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ABSTRACT

The stability of coupled Langmuir and ion-acoustic solitons has been investigated by means of numerical computations. Using the Zakharov equation to describe the envelope of the oscillating electric field, and the Korteweg-de-Vries equation with the ponderomotive driving term, to describe the low-frequency electron density variation, we found that (1) Langmuir waves and short scale sound waves do not affect the soliton, (2) two solitons destroy each other when colliding, and (3) a long scale sound wave or ion-acoustic soliton break up a coupled soliton in the interaction. Moreover, we did not find any initial condition far from a soliton state which would create a coupled soliton.

INTRODUCTION

It is widely recognized that the concept of weak turbulence is inapplicable to the theory of Langmuir turbulence associated with the heating of a plasma by high-current relativistic electron beams or powerful lasers. Rudakov[1] and Kingsep et al.[2] suggested that the strong Langmuir turbulence which arises in this case could be described by randomly distributed Langmuir solitons, which can interact with each other as well as with the plasma particles. These solitons are entities which store high-frequency electric fields in regions of local density depression. The latter are produced by the low-frequency ponderomotive force on electrons expelling them (and hence ions as well by ambipolar effects) from the region of strong field intensity. Formally, the solitons are one-dimensional localized stationary solutions of a Schrödinger-like equation for the envelope of the oscillating electric field, with an effective potential proportional to the low-frequency electron density perturbation. This perturbation in turn obeys an equation for linear ion-acoustic waves driven by the ponderomotive force[3].

It was pointed out by Nishikawa et al.[4] and Makhankov[5] that for solitons moving with a group velocity close to the ion-acoustic speed the nonlinearities and the dispersion of the ion-acoustic waves become important. In fact, the results of particle simulations performed by Pereira et al.[6] indicate that the ion dispersion and nonlinearities may have a significant effect even for slower solitons. In the simplest way, this effect can be taken into account if the low-frequency electron density perturbation satisfies a Korteweg-de-Vries[7] or Bousinesq[5] equation with the pondero-motive driving term, while the envelope of the electric field obeys the same Schrödinger-like equation as in the previous case. The equations of this model admit new types of soliton-like solutions which correspond to coupled Langmuir and ion-acoustic solitary waves - C solitons[4].

The main difference between a Langmuir soliton and a C soliton consists in the magnitude of their density depressions. The density depressions are proportional to the second and to the first power of the electric field amplitude, in the former and latter case respectively. It follows that for the same amount of trapped high-frequency energy the density hole of a C soliton is much deeper than that of a Langmuir soliton. Thus, the C soliton is actually a negative sound pulse loaded with a high-frequency field. In our opinion, this is a reason why the C soliton is a relatively "fragile" entity, since a negative sound pulse is no stationary solution of either the Korteweg-de-Vries or Bousinesq equations[8].

Although the dynamics of the formation and interaction of Langmuir solitons has been extensively investigated (see e.g., Degtyarev et al.[9], Pereira et al.[6]), at present little is known about the dynamical behaviour of C solitons. This problem is of great importance in connection with the previously mentioned model of strong turbulence. The solitons should be reasonably stable entities in order to be used as "quasi-particles" in that model. Recently it has been shown[10] that the C solitons are unstable against perturbations transverse to their motion.

The present paper is an attempt to answer, by means of computer calculations, a number of questions related to the dynamics and stability of one-dimensional C solitons. The plan of the paper is as follows : Section II gives a brief mathematical introduction leading to the numerical method discussed in Section III. Section IV discusses the results of the computations of different elementary processes, viz. the formation of a C soliton, the collision of two C solitons, the collision of a C soliton with an ion-acoustic soliton, and the interaction of a C soliton with ion-acoustic and Langmuir wave packets. To a great extent our conjecture concerning the fragility of the C solitons appears to be confirmed by the computations.

DYNAMICAL EQUATIONS AND CONSERVATION LAWS

The basic equations describing the nonlinear dynamics of one-dimensional Langmuir and ion-acoustic waves, in a system of coordinates moving at the ion-acoustic speed, can be given in the form[7]

$$i\varepsilon\left(\frac{\partial E}{\partial t} - \frac{\partial E}{\partial x}\right) + \frac{3}{2}\frac{\partial^2 E}{\partial x^2} - \frac{N}{2}E = 0\,, \tag{1}$$

$$\frac{\partial N}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial^2 N}{\partial x^2} + N^2 + |E|^2\right) = 0\,, \tag{2}$$

where N is the low-frequency density perturbation and E is the electric field of Langmuir oscillations with $\exp(-i\omega_{pe}t)$ factored out. Equations (1) and (2) are in dimensionless units; the units of time, space, density perturbation, and electric field are, respectively, $(\varepsilon\omega_{pe})^{-1}$, λ_D, n_o, and $4(\pi n_o T_e)^{1/2}$. Here ω_{pe}, λ_D, n_o, T_e, and ε^2 are, respectively, the plasma frequency, Debye length, average density, electron temperature, and the electron-to-ion mass ratio. Equation (1) is just the Schrödinger-like equation derived by Zakharov[3] whereas Eq. (2) is the Korteweg-de-Vries equation with the ponderomotive driving term.

The convective term in Eq. (1) is removed by the simple transformation $E \to E \exp\{i\varepsilon(x+t/2)/3\}$. Henceforth, instead of Eq. (1) we shall consider the equation

$$i\varepsilon \frac{\partial E}{\partial t} + \frac{3}{2}\frac{\partial^2 E}{\partial x^2} - \frac{N}{2}E = 0 . \tag{3}$$

It can easily be verified that the set comprising equations (2) and (3) has the following integrals of motion :

the number of plasmons

$$I_1 = \int |E|^2 dx , \tag{4}$$

the momentum of the oscillations

$$I_2 = \int\left\{N^2 + i\varepsilon\left(E\frac{\partial E^*}{\partial x} - E^*\frac{\partial E}{\partial x}\right)\right\}dx , \tag{5}$$

the energy of the oscillations

$$I_3 = \int\left\{3\left|\frac{\partial E}{\partial x}\right|^2 + N|E|^2 + \frac{1}{3}N^3 - \frac{1}{2}\left(\frac{\partial N}{\partial x}\right)^2\right\}dx , \tag{6}$$

and the number of particles

$$I_4 = \int N dx . \tag{7}$$

Equations (2) and (3) admit a stationary travelling localized solution - C soliton - given by[4]

$$E = -\frac{6}{5}3^{1/2}a \,\mathrm{sech}\,\xi \tanh\xi \exp\left\{ia\left[\left(\frac{3}{20\varepsilon} - \frac{\varepsilon a}{6}\right)t - \frac{\varepsilon x}{3}\right]\right\} \tag{8}$$

and

$$N = -\frac{9}{5} a \, \mathrm{sech}^2 \xi , \tag{9}$$

where $\xi = (a/10)^{\frac{1}{2}}(x+at)$, and a is a free positive parameter. Evidently, if E=0 equation (2) has a stationary solution in the form of a localized density hump - ion-acoustic (S) soliton :

$$N = 3a \, \mathrm{sech}^2 \left[(a/2)^{1/2} (x-at) \right] . \tag{10}$$

The substitution of the C soliton solution (8) and (9) into Eqs. (4) - (7) yields the following relations :

$$I_1^C \sim I_2^C \sim a^{3/2} , \; I_3^C \sim -a^{5/2} , \; I_4 \sim -a^{1/2} . \tag{11}$$

Similar relations can be obtained for the S soliton, and for ion-acoustic (SW) and Langmuir (LW) wave packets. If we now assume that only the C, S, SW and LW entities exist in the plasma, we can derive the selection rules for the interactions among them requiring that the quantities I_1 - I_4 be conserved in the interactions. An analysis of this kind was first time performed by Gibbons et al.[11] for a system consisting of Langmuir solitons and linear ion-acoustic waves. Considering, for example "two-body" interactions we find that the following processes are forbidden:

$$C_1 + C_2 \longleftrightarrow C_3 + X , \quad X = S, SW, LW. \tag{12}$$

These rules indicate that two C solitons cannot merge while emitting only one entity of type X. Thus, if two solitons collide they should either pass through one another or three entities at least should appear after the collision. Likewise, one soliton cannot be broken up into two solitons in a collision with only one of the X entities. It follows that a soliton and an X entity should either not interact in a collision or three entities at least should appear at the end of the process. In the same way one can analyze three-body interactions. However, the number of possible final states of a process increases in this case so that it is difficult to conclude what will actually happen. In our opinion one can answer these questions only by means of numerical computations.

THE NUMERICAL PROCEDURE

We have solved Eqs. (2) and (3) by a finite-difference method. The explicit three-time-level Zabusky-Kruskal[12] scheme was used for the density equation (2) :

$$(N_\ell^{j+1} - N_\ell^{j-1})\frac{1}{k} + (N_{\ell+2}^j - 2N_{\ell+1}^j + 2N_{\ell-1}^j - N_{\ell-2}^j)\frac{1}{2h^3}$$
$$+\Big[(N_{\ell+1}^j + N_\ell^j + N_{\ell-1}^j)(N_{\ell+1}^j - N_{\ell-1}^j)$$
$$+(|E|_{\ell+1}^j + |E|_\ell^j + |E|_{\ell-1}^j)(|E|_{\ell+1}^j - |E|_{\ell-1}^j)\Big]\frac{1}{3h} = 0, \qquad (13)$$

where ℓ and j number the spatial grid-points and the temporal levels, respectively : $x_\ell = \ell h$, $0 < \ell < L$; $t_j = jk$, $0 < j < J$. The electric field equation (3) was discretized according to a standard Crank-Nicholson[13] scheme

$$\frac{i\varepsilon}{k}\left(E_\ell^{j+1} - E_\ell^j\right) + \frac{3}{4h^2}\left(E_{\ell+1}^{j+1} - 2E_\ell^{j+1} + E_{\ell-1}^{j+1} + E_{\ell+1}^j - 2E_\ell^j + E_{\ell-1}^j\right) - \frac{1}{4}\left(N_\ell^{j+1}E_\ell^{j+1} + N_\ell^j E_\ell^j\right) = 0, \qquad (14)$$

which necessitates the solution of a linear system of algebraic equations.

Ideally one would like to solve equations (13) and (14) in infinite space. Since all the processes considered take place in localized regions of space we have imposed, for numerical purposes, the simplest boundary conditions :

$$N_0^j = N_1^j = N_L^j = E_0^j = E_L^j = 0\,. \qquad (15)$$

As regards the electric field, these conditions correspond to reflecting boundaries, whereas for the density they have no physical meaning since a sound wave or C soliton cannot be reflected at the left boundary, and an S soliton cannot be reflected at the right boundary. For this reason one must take care that the phenomena in question never approach the boundary. Without additional refinement to the numerical code, large grids, wasting the computer space, have to be used for long-lasting runs. Therefore the two following features were built into the code. Firstly, we have used a sliding grid moving with the "center-of-mass-velocity" of the process under consideration. The zero values of N and E were introduced at the front edge of the grid while the actual values of these quantities at the rear edge were discarded. Secondly, in some cases we allowed for an artificial damping in a small region near to the sliding edge, which permitted suppression of less localized, emitted sound or Langmuir waves.

The accuracy of the numerical approximation was checked by following the variations of the integrals of motion (4) - (7). The most pronounced variation was that of the energy integral, and was typically 1% for long runs. An additional test of the scheme was the propagation of a single C soliton. By comparison with the exact analytic solution (8) and (9) we found that the code preserves the amplitude, shape and speed of the soliton, whereas the description of the phase of the electric field is less accurate due to its fast variation.

RESULTS OF THE CALCULATIONS

In the first study we considered the collision of two solitons. We varied their relative velocities, and consequently the ratios of their amplitudes and widths. In all sets of computations we have observed essentially the same behaviour. A typical example is shown in Fig. 1 for the case where the soliton velocities are $a_1 = 0.45$ and $a_2 = 0.15$. One can see that as soon as the solitons start to overlap at $t = 160$ the Langmuir field flows out of the density well of the larger soliton. Since the effects of the ion dispersion ans nonlinearities are no longer balanced by the ponderomotive force the density hole of the larger soliton begins to emit sound waves. Precisely the same process occurs (Fig.2) when an isolated negative sound pulse breaks up into wave trains as known from the theory of the Korteweg-de-Vries equation[14]. Inasmuch as the characteristic time of this process is much shorter than that of the interpenetration of the solitons, in a later stage of the collision the emitted sound waves interfere with the density depression of the smaller soliton in a destructive way. In turn, the electric field leaks out also from this soliton. Thus, at the end of the observation time both solitons are broken up into series of sound trains, positive and negative sound pulses, and the Langmuir wave packets.

Next, we studied the collision of a C soliton with an S soliton. This process proceeds differently for different signs of the total density integral (7). Figure 3 displays the course of the collision in the case when $I_4 > 0$. We observe that as the S soliton passes through the C soliton the Langmuir field becomes untrapped due to a deformation of the density well. The S soliton reappears unchanged whereas the resulting isolated density hole spreads out into a sound train as was already discussed. If the total density integral is negative, both the C soliton and the S soliton are destroyed in the collision. These results provide additional evidence that the dynamics of a C soliton are essentially governed by the ion nonlinearities and dispersion as described by the Korteweg-de-Vries equation.

The interaction of a C soliton with a sound wave appears to be very much dependent on the ratio of the spatial scales of these entities. The sound wave packet with width smaller or comparable to that of the C soliton passes through the latter without affecting it. On the other hand, a large scale sound wave destroys the C soliton in a way very similar to that in which the S soliton does (Fig. 4).

Also we have investigated the collision between a C soliton and Langmuir wave packet. As can be seen from Fig. 5 these entities do not interact with each other. This fact indicates again that the sole of the Langmuir field is not very important in the dynamical behaviour of the C soliton.

Finally, having observed the fragility of the C soliton, we tried to establish under which initial conditions this entity can be formed at all. It turned out that the initial conditions must be "tailored" in a very special way. Figure 6 shows such an example. Here the initial state was chosen in the following manner. The electric field was given by formula (8), and the density by formula (9) where the amplitude "a" was replaced by "1.25a". We notice that the self-consistent shape of the soliton is established after the excess of the density depression is emitted in the form of a sound train. A similar behaviour was observed in the case when the initial state was represented by a C soliton with a slightly-deformed electric field. On the other hand, if the density well of the soliton is made shallower the barely-trapped Langmuir field starts to flow out almost immediately, and consequently the well decays into a wave train. As demonstrated in Fig. 7, the soliton is not restored in this case.

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REFERENCES

[1] L.I. Rudakov, Dokl. Akad. Nauk. SSSR 207, 821 (1972) {Sov.Phys.Dokl. 17, 1166 (1973)}.

[2] A.S. Kingsep, L.I. Rudakov and R.N. Sudan, Phys.Rev.Lett. 31, 1482 (1973).

[3] V.E. Zakharov, Zh.Eksp.Teor.Fiz. 62, 1745 (1972) {Sov.Phys. JETP 35, 908 (1972}.

[4] K. Nishikawa, H. Hojo, K. Mima, and H. Ikezi, Phys.Rev.Lett. 33, 149 (1974).

[5] V.G. Makhankov, Phys.Lett. 50A, 42 (1974).

[6] N.R. Pereira, R.N. Sudan, and J. Denavit, Phys.Fluids 20, 271 (1977).

[7] H. Ikezi, K. Nishikawa, H. Hojo, and K. Mima, Proc. 5th Intern.Conf. on Plasma Phys. and Contr.Nucl.Fusion Res., Tokyo (1974) IAEA-CN-33/H4-3.

[8] G.B. Whitham, Linear and nonlinear waves (Wiley, New York, 1974)

[9] L.M. Degtyarev , V.G. Makhankov, and L.I. Rudakov, Zh.Eksp.Teor.Fiz. 67, 533 (1974) {Sov.Phys.JETP 40, 264 (1975)}.

[10] K. Appert and J. Vaclavik, to be published.

[11] J. Gibbons, S.G. Thornhill, M.J. Wardrop, and D. ter Haar, J. Plasma Phys. (1977), in press.

[12] N.J. Zabusky and M.D. Kruskal, Phys.Rev.Lett. 15, 240 (1965)

[13] G.D. Smith, Numerical Solution of Partial Differential Equations (Oxford University Press, London, 1965).

[14] C.S. Gardner, J.M. Greene, M.D. Kruskal, and R.M. Miura, Phys.Rev.Lett. 19, 1095 (1967).

FIGURE CAPTIONS

Fig. 1 The evolution of the density perturbation (solid line) and the amplitude of the electric field (broken line) for a collision between two C solitons, as a function of x at different times. The initial parameters are : $a_1 = 0.45$ and $a_2 = 0.15$.

Fig. 2 The evolution of the density perturbation for an isolated negative sound pulse of the form given by Eq. (9), as a function of x at different times. The initial parameter is $a = 0.45$.

Fig. 3 The evolution of the density perturbation (solid line) and the amplitude of the electric field (broken line) for a collision between a C soliton and S soliton, as a function of x at different times. The initial parameters are : $a_c = 0.15$ and $a_s = 0.3$.

Fig. 4 The evolution of the density perturbation (solid line) and the amplitude of the electric field (broken line) for an interaction between a C soliton and large scale sound wave, as a function of x at different times. The initial amplitude and wave number of the sound wave are : $N_o = 0.2$ and $k_o = 0.05$, respectively, with $a_c = 0.45$.

Fig. 5 The evolution of the density perturbation (solid line) and the amplitude of the electric field (broken line) for a collision between a C soliton and Langmuir wave packet, as a function of x at different times. The initial amplitude, width, and wave number are $|E|_o = 0.025$, $\Delta = 40$, and $k_o = 0.33$, respectively, with $a_c = 0.15$.

Fig. 6 The evolution of the density perturbation (solid line) and the amplitude of the electric field (broken line) for a C soliton with a deeper density well (by the factor 1.25), as a function of x at different times. The initial parameter is a = 0.45.

Fig. 7 The evolution of the density perturbation (solid line) and the amplitude of the electric field (broken line) for a C soliton with a shallower density well (by the factor 0.8), as a function of x at different times. The initial parameter is a = 0.45.

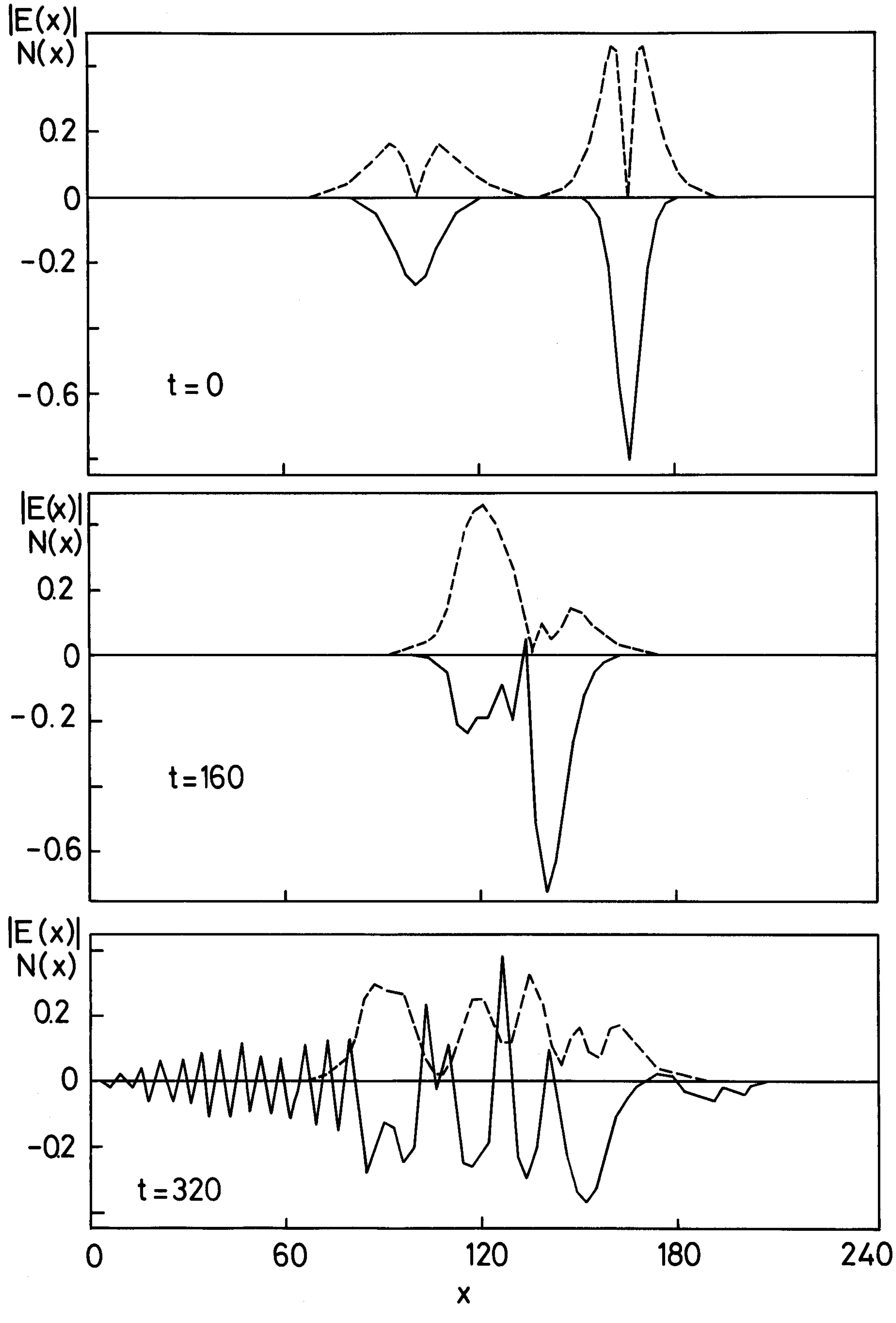

|E(x)|
N(x)
0.2
0
-0.2
-0.6
t = 0
|E(x)|
N(x)
0.2
0
-0.2
-0.6
t=160
|E(x)|
N(x)
0.2
0
-0.2
t=320
0
60
120
180
240
x

FIG. 1

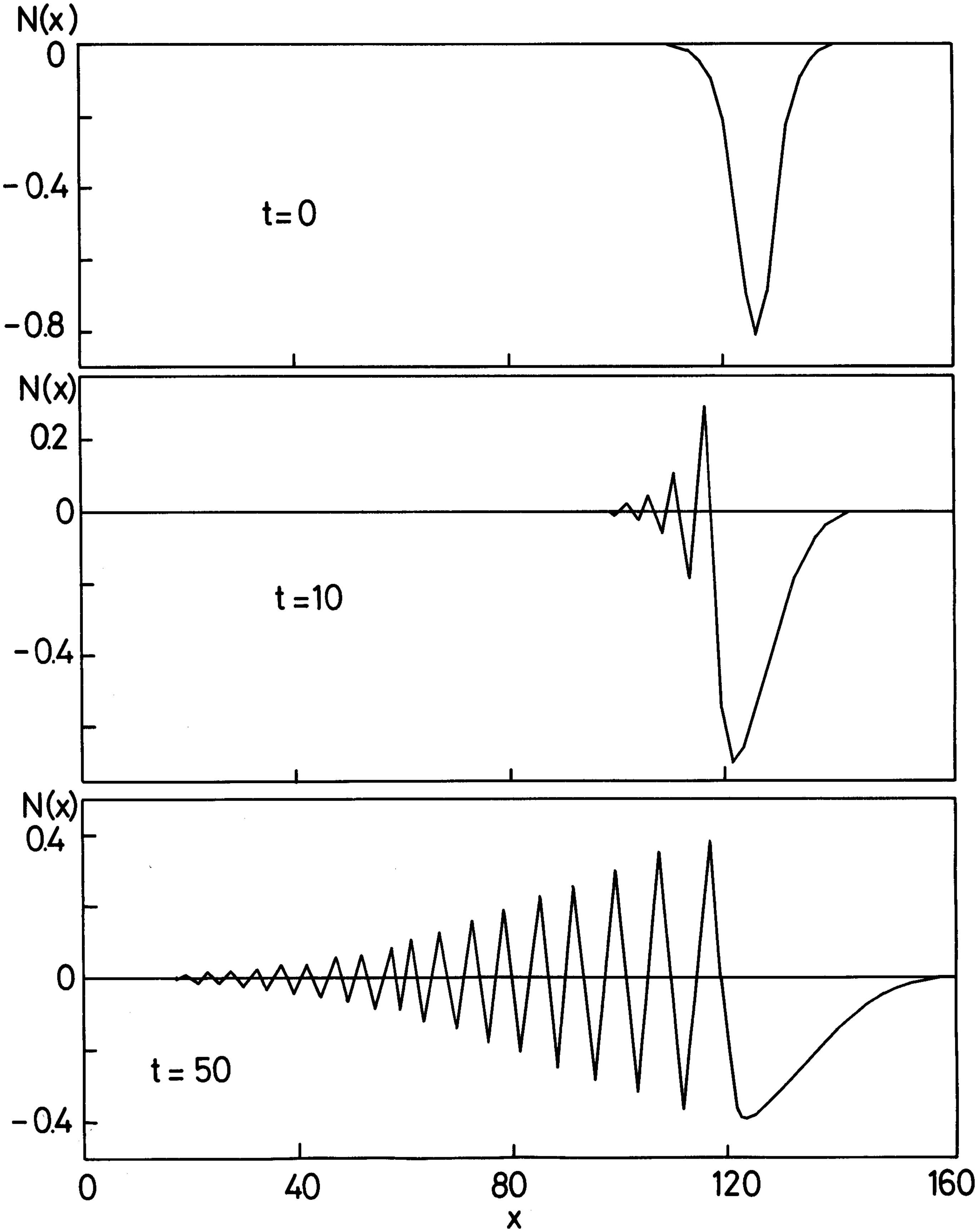
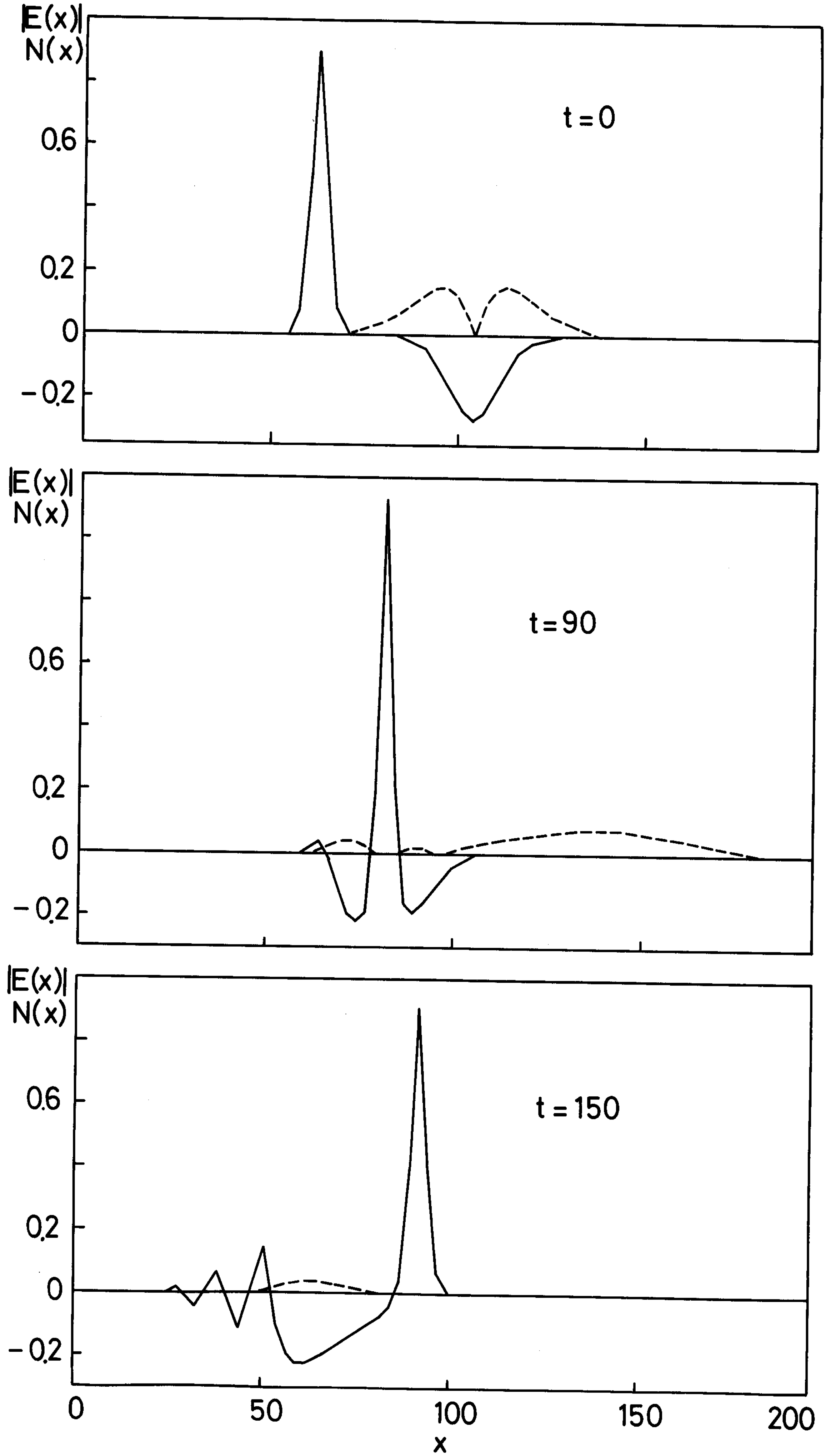


FIG. 2


|E(x)|
N(x)
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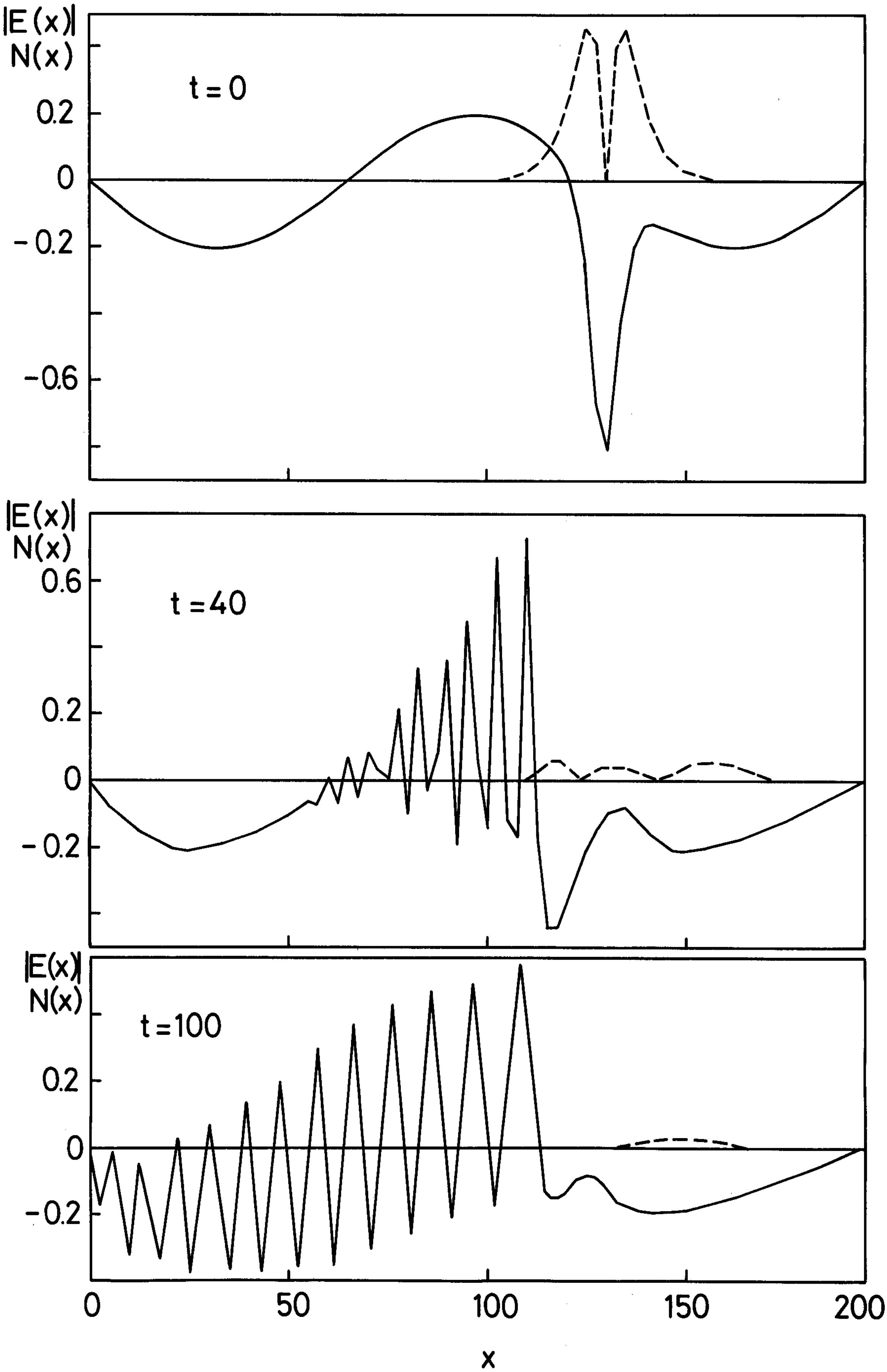
0
-0.2
t=0
|E(x)|
N(x)
0.6
0.2
0
-0.2
t=90
|E(x)|
N(x)
0.6
0.2
0
-0.2
t = 150
0
50
100
150
200
x

FIG. 3


|E(x)|
N(x)
t = 0
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0
- 0.2
-0.6
|E(x)|
N(x)
0.6
t = 40
0.2
0
- 0.2
|E(x)|
N(x)
t = 100
0.2
0
-0.2
0
50
100
150
200
x

FIG. 4

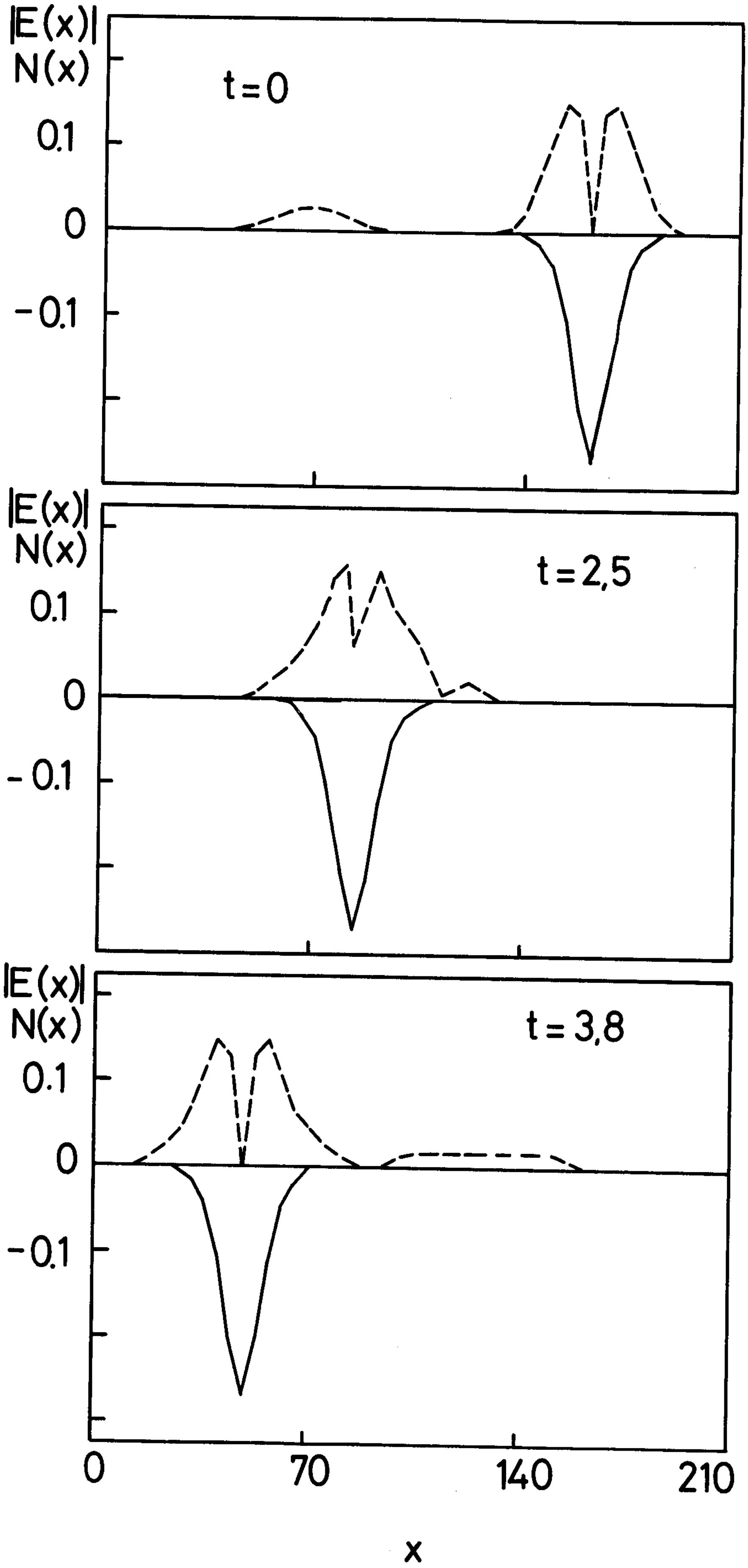
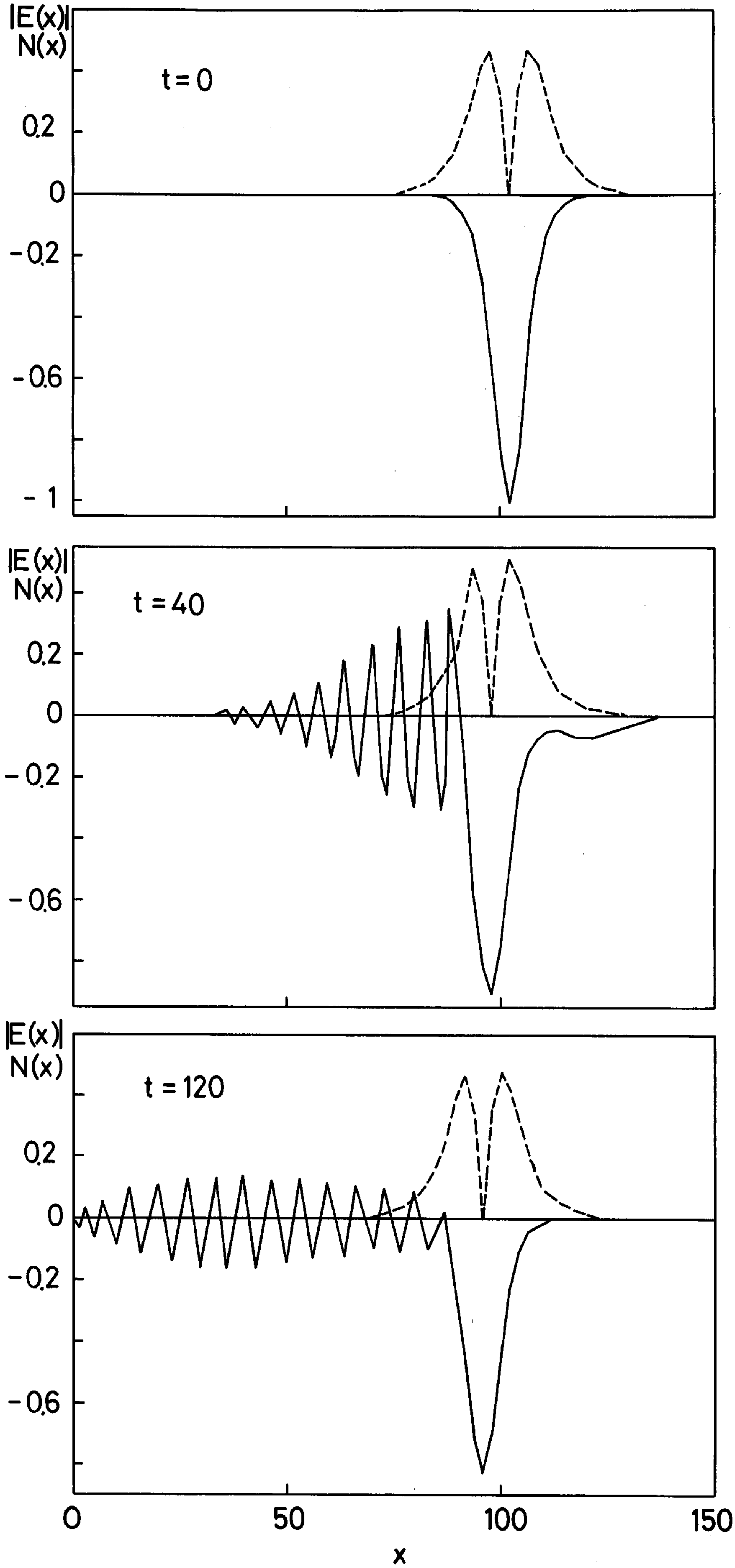


FIG. 5


|E(x)|
N(x)
t = 0
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0
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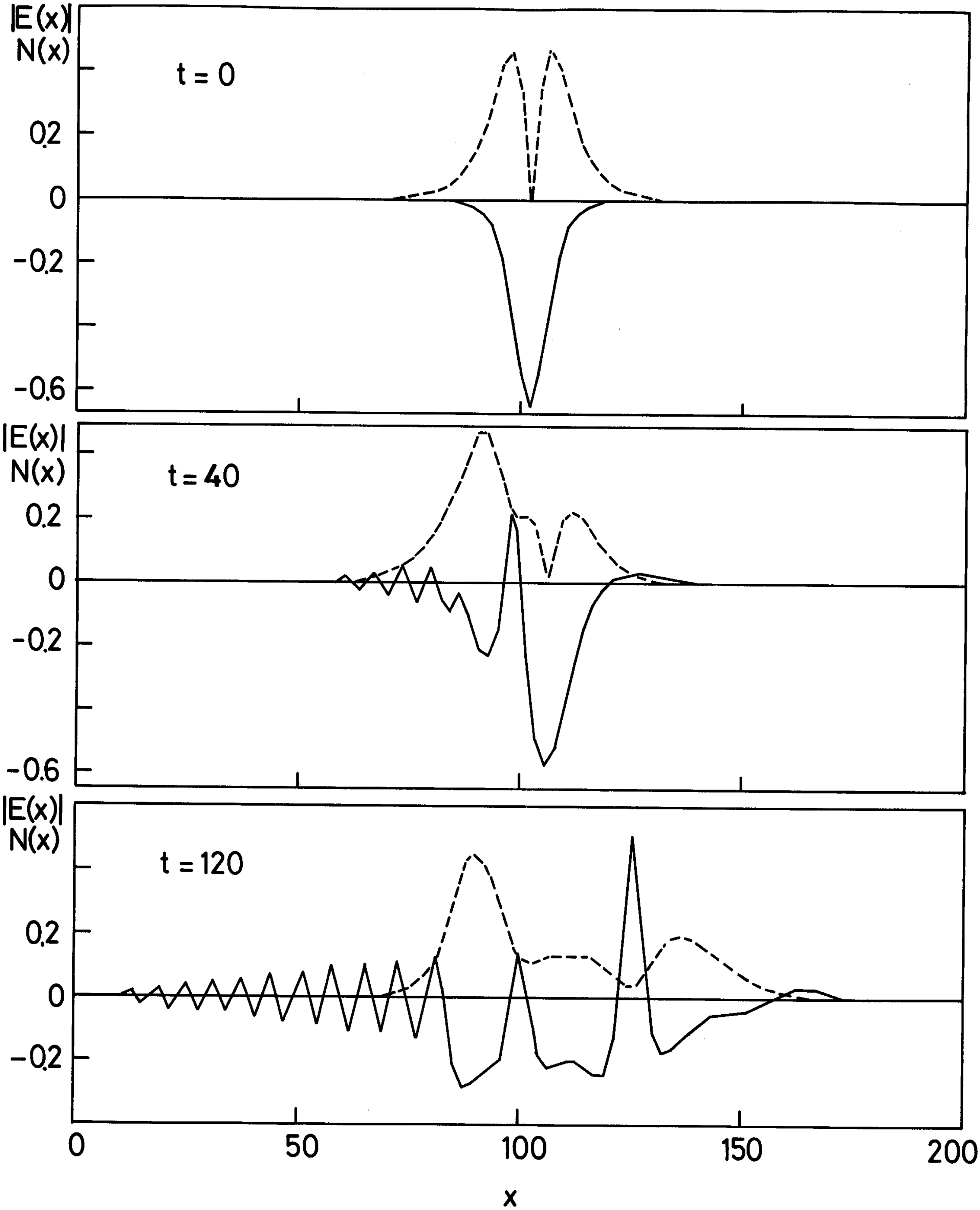
-0.6
-1
|E(x)|
N(x)
t = 40
0.2
0
-0.2
-0.6
|E(x)|
N(x)
t = 120
0.2
0
-0.2
-0.6
0
50
100
150
x
FIG. 6



|E(x)|
N(x)
t = 0
0.2
0
−0.2
−0.6
|E(x)|
N(x)
t = 40
0.2
0
−0.2
−0.6
|E(x)|
N(x)
t = 120
0.2
0
−0.2
0
50
100
150
200
x

FIG. 7