO}(x,t)$ and an oscillating fluid motion $v_{LO}(x,t)$ where we have used the subscript L to distinguish n_{LO} from the equilibrium density of the plasma n_o. We assume that ω_o and k_o are related by the linear dispersion relation

$$\omega_o^2 = \omega_{pe}^2 + \gamma_e\, k_o^2\, v_{Te}^2 \tag{2}$$

where ω_{pe}, γ_e and v_{Te} are the electron plasma frequency, the ratio of specific heats of the electron fluid and the electron thermal velocity respectively.

The plasma model we use to analyse the problem is the two fluid isothermal approximation. We have chosen this model in the interests of simplicity. It gives an adequate description of the phenomena, at least for the initial stages of the non-linear behaviour, since the basic phenomena are non-resonant. However, it does leave out the important effect of particle trapping. We shall return to this point later.

The equations are as follows

$$\frac{\partial v_j}{\partial t} + v_j \frac{\partial}{\partial x} v_j + \gamma_j \frac{\kappa T_j}{n_j m_j} \frac{\partial}{\partial x} n_j = \frac{q_j}{m_j} E - \nu_j u_j \tag{3}$$

$$\frac{\partial}{\partial t} n_j + \frac{\partial}{\partial x} (n_j v_j) = 0 \qquad (4)$$

$$\frac{\partial E}{\partial x} - \frac{1}{\epsilon_o} \sum_j n_j q_j = 0 \qquad (5)$$

where $j = i$ or e, and we have immediately specialised to one spatial dimension - the x coordinate. Since we consider only longitudinal perturbations all the fields are also in the x direction. v_j, n_j, q_j, m_j, ν_j, γ_j and T_j are, respectively, the fluid velocity, density, charge, mass, phenomenological damping coefficient, ratio of specific heats and the temperature of the j^{th} species. E is of course the elctric field and k and ϵ_o are Boltzmann's constant and the dielectric coefficient of a vacuum (we use MKS units). As equations (3)-(5) are written above, they are fully non-linear and contain all the fields and perturbations of interest. For example, the electric field consists of a sum of the initial finite amplitude Langmuir wave plus both high and low frequency electric field perturbations. We now wish to consider the effect of the finite amplitude Langmuir wave on perturbations which may arise.

Suppose, for example, there is a low frequency density perturbation, of frequency and wave number (Ω, k_s), which are as yet unspecified. This low frequency density perturbation, which involves both ions and electrons, will beat with the initial Langmuir wave to produce Langmuir wave perturbations with wave numbers $k_o \pm k_s$. These Langmuir perturbations can in turn beat with the initial Langmuir wave to regenerate the assumed low frequency perturbation thus closing the feedback loop for the perturbations and giving rise to the possibility of instability. The above scheme is illustrated in Fig.1 which shows these perturbations related to the linear dispersion relation. This diagram is not to be interpreted too literally since although the Langmuir waves are only weakly perturbed by the interaction the low frequency waves can be strongly perturbed. As already mentioned in the introduction, the wave number of the initial Langmuir wave is chosen to satisfy the condition $k_o \lambda_{De} < \gamma_e^{-1}(m_e/m_i)^{\frac{1}{2}}$ such that the decay instability is forbidden. Under these conditions it is essential to allow for the coupling of both side bands ($k_o - k_s$ and $k_o + k_s$). We shall refer to these perturbations as the Stokes and anti-Stokes Langmuir waves.

We must now obtain the equations for the Stokes and anti-Stokes Langmuir waves. Since the initial finite amplitude wave is required to satisfy the condition $\epsilon_o|E_o|^2/n_o\kappa T_e \ll 1$, we may use a perturbation procedure to obtain these equations. The details of this are given elsewhere (Bingham and Lashmore-Davies (1976),(1977)). The method consists in expanding the equations for the Langmuir perturbations about the linear solutions $(\omega_{1,2}, k_o \mp k_s)$ where

$$\omega_{1,2}^2 = \omega_{pe}^2 + \gamma_e(k_o \mp k_s)^2 v_{Te}^2 \tag{6}$$

Introducing slowly varying amplitudes to describe the non-linear behaviour of both the Stokes and anti-Stokes Langmuir waves and the low frequency density perturbation

$$E_{L1,2}(x,t) = \mathrm{Re}\,\mathcal{E}_{L1,2}(x,t)e^{i(k_{1,2}x - \omega_{1,2}t)} \tag{7}$$

where $k_{1,2} = k_o \mp k_s$ and,

$$n_{es}(x,t) = \mathrm{Re}N_s(x,t)e^{ik_s x} \tag{8}$$

where n_{es} is the low frequency density perturbation of the electron fluid (we use this variable to describe the low frequency perturbation). For the long wavelengths $(k^2\lambda_{De}^2 \ll 1)$ with which we shall be concerned $n_{es} \approx n_{is}$. The amplitudes $\mathcal{E}_{L1,2}$ and N_s are required to vary slowly compared with the high frequency fields whose time scales are $\sim \omega_{pe}^{-1}$. Notice that we have not separated the linear time scale from the total time variation for the low frequency perturbation since, as already mentioned, this mode can be strongly perturbed. The equations for $\mathcal{E}_{L1,2}$ are then

$$\left(\frac{\partial}{\partial t} + v_1\frac{\partial}{\partial x} + \gamma_L\right)\mathcal{E}_{L1}(x,t) = \frac{-i\omega_o^2}{4n_o\omega_1} N_s^* \mathcal{E}_{Lo}\, e^{-i\delta_1 t} \tag{9}$$

$$\left(\frac{\partial}{\partial t} + v_2\frac{\partial}{\partial x} + \gamma_L\right)\mathcal{E}_{L2}(x,t) = \frac{-i\omega_o^2}{4n_o\omega_2} N_s \mathcal{E}_{Lo}\, e^{-i\delta_2 t} \tag{10}$$

where $\delta_{1,2} \equiv \omega_o - \omega_{1,2}$, $v_{1,2} \equiv \gamma_e k_{1,2} v_{Te}^2/\omega_{pe}$, $\gamma_L \equiv \nu_e/2$ and n_o is the equilibrium plasma density in the absence of all fields.

Equations (9) and (10) have been obtained by choosing only those fields which satisfy perfect k-matching. This results in the mismatch parameters δ_1 and δ_2. Only the dominant non-linear coupling term has been included in these equations. This term comes from the continuity equation and is $\frac{\partial}{\partial x}(n_{es} v_{Lo})$ where v_{Lo} is the electron fluid velocity associated with the initial Langmuir wave.

The equation for N_s can be obtained from equations (3)-(5) without expanding about the linear mode. Again, only the dominant non-linear term is included, which comes from the momentum equation for the electrons (the ion non-linearities are all negligible) and is $v_e \frac{\partial}{\partial x} v_e$. Again only those terms are included which satisfy perfect k-matching. The equation for N_s is

$$\left\{\frac{\partial^2}{\partial t^2} + c_s^2 \left(k_s - i\frac{\partial}{\partial x}\right)^2 + \gamma_s \frac{\partial}{\partial t}\right\} N_s(x,t) = - \frac{\epsilon_o k_s^2}{2m_i} \frac{\omega_o}{\omega_{pe}^2} \left(\omega_1 \mathcal{E}_{Lo} \mathcal{E}_{L1}^* e^{-i\delta_1 t} + \omega_2 \mathcal{E}_{Lo}^* \mathcal{E}_{L2} e^{i\delta_2 t}\right) \qquad (11)$$

where $c_s^2 \equiv \kappa T_e/m_i$.

Before considering the non-linear development of these perturbations and the initial wave let us first obtain the initial behaviour of the perturbations assuming the pump wave amplitude $\mathcal{E}_{Lo}$ remains constant. Using $\mathcal{E}_{L1}^* e^{-i\delta_1 t}$, $\mathcal{E}_{L2} e^{i\delta_2 t}$ and N_s as the amplitude variables, equations (9)-(11) become a set of linear differential equations with constant coefficients and can be solved in the usual way assuming a variation $\exp i(qx - \Omega t)$. The resulting dispersion relation is a quartic in Ω. However, we shall be concerned with the simpler case when $\Omega^2 \ll k_s^2 c_s^2$. If we also take $q = 0$ then the dispersion relation for Ω reduces to the simpler form

$$(\Omega - \delta_1 + i\gamma_L)(\Omega + \delta_2 + i\gamma_L) - \left(\frac{\delta_1 + \delta_2}{2}\right) \omega_o K = 0 \qquad (12)$$

where

$$K \equiv \frac{\epsilon_o |\mathcal{E}_{Lo}|^2}{4 n_o \kappa T_e}$$

Solving this equation we obtain the following threshold for instability

$$K = -[\gamma_L^2 + \Delta^2]/\omega_o\Delta \tag{13}$$

where $\Delta \equiv (\delta_1 + \delta_2)/2$, i.e. we have instability only when $\Delta < 0$. However, from the definitions of δ_1 and δ_2 we find that

$$\Delta \approx -\gamma_e k_s^2 v_{Te}^2/\omega_o \tag{14}$$

and so Δ is negative definite. When the threshold for instability is exceeded the real part of Ω is given by

$$\mathrm{Re}\Omega = (\delta_1 - \delta_2)/2 \tag{15}$$

Again, using the definitions of δ_1 and δ_2 we can write this as

$$\mathrm{Re}\Omega = \gamma_e k_o k_s v_{Te}^2/\omega_o \tag{16}$$

Since $\mathrm{Re}\Omega$ is the frequency of oscillation of the low frequency density perturbation we see that in the limit of an infinite wavelength pump wave the density perturbation is purely growing or non-oscillatory. This instability then becomes indistinguishable from the oscillating two stream mode as already noted by Lashmore-Davies (1975).

In the general case of finite k_o the instability excites a low frequency wave of frequency given by equation (15) and two high frequency Langmuir waves whose frequencies are shifted from their unperturbed values $\omega_{1,2}$ to $\omega_o \mp \frac{1}{2}(\omega_2 - \omega_1)$.

The growth rate γ resulting from equation (12) can be expressed as

$$\frac{\gamma}{\omega_o} = -\frac{\gamma_L}{\omega_o} + \kappa_s(K - \kappa_s^2)^{\frac{1}{2}} \tag{17}$$

where $\kappa_s \equiv (\gamma_e/2)^{\frac{1}{2}}(k_s v_{Te}/\omega_o)$ and we have used equation (14) in deriving equation (17). We can see from the expression for the growth rate that there will be a wave number k_{sm} at which the growth is a maximum. The maximum growth rate occurs for

$$\kappa_{sm} = \left(\frac{K}{2}\right)^{\frac{1}{2}} \tag{18}$$

where κ_s is defined below equation (17). The maximum growth rate is

$$\frac{\gamma_m}{\omega_o} = \frac{K}{2} - \frac{\gamma_L}{\omega_o} \tag{19}$$

It is of interest to calculate the range of unstable wave numbers k_s corresponding to γ falling to $\gamma_m/2$. For the pump wave only slightly above the minimum threshold

$$\frac{\Delta\kappa_s}{\kappa_{sm}} \approx \left(\frac{\delta K}{K_{min}}\right)^{\frac{1}{2}} \tag{20}$$

where $K = K_{min} + \delta K$ and $\delta K \ll K_{min}$. For K sufficiently close to threshold the range of unstable wave numbers will clearly be very small.

For pump amplitudes well above threshold (but not so far above to violate the condition $\gamma < k_s c_s$) the range of unstable wave numbers is given by

$$\frac{\Delta\kappa_s}{\kappa_{sm}} = \frac{\sqrt{3}}{2} \tag{21}$$

In this case, the band of unstable wave numbers is much broader but even so the spread of κ_s is only of the order $0.4\ \kappa_{sm}$.

Let us now calculate the ratio of k_{sm} to k_o. This is an important quantity since it determines whether all the high frequency waves occur in a narrow band centred on k_o or whether they occur in well separated regions of k_s-space.

Using equation (18) we obtain

$$\frac{k_{sm}}{k_o} = \left(\frac{2}{\gamma_e}\right)^{\frac{1}{2}} \frac{\omega_o}{k_o v_{Te}} \left(\frac{\epsilon_o|\mathcal{E}_{Lo}|^2}{8 n_o \kappa T_e}\right)^{\frac{1}{2}} \tag{22}$$

We can obtain an estimate of k_{sm}/k_o by using the condition on k_o, namely $k_o\lambda_{De} < \gamma_e^{-1}(m_e/m_i)^{\frac{1}{2}}$ and the threshold condition

$$\epsilon_o|\mathcal{E}_{Lo}|^2/4n_o\kappa T_e > \nu_e/\omega_o$$

For a neodymium laser created plasma at a temperature of 1keV, $k_{sm}/k_o > 1$ for all k_o and the ratio can be several times greater than unity. Similarly, for a carbon dioxide laser created plasma also at a temperature of 1 keV, $k_{sm}/k_o > 1$. Finally, for the experimental conditions of Wong and Quon (1975) $n_o \approx 5 \times 10^8\ cm^{-3}$, $T_e = 2$ eV and $E_o = 5$ V/cm, we also obtain $k_{sm}/k_o > 1$. This means that for many situations of practical importance the Stokes and anti-Stokes Langmuir waves are excited in bands of wave numbers which are physically quite distinct from k_o and each other. The Stokes and anti-Stokes waves will propagate at very different group velocities from the initial Langmuir wave and from each other. In particular the anti-Stokes wave will propagate in the opposite direction from the pump wave whereas the anti-Stokes wave will propagate in the same direction as the pump wave but significantly faster. Under these conditions a single envelope description of the high frequency waves (the non-linear Schrodinger equation) does not appear to be the most natural treatment. Instead, we shall use a model which describes the high frequency waves in terms of three physically distinct envelopes, one for the pump wave and one each for the Stokes and anti-Stokes Langmuir waves. We shall now consider the fully non-linear development of the initial Langmuir pump wave and the waves it excites.

3. NON-LINEAR SOLUTIONS

In order to close the non-linear system described by equations (9)-(11) we must add one more equation namely the equation for the pump wave. The pump wave is now treated on the same footing as the Stokes and anti-Stokes perturbations so that we no longer assume that it is large compared with the other high frequency waves. The equation for the pump wave is obtained in a similar manner to equations (9) and (10). The equation for $\mathcal{E}_{Lo}(x,t)$ is

$$\left(\frac{\partial}{\partial t} + v_o \frac{\partial}{\partial x} + \gamma_L\right) \mathcal{E}_{Lo}(x,t) = \frac{-i\omega_1^2}{4n_o\omega_o} N_s \mathcal{E}_{L1} e^{i\delta_1 t} - \frac{i\omega_2^2}{4n_o\omega_o} N_s^* \mathcal{E}_{L2} e^{i\delta_2 t} \tag{23}$$

where $v_o = \gamma_e k_o v_{Te}^2/\omega_{pe}$.

Equations (9)-(11) and (23) now form the non-linear system of equations we wish to solve. We again emphasize the idea behind this set of equations. $\mathcal{E}_{L1}$ and $\mathcal{E}_{L2}$ are the high frequency waves which are excited by the pump wave. These waves are chosen by imposing perfect k-matching and we take $k_s = k_{sm}$ so that we concentrate on those waves with the maximum growth rate. In practice other Stokes and anti-Stokes waves would be excited for neighbouring values of k_s. These waves could be described either by the addition of other pairs of waves $\mathcal{E}_{L3,4}$ and additional pairs of equations corresponding to these neighbouring values of k_s. Alternatively, these additional waves could be described by including a term in $\partial^2/\partial x^2$ in the equations for $\mathcal{E}_{L1}$ and $\mathcal{E}_{L2}$. This dispersive term would describe the spreading of the Stokes and anti-Stokes wave envelopes. Either of these additions would complicate our non-linear system enormously. Since $k_{sm}/k_o > 1$ and often $\gg 1$ we assume that the most important effect of dispersion has already been allowed for by the inclusion of three significantly different group velocities v_o, v_1 and v_2 for the three wave envelopes. Since the spread in k_s about k_{sm} is fairly small and $k_{sm}/k_o > 1$ this appears to be a reasonable first approximation.

In order to solve equations (9)-(11) and (23) we must make one further approximation. Equation (11) can be solved easily if we assume that $\partial^2/\partial t^2 \ll k_s^2 c_s^2$. This is equivalent to our previous assumption in obtaining the dispersion relation, given in equation (12), that $\Omega^2 \ll k_s^2 c_s^2$ and is referred to as the static approximation for the ions i.e. the ion response time is assumed infinitely fast and the low frequency density perturbation is a driven response given by

$$N_s(x,t) \approx -\frac{\epsilon_o \omega_o}{2\kappa T_e \omega_{pe}^2} \left(\omega_1 \mathcal{E}_{Lo} \mathcal{E}_{L1}^* e^{-i\delta_1 t} + \omega_2 \mathcal{E}_{Lo}^* \mathcal{E}_{L2} e^{i\delta_2 t}\right) \quad (24)$$

Substituting this expression into equations (9), (10) and (23) we obtain a reduced non-linear system

$$\left(\frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial x} + \gamma_L\right) \mathcal{E}_{L1}(x,t) = \frac{i\,\epsilon_o}{8 n_o \kappa T_e} \frac{\omega_o^2}{\omega_{pe}^2} \frac{\omega_o}{\omega_1} \Big(\omega_1 |\mathcal{E}_{Lo}|^2 \mathcal{E}_{L1} + \omega_2 \mathcal{E}_{Lo}^2 \mathcal{E}_{L2}^* e^{-i(\delta_1+\delta_2)t}\Big) \quad (25)$$

$$\left(\frac{\partial}{\partial t} + v_2 \frac{\partial}{\partial x} + \gamma_L\right) \mathcal{E}_{L2}(x,t) = \frac{i\,\epsilon_o}{8 n_o \kappa T_e} \frac{\omega_o^2}{\omega_{pe}^2} \frac{\omega_o}{\omega_2} \Big(\omega_1 \mathcal{E}_{Lo}^2 \mathcal{E}_{L1}^* e^{-i(\delta_1+\delta_2)t} + \omega_2 |\mathcal{E}_{Lo}|^2 \mathcal{E}_{L2}\Big) \quad (26)$$

$$\left(\frac{\partial}{\partial t} + v_o \frac{\partial}{\partial x} + \gamma_L\right) \mathcal{E}_{Lo}(x,t) = \frac{i\,\epsilon_o}{8n_o \kappa T_e} \frac{\omega_1^2}{\omega_{pe}^2} \left(\omega_1 |\mathcal{E}_{L1}|^2 \mathcal{E}_{Lo} + \omega_2 \mathcal{E}_{L1} \mathcal{E}_{L2} \mathcal{E}_{Lo}^* e^{i(\delta_1+\delta_2)t}\right)$$

$$+ \frac{i\,\epsilon_o}{8n_o \kappa T_e} \frac{\omega_2^2}{\omega_{pe}^2} \left(\omega_1 \mathcal{E}_{L1} \mathcal{E}_{L2} \mathcal{E}_{Lo}^* e^{i(\delta_1+\delta_2)t} + \omega_2 |\mathcal{E}_{L2}|^2 \mathcal{E}_{Lo}\right) \tag{27}$$

This is the set of equations we use to describe the non-linear development of the initial Langmuir wave and the perturbations it excites. It should be noted that similar coupling terms to those appearing in the above equations also arise from the second harmonic of the pump wave and the other high frequency fields. We have compared these terms with those arising from the low frequency density perturbation and have found them to be negligible. This result has also been noted by Nishikawa and Watanabe (1976). Equations (25)-(27) can be written in a simpler form in terms of the amplitudes

$$\alpha_o = \left(\frac{\epsilon_o}{2}\right)^{\frac{1}{2}} \frac{\omega_o}{\omega_{pe}} \mathcal{E}_{Lo}$$

$$\alpha_1 = \left(\frac{\epsilon_o}{2}\right)^{\frac{1}{2}} \frac{\omega_1}{\omega_{pe}} \mathcal{E}_{L1}\, e^{i\delta_1 t}$$

$$\alpha_2 = \left(\frac{\epsilon_o}{2}\right)^{\frac{1}{2}} \frac{\omega_2}{\omega_{pe}} \mathcal{E}_{L2}.\, e^{i\delta_2 t}$$

The significance of these new amplitudes is that $|\alpha_o|^2$ is the total wave energy density of the pump wave and $|\alpha_1|^2$ and $|\alpha_2|^2$ have the dimensions of energy density. In terms of α_o, α_1 and α_2 equations (25)-(27) become

$$\left(\frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial x} - i\delta_1 + \gamma_L\right) \alpha_1(x,t) = i\, \Gamma\omega_o(|\alpha_o|^2 \alpha_1 + \alpha_o^2 \alpha_2^*) \tag{28}$$

$$\left(\frac{\partial}{\partial t} + v_2 \frac{\partial}{\partial t} - i\delta_2 + \gamma_L\right) \alpha_2(x,t) = i\, \Gamma\omega_o(\alpha_o^2 \alpha_1^* + |\alpha_o|^2 \alpha_2) \tag{29}$$

$$\left(\frac{\partial}{\partial t} + v_o \frac{\partial}{\partial x} + \gamma_L\right)\alpha_o(x,t) = i\, \Gamma\omega_1(|\alpha_1|^2 \alpha_o + \alpha_1 \alpha_2 \alpha_o^*)$$

$$+ i\, \Gamma\omega_2(\alpha_1\alpha_2\alpha_o^* + |\alpha_2|^2 \alpha_o) \tag{30}$$

where $\Gamma \equiv 1/4n_o \kappa T_e$.

4. CONSERVATION RELATIONS

The total energy density of a wave in a medium, averaged over the period of oscillation and a wavelength depends on the values of ω and k in the medium. For a Langmuir wave in a uniform isotropic plasma we use equations (3)-(5) for the electron fluid to obtain the following expression for the total wave energy density

$$\epsilon_L = \left\{ \frac{1}{4} \epsilon_o |E_L|^2 + \frac{1}{4} n_o m_e |v_L|^2 + \frac{1}{4} \gamma_e \frac{\kappa T_e}{n_o} |n_L|^2 \right\}_{\omega,k} \tag{31}$$

Expressing n_L in terms of E_L from Poisson's equation and v_L in terms of E_L with the aid of the momentum equation we obtain

$$\epsilon_L = \frac{\epsilon_o}{4} \left\{ 1 + \frac{\omega_{pe}^2}{\omega^2} \frac{1}{\left(1 - \gamma_e \frac{k^2 v_{Te}^2}{\omega^2}\right)^2} + \gamma_e \frac{k^2 v_{Te}^2}{\omega_{pe}^2} \right\} \Bigg|_{\omega,k} |E_L|^2 \tag{32}$$

For the pump wave, we substitute the unperturbed values ω_o, k_o in equation (32) to obtain the usual result

$$\epsilon_{Lo} = \frac{1}{2} \epsilon_o |E_{Lo}|^2 \frac{\omega_o^2}{\omega_{pe}^2} \tag{33}$$

However, for the Stokes and anti-Stokes waves we must take account of the fact that the wave frequency is shifted by the interaction from $\omega_{1,2}$ to $\omega_o \mp \frac{1}{2}(\omega_2 - \omega_1)$. When this is done equation (32) yields the following expression for the total energy density of the Stokes and anti-Stokes waves

$$\epsilon_{L1,2} = \frac{\epsilon_o}{2} |E_{L1,2}|^2 \frac{\omega_{1,2}^2}{\omega_{pe}^2} \left[1 - \frac{(\delta_1 + \delta_2)}{2\omega_{pe}} \right] \tag{34}$$

We may now use these expressions to derive a conservation relation for the wave energy density. For the sake of simplicity we neglect the damping and the spatial derivatives in equations (28)-(30). It is then straightforward to obtain the following expression which shows that the wave energy density is conserved by the interaction

$$\frac{\partial}{\partial t}\left\{|\alpha_o|^2 + |\alpha_1|^2\left(1 - \left(\frac{\delta_1 + \delta_2}{2\omega_{pe}}\right)\right) + |\alpha_2|^2\left(1 - \left(\frac{\delta_1 + \delta_2}{2\omega_{pe}}\right)\right)\right\} = 0 \tag{35}$$

Using equation (35) we can then obtain the equation for conservation of wave action density

$$\frac{\partial}{\partial t}\left\{\frac{|\alpha_o|^2}{\omega_o} + \frac{|\alpha_1|^2}{\omega_1'}\left(1 - \left(\frac{\delta_1 + \delta_2}{2\omega_{pe}}\right)\right) + \frac{|\alpha_2|^2}{\omega_2'}\left(1 - \left(\frac{\delta_1 + \delta_2}{2\omega_{pe}}\right)\right)\right\} = 0 \tag{36}$$

where ω_1' and ω_2' are the perturbed frequencies defined above. Equation (36) illustrates the fact that the basic instability results from a four-wave interaction. In other words, if the Stokes and anti-Stokes waves both increase by one unit the pump wave will decrease by two units, i.e., two pump 'quanta' produce two excited 'quanta'.

The conservation of wave momentum density follows automatically from equation (36) since we have imposed perfect k-matching. We note that analogous conservation relations are obtained when we include the spatial derivatives and damping terms in equations (28)-(30).

5. TIME DEPENDENT, SPATIALLY INDEPENDENT SOLUTION

We will now give some exact analytic solutions of equations (28)-(30) for a number of special cases. So far we have been unable to solve these equations for their simultaneous evolution in space and time. At the present time no inverse scattering transform for a four wave interaction appears to exist. The first special case we consider is when all the waves are uniform in space and we solve for the time evolution of the system neglecting the spatial derivatives and also the damping terms (these are included later). We follow the method of Armstrong et al (1962). Writing the complex amplitudes $\alpha_n(t)$ as follows

$$\alpha_n(t) = a_n(t)e^{i\varphi_n(t)}; \quad n = 0, 1, 2$$

where $a_n(t)$ and $\varphi_n(t)$ are real functions of time, substituting into equations (28)-(30) and separating real and imaginary terms we obtain

$$\frac{\partial a_o}{\partial t} = \Gamma\,(\omega_1 + \omega_2) a_o\, a_1\, a_2 \sin\theta \tag{37}$$

$$\frac{\partial a_1}{\partial t} = -\,\Gamma\,\omega_o\, a_o^2\, a_2 \sin\theta \tag{38}$$

$$\frac{\partial a_2}{\partial t} = -\,\Gamma\,\omega_o\, a_o^2\, a_1 \sin\theta \tag{39}$$

$$\frac{\partial \varphi_o}{\partial t} = \Gamma\,(\omega_1\, a_1^2 + \omega_2\, a_2^2 + (\omega_1 + \omega_2)\, a_1\, a_2 \cos\theta) \tag{40}$$

$$\frac{\partial \varphi_1}{\partial t} = \Gamma\,\omega_o (a_o^2 + \frac{a_o^2\, a_2}{a_1} \cos\theta) + \delta_1 \tag{41}$$

$$\frac{\partial \varphi_2}{\partial t} = \Gamma\,\omega_o (a_o^2 + \frac{a_o^2\, a_1}{a_2} \cos\theta) + \delta_2 \tag{42}$$

where $\theta(t) \equiv 2\varphi_o(t) - \varphi_1(t) - \varphi_2(t)$. Equations (37)-(39) give rise to the following constants of the motion

$$2a_o^2 + \left(\frac{\omega_1 + \omega_2}{\omega_o}\right) a_1^2 + \left(\frac{\omega_1 + \omega_2}{\omega_o}\right) a_2^2 = W \tag{43}$$

$$a_o^2 + \left(\frac{\omega_1 + \omega_2}{\omega_o}\right) a_1^2 = n_1 \tag{44}$$

$$a_o^2 + \left(\frac{\omega_1 + \omega_2}{\omega_o}\right) a_2^2 = n_2 \tag{45}$$

$$a_2^2 - a_1^2 = n_3 \tag{46}$$

where W, n_1, n_2 and n_3 are constants defined by the above equations. Using equations (40)-(42) we obtain the following equation for $\theta(t)$

$$\frac{d}{dt}\,\theta(t) = -(\delta_1+\delta_2) + 2\Gamma\,(\omega_1 a_1^2 + \omega_2 a_2^2 - \omega_o a_o^2) + \cot\theta\,\frac{d}{dt}\,\ell n(a_o^2\, a_1 a_2) \tag{47}$$

Next, we make use of equations (38), (44), (46) and (47) to obtain the result

$$a_o^2\, a_1\, a_2 \cos\theta + \frac{(\delta_1+\delta_2)}{2\Gamma\omega_o}\, a_1^2 - \left(n_1 + \frac{\omega_2}{\omega_o}\, n_3\right) a_1^2 - \frac{\omega_o}{(\omega_1+\omega_2)}\, a_o^4 = \Lambda \tag{48}$$

where again Λ is a constant which is defined by the above equation. We shall take as the initial conditions $a_o(o) \gg a_2(o) \neq 0$ and $a_1(o) = 0$ we then find

$$\Lambda = -\frac{\omega_o}{(\omega_1+\omega_2)}\, n_1^2 \tag{49}$$

Calculating $\sin\theta(t)$ from equation (48), substituting it into equation (38) and using equations (44) and (46) we finally obtain an equation involving only a_1

$$\frac{da_1}{dt} = \Gamma\omega_o\,(|a|\; A^2B^2)^{\frac{1}{2}}\left[\left(1 - \frac{a_1^2}{A^2}\right)\left(1 + \frac{a_1^2}{B^2}\right)\right]^{\frac{1}{2}} \tag{50}$$

where

$$A^2 = \frac{b}{2|a|} + \frac{1}{2|a|}\,(b^2 + 4|a|c)^{\frac{1}{2}}$$

$$B^2 = -\frac{b}{2|a|} + \frac{1}{2|a|}\,(b^2 + 4|a|c)^{\frac{1}{2}}$$

and

$$a = \frac{(\omega_1^2 - \omega_2^2)}{\omega_o^2}\, n_3 + \frac{(\omega_1+\omega_2)}{\Gamma\omega_o^2}\,(\delta_1+\delta_2)$$

$$b = -\left[\frac{\omega_2}{\omega_o}\, n_3\left(2n_1\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_o}\, n_3\right) + \frac{(\delta_1+\delta_2)}{\Gamma\omega_o}\left(n_1 - \frac{\omega_2}{\omega_o}\, n_3 + \frac{\delta_1+\delta_2}{4\Gamma\omega_o}\right)\right]$$

$$c = n_3 n_1^2$$

The solution of equation (50) satisfying the initial condition $a_1(o) = 0$ is now obtained in terms of a Jacobi Elliptic function

$$a_1(t) = A\ \mathrm{cn}(K(k) - \beta t, k) \qquad (51)$$

where $K(k)$ is the complete elliptic integral of the first kind, $k(= A^2/(A^2 + B^2))$ is its modulus and $\beta = \{(A^2 + B^2)|a|\}^{\frac{1}{2}}\Gamma\omega_o$.

The corresponding solutions for the pump wave and anti-Stokes wave can be obtained using equation (51) and equations (44) and (46). The solutions are periodic in time and the pump and excited waves are one half a period out of phase. The Stokes and anti-Stokes waves reach their maximum amplitudes in a time given by $\tau_{max} = K(\kappa)/\beta$. This time dependent (spatially independent) solution is very similar to the one recently obtained by the present authors for the filamentation of an electromagnetic wave (Bingham and Lashmore-Davies (1977)).

To complete this solution we must calculate expressions for the phases. These can also be obtained explicitly and are

$$\varphi_1(t) = \left(\Gamma\omega_2 n_3 + \frac{(\delta_1 - \delta_2)}{2}\right) t + \varphi_1(o) \qquad (52)$$

$$\varphi_o(t) = \frac{\Gamma(\omega_o + \omega_1 + \omega_2)}{\omega_o} n_3\, t - \frac{(\delta_1 + \delta_2)}{2\omega_o} \frac{A^2 k'^2}{\beta n_1 \alpha^2} \left\{\pi(\varphi, \alpha^2, k) - F(\varphi, k)\right\} + \varphi_o(o) \qquad (53)$$

where

$$k' = \frac{1}{1 - k} \quad , \quad \alpha^2 = \left(\frac{\omega_o n k^2}{(\omega_1 + \omega_2)} + A^2 k'^2\right) \Big/ \left(\frac{\omega_o n_1}{\omega_1 + \omega_2}\right)$$

and $F(\varphi,k)$, $\pi(\varphi,\alpha^2,k)$ are the incomplete elliptic integrals of the first and third kind respectively. $\varphi_2(t)$ can now be obtained from equation (48).

We conclude this section by briefly noting that the density perturbation can be obtained using equation (24) and the above solutions and is

$$\frac{N_s(t)}{n_o} = - \frac{1}{n_o \kappa T_e} (a_o a_1 \cos(\varphi_o - \varphi_1) + a_o a_2 \cos(\varphi_o - \varphi_2) \quad (54)$$

By analogy with the filamentation calculation (Bingham and Lashmore-Davies (1977)) there will be a maximum density depletion when the Stokes and anti-Stokes waves reach a maximum and half a period later when the Stokes and anti-Stokes waves interfere destructively the density perturbation will pass through a maximum accumulation.

6. TIME DEPENDENT SOLUTION WITH DAMPING

If the initial value of the pump wave is close to the threshold for instability then it will not be a good approximation to neglect damping. However, for the spatially independent case discussed in the previous section, we have been able to include the effect of damping as follows. [Bingham and Lashmore-Davies (1977))]. Introducing the new amplitudes G_o, G_1 and G_2 into equations (28)-(30) through the transformation

$$\alpha_n = G_n e^{-\gamma_L t} \quad ; \quad n = 0, 1, 2$$

and transforming to the new time variable τ where $\tau \equiv \{1 - \exp(-2\gamma_L t)\}/2\gamma_L$ (Armstrong et al (1962)) the non-linear equations for G_o, G_1 and G_2 appear in exactly the same form as the equations describing the time dependent (spatially independent) problem without damping. It is only possible to solve the problem with damping in this manner because we have assumed that the damping rate for all high frequency waves is the same. This is a resonable approximation however, since in the long wavelength regime ($k_L \lambda_{De} \ll 1$) we consider, damping is due to collisions.

Having cast the problem with damping into the same form as in the previous section we may immediately write down the solution. To illustrate the effect that dissipation has we shall just give the solution for the Stokes wave $\mathcal{E}_{L1}$

$$|\mathcal{E}_{L1}| = A \, \mathrm{cn}\{K(k) - \frac{\beta}{2\gamma_L} (1 - e^{-2\gamma_L t}), k\} e^{-\gamma_L t} \quad (55)$$

where A, β, k and $K(k)$ have been defined in the previous section.

The solutions for the anti-Stokes and pump waves can be obtained, as before, with the aid of the conservation relations. The physical significance of equation (55) is that, the amplitude of the Stokes wave decays exponentially in time and simultaneously there is a lengthening of the non-linear period of oscillation. The period of oscillation tends towards infinity as the amplitude decays to zero. This behaviour has already been noted in another problem by the present authors (Bingham and Lashmore-Davies (1977)). A similar result has also been obtained by Nicholson and Goldman (1976) as a solution of the damped non-linear Schrodinger equation.

7. SPATIALLY DEPENDENT STATIONARY SOLUTIONS

We now look for stationary solutions to equations (28)-(30). In order to obtain the solutions we neglect the damping terms in the equations and assume the existence of a velocity u (to be determined) such that the complex wave amplitudes depend only on the new coordinate

$$\xi \equiv x - ut$$

These stationary solutions are obtained using the method already described for the time dependent problem discussed in Section 5. Thus, transforming equations (28)-(30) from x and t to ξ and t and introducing

$$\alpha_n(\xi,t) = a_n(\xi,t)e^{i\varphi_n(\xi,t)} \quad ; \quad n = 0,1,2$$

the stationarity condition is then imposed by requiring that

$$\frac{\partial a_n}{\partial t} = \frac{\partial \varphi_n}{\partial t} = 0$$

where a_n and φ_n are real functions of ξ. The set of equations for the a_n's and φ_n's is similar to equations (37)-(42) of Section 5, the only difference being a factor $V_n \equiv v_n - u$ which multiplies the first order derivatives with respect to ξ on the left hand side of the equations (v_n is the group velocity where $n = 0, 1,2$).

The following constants of the motion result from the equations for the a_n's

$$a_o^2 + \frac{V_1}{V_o} \frac{(\omega_1 + \omega_2)}{\omega_o} a_1^2 = \text{const} . \tag{56}$$

$$a_o^2 + \frac{V_2}{V_o} \frac{(\omega_1 + \omega_2)}{\omega_o} a_2^2 = \text{const} . \tag{57}$$

$$\frac{V_2}{V_o} a_2^2 - \frac{V_1}{V_o} a_1^2 = \text{const} \tag{58}$$

where the V_n's are defined above. Again, the phase functions always appear in the same combination, namely

$$\theta(\xi) \equiv 2\varphi_o(\xi) - \varphi_1(\xi) - \varphi_2(\xi)$$

Proceeding as in Section 5 we obtain the analogous equation to (48) involving $\cos\theta$

$$a_o^2\, a_1\, a_2 \cos\theta - \frac{1}{4}\frac{V_1}{V_o}\left[\frac{\omega_1}{\omega_o}\left(3 + \frac{V_1}{V_2}\right) + \frac{\omega_2}{\omega_o}\left(3\frac{V_1}{V_2} + 1\right)\right] a_1^4$$

$$- \frac{1}{2}\frac{V_1}{V_o}\left[2\frac{\omega_2}{\omega_o}\left(m_2 - \frac{V_1}{V_2} m_1\right) - \left(1 + \frac{V_1}{V_2}\right)\frac{V_o}{V_1} m_o\right.$$

$$\left. - \left(1 + \frac{V_1}{V_2}\right)\frac{(\omega_1 + \omega_2)}{\omega_o} m_1\right] a_1^2 + \frac{1}{2}\left(\frac{\delta_1}{V_1} + \frac{\delta_2}{V_2}\right)\frac{V_1}{\Gamma\omega_o} a_1^2 = \Lambda \tag{59}$$

where the constant Λ is defined by the above equation and the m_n are the values of the a_n^2 at $\xi = 0$ (where, of course, $n = 0, 1, 2$). We have chosen the origin in the ξ-coordinate to coincide with an extremum in a_1^2 (usually a maximum). This is obtained by choosing $\sin\theta(o) = 0$ and we have therefore chosen $\theta(o) = 0$. Λ can now be expressed in terms of the values the various quantities have at the origin. With the aid of equations (56), (58), (59) and the equation for a_1 (which is similar to equation (38) in Section 5) we finally obtain a single non-linear equation involving only a_1

$$\frac{du}{d\xi} = \pm \frac{2\Gamma\omega_o}{V_1} (\alpha\, u^4 + \beta\, u^3 + \gamma\, u^2 + \eta\, u + \epsilon)^{\frac{1}{2}} \tag{60}$$

where we have put $u \equiv a_1^2(\xi)$. The coefficients α, β, γ, η and ϵ can be expressed in terms of m_o, m_1, m_2, Λ, V_o, V_1, V_2 and δ with the aid of some straightforward but rather tedious algebra. For the sake of completeness we give these expressions which are as follows:

$$\alpha = D_4 + C_2^2$$

$$\beta = D_3 + 2C_1C_2$$

$$\gamma = D_2 + C_1^2 + 2\Lambda C_2$$

$$\eta = D_1 + 2\Lambda C_1$$

$$\epsilon = \Lambda^2$$

where

$$C_1 = \frac{V_1}{V_o}\frac{\omega_2}{\omega_o}\left(m_2 - \frac{V_1}{V_2} m_1\right) - \frac{1}{2}\left(1 + \frac{V_1}{V_2}\right) X - \left(\frac{\delta_1}{V_1} + \frac{\delta_2}{V_2}\right)\frac{V_1}{2\Gamma\omega_o}$$

$$C_2 = \frac{V_1}{4V_o}\left[\frac{\omega_1}{\omega_o}\left(3 + \frac{V_1}{V_2}\right) + \frac{\omega_2}{\omega_o}\left(3\ \frac{V_1}{V_o} + 1\right)\right]$$

$$D_1 = X^2 Y$$

$$D_2 = \frac{V_1}{V_o} X^2 - 2\frac{V_1}{V_o}\frac{(\omega_1 + \omega_2)}{\omega_o} XY$$

$$D_3 = \frac{V_1^2}{V_o^2}\frac{(\omega_1 + \omega_2)^2}{\omega_o^2} Y - \frac{2V_1^2}{V_o^2}\frac{(\omega_1 + \omega_2)}{\omega_o} X$$

$$D_4 = \frac{V_1^3}{V_o^3}\frac{(\omega_1 + \omega_2)^2}{\omega_o^2}$$

and

$$X \equiv m_o + \frac{V_1}{V_o}\frac{(\omega_1 + \omega_2)}{\omega_o} m_1$$

$$Y \equiv \frac{V_2}{V_o} m_2 - \frac{V_1}{V_o} m_1$$

Inoue (1975) has considered a model four-wave interaction for which he obtained an equation of the same form as (60). Inoue found four basic types of solution which we give below.

As noted by Inoue and many other authors, an equation of the form given by (60) is analogous to the motion of a particle subject to some non-linear force field. The amplitude u represents the position of the particle, the coordinate ξ the time and the potential energy of the particle is equivalent to the polynominal

$$P(u) \equiv \alpha u^4 + \beta u^3 + \gamma u^2 + \eta u + \epsilon$$

We will consider the solution of equation (60) when $\alpha < 0$ and when $\alpha > 0$.

Case I $\alpha < 0$

Equation (60) can be written in the form

$$\frac{du}{d\xi} = \pm K[(a - u)(u - b)(u - c)(u - d)]^{\frac{1}{2}} \tag{61}$$

where $K \equiv 2\Gamma\omega_o(|\alpha|)^{\frac{1}{2}}/V_1$ and a, b, c, d are the real roots of the equation

$$P(u) = 0 \tag{62}$$

The roots are ordered such that $a > b > c > d$. Equation (61) can be solved exactly, in terms of Jacobi Elliptic functions. The integrals arising in the solution of equation (61) are given by Byrd and Friedman (1971). We obtain

$$a_1^2(\xi) = \frac{a + \frac{d(a - b)}{(b - d)} \operatorname{sn}^2(G\xi,k)}{1 + \frac{(a - b)}{(b - d)} \operatorname{sn}^2(G\xi,k)} \tag{63}$$

where $G \equiv \Gamma\omega_o(|\alpha|)^{\frac{1}{2}}[(a - c)(b - d)]^{\frac{1}{2}}/V_1$ and $k^2 \equiv (a - b)(c - d)/(a - c)(b - d)$ is the square of the modulus of the Elliptic function. Clearly $a_1^2(\xi)$ is a periodic function which oscillates between its maximum value $a_1^2(0) = a$ at the origin and $a_1^2(\xi_{min}) = b$. This is the first of Inoue's four types of solution. $a_o^2(\xi)$ and $a_2^2(\xi)$ are also periodic in ξ and can be obtained from the solution for $a_1^2(\xi)$ and the two conservation equations (56) and (58.). The periodicity length ℓ of these solutions is given by $\ell = K(k)/G$.

In the limit when $\ell \to \infty$ these periodic solutions become solitary wave or pulse-like envelopes. This is the second type of solution of equation (61). It occurs when $k^2 \to 1$ i.e. when the roots b and c, of equation (62), merge into a double root. If, further $b = c = 0$, then a_1^2 tends asymptotically to zero as $\xi \to \pm \infty$.

The conditions for $u = 0$ to be a double root of equation (62) are clearly

$$\eta = 0 \quad \text{and} \quad \epsilon = 0$$

These conditions imply that

$$\Lambda = 0 \tag{64}$$

and either

$$\frac{m_2}{m_1} = \frac{V_1}{V_2} \tag{65}$$

or

$$\frac{V_1}{V_o} = -\frac{\omega_o}{(\omega_1 + \omega_2)} \frac{m_o}{m_1} \tag{66}$$

The physical significance of equations (64)-(66) is as follows. Firstly, the solitary wave solution for $a_1^2(\xi)$ follows from equation (63) if we put $b = c = 0$. The solution becomes

$$a_1^2(\xi) = \frac{ad}{[a - (a - d)\cosh^2 G\xi]} \tag{67}$$

We now have two types of solitary wave solutions according to whether we choose to impose the condition given by equation (65) or (66). The difference between these two types of solution is demonstrated when we use equations (56) and (58) to obtain the solutions for $a_o^2(\xi)$ and $a_2^2(\xi)$. When we impose equation (65) we obtain

$$a_o^2(\xi) = m_o + \frac{V_1}{V_o}\frac{(\omega_1 + \omega_2)}{\omega_o}[m_1 - a_1^2(\xi)] \qquad (68)$$

$$a_2^2(\xi) = \frac{m_2}{m_1} a_1^2(\xi) \qquad (69)$$

We see that, for this solution, the Stokes and anti-Stokes wave amplitudes have solitary wave envelopes which tend to zero as $\xi \to \pm \infty$. Correspondingly the pump wave forms an envelope hole which tends asymptotically to its maximum value as the solitary waves vanish. Using equation (65), we can obtain the following expression for V_1/V_o

$$\frac{V_1}{V_o} = \frac{2\frac{m_2}{m_1}}{\left(1 + \frac{m_2}{m_1}\right)} \qquad (70)$$

We can now obtain a relation between the maximum and minimum values of $a_o^2(\xi)$ with the aid of equations (68) and (70).

$$[a_o^2(\xi)]_{max} = m_o + \frac{2\frac{m_2}{m_1}}{\left(1 + \frac{m_2}{m_1}\right)} m_1 \qquad (71)$$

To complete this solution we choose a value for the ratio m_2/m_1 and then use equation (64) to determine m_o/m_1. The normalization of the amplitudes results from the conservation of energy. This gives

$$[a_o^2(\xi)]_{max} = \epsilon_o E_{Lo}^2/2n_o \kappa T_e \qquad (72)$$

where E_{Lo} is the initial pump wave amplitude. Thus, we obtain a family of solutions depending on the parameter m_2/m_1. The choice of m_2/m_1 immediately determines the value of the velocity u.

Now consider the second type of solitary wave solution as determined by equation (66). For this case, we obtain the following expressions for $a_o^2(\xi)$ and $a_2^2(\xi)$

$$a_o^2(\xi) = \frac{m_o}{m_1} a_1^2(\xi) \tag{73}$$

$$a_2^2(\xi) = m_2 + \frac{V_1}{V_2} [a_1^2(\xi) - m_1] \tag{74}$$

This time we see that it is the anti-Stokes wave which forms the envelope hole and the 'pump' develops into a solitary wave envelope. At first sight, this may seem surprising, however, since equations (28-30) treat all the waves on an equal footing and the anti-Stokes wave is the highest frequency wave in the system, it is perfectly reasonable that there should be a steady state solution in which the anti-Stokes wave appears as the pump wave. These two types of solitary wave solution are, of course, mutually exclusive. There is no suggestion that for a given set of physical conditions the pump wave and anti-Stokes wave will interchange roles during the interaction.

Using equation (66) we can calculate the quantity V_1/V_2

$$\frac{V_1}{V_2} = - \frac{\eta}{(\eta + 2)} \tag{75}$$

where

$$\eta \equiv \frac{\omega_o}{(\omega_1 + \omega_2)} \frac{m_o}{m_1}$$

Substituting equation (75) into equation (74) demonstrates that for this second type of solution the anti-Stokes wave always forms an envelope hole.

There is another periodic solution of equation (61), still for $\alpha < 0$. This is

$$a_1^2(\xi) = \frac{\left[c - \frac{b(c - d)}{(b - d)} \mathrm{sn}^2(G\xi,k)\right]}{\left[1 - \frac{(c - d)}{(b - d)} \mathrm{sn}^2(G\xi,k)\right]} \tag{76}$$

where G and k are defined below equation (63). As was noted by Inoue, this solution does not reduce to a solitary wave when $k = 1$. For this case a_1^2 is constant corresponding to a phase modulation of the envelope with the phase proportional to ξ. The behaviour of a_o^2 and a_2^2 is similar and can be obtained as before. Figures 2 and 3 represent the amplitudes and corresponding density perturbation using the solitary wave type solutions for the amplitudes and equation 24 for the density perturbation.

Case II, $\alpha > 0$

For this case, equation (60) can be written

$$\frac{du}{d\xi} = \pm K'[(a - u)(b - u)(u - c)(u - d)]^{\frac{1}{2}} \tag{77}$$

where $K' \equiv 2\Gamma\omega_o \alpha^{\frac{1}{2}}/V_1$ and the roots a, b, c and d are ordered in the same way as for Case I. Again, with the aid of Byrd and Friedman (1971) we can write down the solution of equation (77)

$$a_1^2(\xi) = \frac{\left[b - \frac{a(b - c)}{(a - c)}\, sn^2(G\xi,k)\right]}{\left[1 - \frac{(b - c)}{(a - c)}\, sn^2(G\xi,k)\right]} \tag{78}$$

where G and k are not the same as for case I and are given by $G \equiv \Gamma\omega_o\alpha^{\frac{1}{2}}[(a - c)(b - d)]^{\frac{1}{2}}/V_1$, $k^2 = (b - c)(a - d)/(a - c)(b - d)$. This solution is again periodic and when $k^2 = 1$ (i.e. $c = d$) it becomes a solitary wave, which for the special case $c = d = 0$ reduces to

$$a_1^2(\xi) = \frac{a\ b\ sech^2 G\xi}{(a - b\ tanh^2 G\xi)} \tag{79}$$

If the double root $c = d$ is finite, then the solitary wave amplitude tends azymptotically to a constant value, rather than zero. We again find the two types of solitary wave solutions as discussed for the case $\alpha < 0$.

We have so far obtained two general types of solution - periodic and solitary waves. As already mentioned, Inoue (1975) noted four general types. The third possibility occurs when the four roots of equation (62) occur as a pair of double roots one of which is $u = 0$. The conditions for this to occur are

$$\eta = \epsilon = 0$$

and

$$\beta^2 - 4\alpha\gamma = 0 \ .$$

Putting $a = b$ and $c = d = 0$ in equation (77) we then obtain

$$\frac{du}{d\xi} = - K'u(u - a) \tag{78}$$

where we have chosen the negative square root. Integrating this equation we obtain

$$a_1^2(\xi) = \frac{a\, e^{\kappa' a\xi}}{1 + e^{\kappa' a\xi}} \tag{79}$$

This time the envelope of the Stokes wave has a shock-like structure, since as ξ varies between $\pm\infty$ a_1^2 changes from one state to another one. We should emphasize that this shock-like structure, which is usually associated either with dissipative or turbulent processes, has been obtained in the absence of dissipation (the damping terms have been neglected) and for a coherent wave interaction. A similar result has also been obtained by Berkhoer and Zakharov (1970) for the interaction of electromagnetic waves with different polarizations.

We can again distinguish two types of solution according to whether a_o or a_2 behaves as the pump wave. The conditions for this are given, as before, by equations (65) and (66). We also note that the maximum value of $a_1^2(\xi)$ does not occur at the orgin, as was the case for the previous solutions. This means that Λ will be different although still determined by equation (59) ($\theta(o)$ will take on some value other than 0 or $n\pi$).

The fourth and final type of solution of equation (60) is obtained when the smallest root of equation (62) is $u = 0$ and the remaining three roots form a triple root. Under these conditions equation (77) becomes

$$\frac{du}{d\xi} = K'[u(a - u)^3]^{\frac{1}{2}} \tag{80}$$

The solution of this equation in terms of $a_1(\xi)$ is

$$a_1(\xi) = \pm \frac{K'\, a^{3/2}\, \xi}{(K'^2 a^2 \xi^2 + 4)^{\frac{1}{2}}} \tag{81}$$

Inoue (1975) has called this a phase jump solution, since $a_1(\xi)$

changes sign as ξ passes through the origin. $a_2(\xi)$ has a similar profile to $a_1(\xi)$ when equation (65) is satisfied while $a_0(\xi)$ forms an envelope solitary wave with its maximum at the origin. When equation (66) is satisfied a_0 and a_2 exchange roles.

8. DISCUSSION

We have presented a non-linear analysis of the development of a long wavelength finite amplitude Langmuir wave in a uniform isotropic plasma. The wave number of the driving (or pump) Langmuir wave was chosen such that only the modulational instability could occur. The analysis given is complementary to a similar study by Morales and Lee (1976). They also considered the perturbations which can be generated by a propagating Langmuir wave. Morales and Lee (1976) solved the space and time evolution of the excited waves numerically allowing for the non-linear interaction between the excited waves but neglecting the non-linear reaction of these fields on the driving wave. In this paper, we have given exact non-linear solutions for the system of excited waves and the pump wave. These solutions were obtained under the following conditions :

(a) spatially independent solutions with and without damping,
(b) stationary spatially varying solutions without damping.

Whereas Morales and Lee (1976) imposed a fixed spatial variation on the amplitude of their pump wave, chosen to match the experiment of Wong and Quon (1975), the variation (both temporal and spatial) in the pump amplitude in our case resulted from the non-linear interaction with the excited waves i.e. it was a result of pump depletion.

The model we have analysed in this paper appears to have some bearing on a recent numerical simulation by Matochkin and Buchelnikova (1977) who also analysed the evolution of a large amplitude Langmuir wave in a uniform plasma in one dimension. Matochkin and Buchelnikova (1977) found that the low frequency density perturbation changed from a moving to a stationary disturbance during the evolution of their system. This lead to the build-up of the excited high frequency fields to large values and later to the almost complete dissipation of these fields due evidently to particle trapping. These authors have suggested that the non-linear excitation of backward propagating Langmuir waves is one of the most important effects in the evolution of their model. This feature is explicitly taken account of in our calculation. Furthermore it is possible that particle trapping could be simultated in our model by the inclusion of a wave number dependent damping term. It would be of interest to discover whether this addition lead to behaviour of the kind observed by Matochkin and Buchelnikova. Another conclusion of Matochkin and Buchelnikova's simulation was that the electron non-linearities (which give rise to harmonics of the high frequency fields) did not affect the development of the instability. As already mentioned, this is an agreement with our analysis and with that of Watanable and Nishikawa (1976) but it is in conflict with some results of Khakimov and Tsytovich (1976) who found that electron non-linearities could prevent Langmuir wave collapse. These authors stress that the electron non-linearities they considered were essentially kinetic effects.

Wong and Quon (1975) have also observed the generation of stationary density perturbations from a travelling electrostatic field and resulting localized high frequency fields. Morales and Lee (1976) have accounted for these structures by assuming the existence of a pump wave whose amplitude is constant in time but varying in some prescribed manner in space. Neither the experiment of Wong and Quon (1975) nor the calculation of Morales and Lee (1976) showed the sudden dissipation of the high frequency fields observed by Matochkin and Buchelnikova (1977).

Finally, we note that Baumgartel and Saur (1977) have carried out a numerical calculation of a model very similar to the one we have analyzed. However, one important difference between their work and ours is that they assume the presence of an initial ion acoustic wave in addition to a high frequency field. In view of this they use two low frequency equations, one for the ion acoustic wave and one for a zero frequency mode. However, we believe that the low frequency response can be described with the aid of a single non-linear equation which will determine the kind of low frequency wave that will occur.

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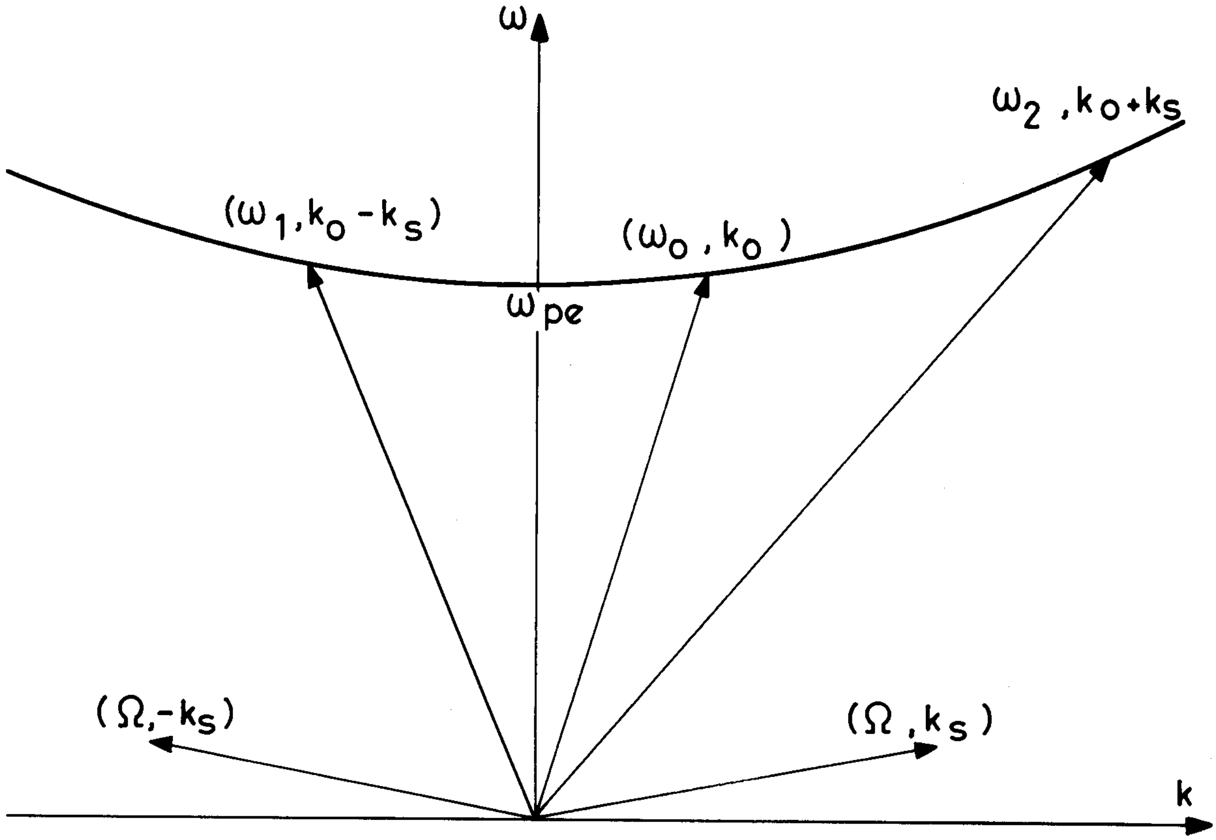

Figure 1. Linear dispersion diagram showing the coupling of the pump wave (ω_0, k_0) to the Stokes (ω_1, k_0-k_s) and the anti-Stokes (ω_2, k_0 + k_s) waves through the low frequency density perturbation (Ω, k_s).

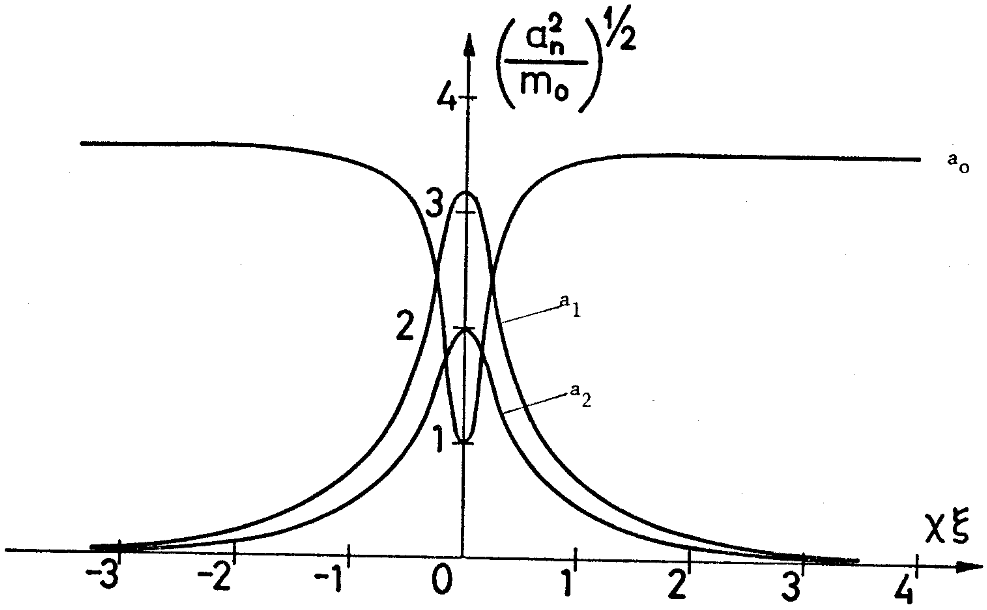

Figure 2. The wave amplitudes obtained from the solitary wave solution of case I for a typical low density unmagnetized plasma ($n_e = 10^{12} cm^{-3}$, $T_e = 10$ eV).

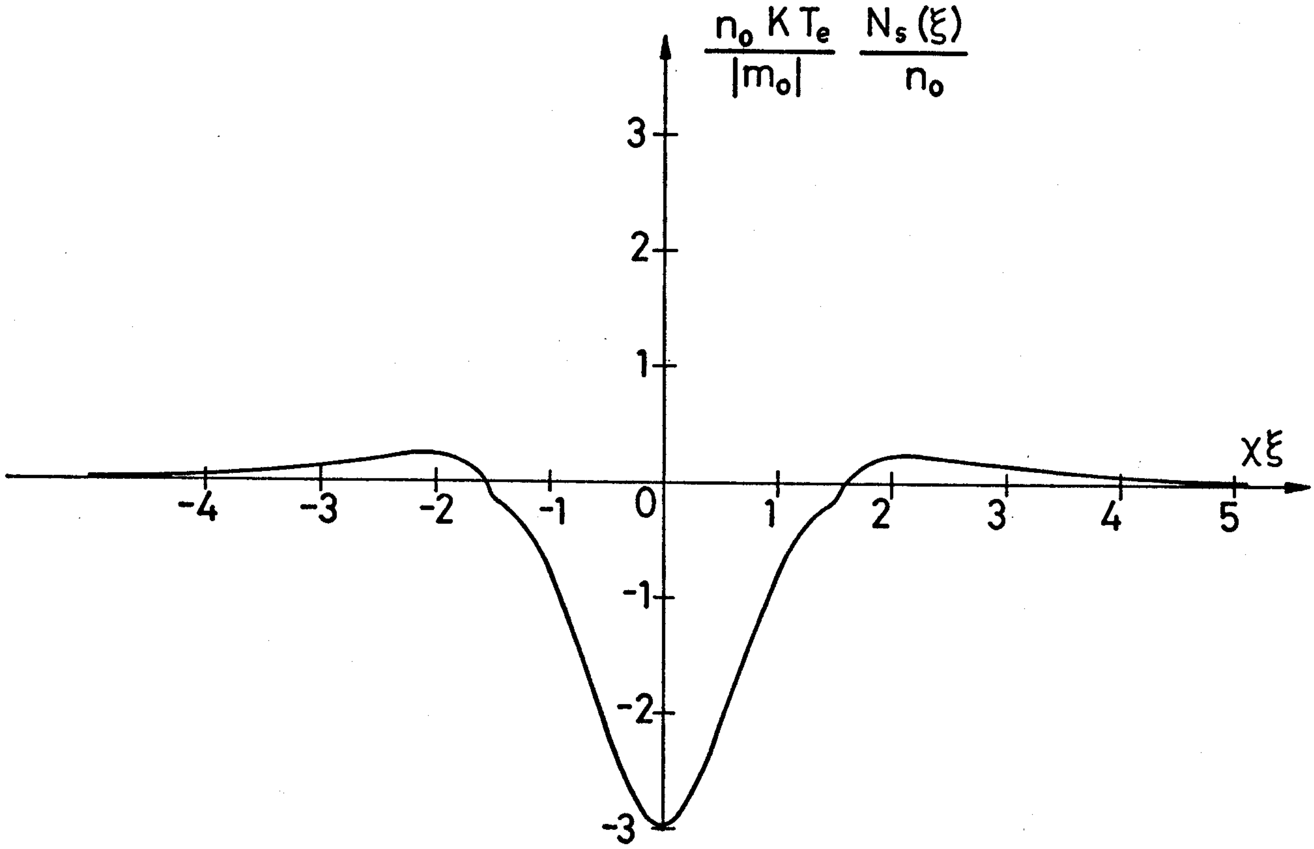

Figure 3. The density perturbation corresponding to the wave amplitudes in Figure 2. Note: $\chi = \Gamma\omega_0(a|\alpha d|)^{\frac{1}{2}}/V_1$.