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NONLINEAR EXCITATION OF ACOUSTIC WAVES IN A DOUBLE PLASMA DEVICE

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ABSTRACT

In the double plasma device ion acoustic waves are generated when the potential difference between the two plasmas is varied as $\Phi_o \sin \omega t$. This perturbation affects at first only the ions since the electrons are repelled from the grid that separates the two plasmas. The analysis of the ion orbits near the grid leads to a source term in Vlasov's equation for the ion distribution function. This source and therefore the ion acoustic wave depends nonlinearly on the exciting potential. Analytic and numerical results are given which show that the nth harmonic of the wave potential is proportional to Φ_o^n but is independent of ω when $\omega \ll \omega_{pi}$.

I. INTRODUCTION

The double plasma device, known as the DP [1] has been used for many years to excite ion-acoustic waves in low density plasma. In spite of this, the mechanism of excitation has never been explained satisfactorily. Various models of wave excitation have been proposed but none of them resembles sufficiently the experimental situation found in the DP.

Gould [2] examines the excitation of waves by a dipole layer whose strength is controlled externally. Hirshfield et al [3], Andersen et al [4], Christoffersen et al [5] and Grésillon [6] analyse the excitation of waves by one, two or three grids. For a pair of grids biased at plasma potential, both ions and electrons are affected by the potential Φ_o applied between the grids. The model is equivalent to the dipole layer and yields an amplitude of the wave potential is proportional to the square of the applied frequency [7]. In most experiments the grids are at floating potential, so that the electrons cannot enter the gap between the grids. They are therefore unaffected by the driving potential. This fact has been taken into account by Andersen et al [4] and by Hirshfield et al [3] in their analyses of a three grid exciter. In this case the amplitude of the wave potential increases linearly with the frequency.

White et al [7] consider the propagation of ion acoustic shocks produced in the DP. These authors are primarily concerned with the formation of the shock during propagation rather than with the kinetic effects near the grid. The excitation of the wave is modelled by assuming that the electrons follow a kind of Boltzmann's law

$$n_e = n_o \exp\left[e(U-\varphi)/T_e\right]$$

where U is the self consistent potential, while φ is the exciting potential, which is assumed to have the form

$$\varphi(x,t) = \frac{1}{2}\left[1 - \tanh(x/L)\right]\Phi_o(t).$$

In this equation $\Phi_o(t)$ is the externally controlled potential difference between the two plasmas. The results are independent of the choice of the width L of the ramp. This model however is unrealistic since in a DP the electrons are repelled from the grid and therefore are not accelerated by the imposed potential. In reality, the applied potential modifies primarily the orbits of the ions.

Taylor [9] and Ikezi [10] have given qualitative descriptions of the excitation mechanism.

The analysis presented here is based on the following picture of the wave generation. The DP is treated as two infinite plasmas filling the half spaces $x > o$ and $x < o$. In the plane $x = o$ lies a grid which is transparent to ions and electrons. This grid is biased at a strong negative potential so that it repels essentially all electrons and thereby prevents them from passing from one plasma to the other.

If the potentials of the two sides of the grid differ, the ions crossing the grid are accelerated or decelerated, while some others remain trapped near the grid. Thus the ion distribution is first disturbed near the grid. The perturbation then propagates away from the grid. We assume the imposed variation of the potential difference to be of the form $\Phi_o \sin\omega t$ with ω much below ω_{pi}. The electrons then follow simply Boltzmann's law. Thus if one wishes to understand the mechanism of excitation one must first investigate the motion of the ions near the grid.

Results

The excitation mechanism is nonlinear even at relatively modest values of

$\lambda = e\,\Phi_0/T_i$. Acoustic waves at the fundamental, ω, and at the harmonic frequencies are generated which have the approximate form

$$U(x,t) = C_n(\lambda)\sin(k_n x - n\omega t + \alpha_n)\exp(-\beta_n x).$$

For $\lambda \leq 1$ the amplitudes of the wave potential, C_n, are small, so that the propagation, as opposed to the generation, of the wave can be considered, and has been treated, as linear. In other words, the nonlinear effects due to the excitation appear first and dominate the nonlinear effects due to propagation. Beyond $\lambda \approx 1$, the results are no longer accurate but may still be of qualitative value.

For very small values of λ, the following analytic expressions for the amplitudes of the wave potentials have been obtained

$$C_1 = i\,2^{-1/2}\,(1+3T_i/T_e)^{-2}\,\Phi_0\,,$$

$$C_2 = -(\sqrt{2}-1)\,(32\pi)^{-1/2}\,(T_i T_e)^{1/2}\,(1+3T_i/T_e)^{-3/2}\,e(\Phi_0/T_i)^2.$$

Not surprisingly the fundamental is proportional to Φ_0 while the harmonics increase as Φ_0^n. The amplitude of the wave potentials are independent of the exciting frequency in agreement with experiment [9], but in contrast to the theories discussed before.

For larger values of λ numerical computation was necessary. Figures 1 and 2 give the potential amplitudes for the first two harmonics for $T_e/T_i = 10$ and 20 respectively. It is surprising to see that the analytic values obtained for small λ are in fact accurate up to $\lambda \approx 1$.

II. ANALYSIS

A. The Perturbation of the Ion Distribution (due to the grid)

The scale of spatial variation, λ_g, of the potential near the grid is of the order of the Debye length, δ. Therefore the time of transit, τ_g, of an ion through this potential is of the order of an ion plasma period, ω_p^{-1}. Since both this length and this time are much smaller than the wavelength λ and period ω^{-1} of the acoustic wave one may treat the ions as moving in a prescribed potential, $\varphi(x)$, as shown in Fig. 3a.

One does not need to know the precise form of the potential $\varphi(x)$ which is in principle determined if the potential difference, Φ, across the two sheaths is given. Arbitrarily we put $\varphi(x) = 0$ for $\delta \ll x \ll \lambda$ and $\varphi(x) = \Phi_o$ for $-\lambda \ll x \ll -\delta$.

As shown in Fig. 3b there are three types of ion orbits: depending on its energy an ion may cross the grid, be reflected or trapped by the grid.

Let $f^+(v)$ and $f^-(v)$ be the distribution function of the ions to the right and to the left of the grid but out-side the sheaths. Since $f(x,v)$ is constant along each orbit it is possible to obtain formulae connecting $f^+(v)$ and $f^-(v)$. Assuming the unperturbed distribution to be

$$f_o = n_o \left(\frac{m_i}{2\pi T_i}\right)^{1/2} \exp\left(-\frac{m_i v^2}{2T_i}\right) \tag{1}$$

one finds

$$f^+(v) - f^-(v) = \begin{cases} \left[\exp(e\Phi/T_i) - 1\right] f_o(v) & , \; v > \max(0, v_\Phi) \\ \left[-\exp(-e\Phi/T_i) + 1\right] f_o(v) & , \; v < \min(0, v_\Phi) \\ 0 & , \; \text{elsewhere} \end{cases} \tag{2}$$

The velocity v_Φ is defined as

$$v_\Phi = \mathrm{sign}(\Phi)\,(2e|\Phi|/m_i)^{1/2}\,. \tag{3}$$

In the following we assume a sinusoidal variation of

$$\Phi(t) = \Phi_o \sin\omega t \tag{4}$$

and introduce the function

$$h(t) = \exp\left[e\Phi(t)/T_i\right] - 1 \tag{5}$$

and

$$g(v,t) = \begin{cases} 1 \;, & v < \max(o, v_\Phi) \\ o \;, & \text{elsewhere}. \end{cases} \tag{6}$$

With the help of these functions the relation (2) can be written in the simple form

$$f^+(v) - f^-(v) = \begin{cases} f_o(v)\; h(t)\; g(v,t) \;, & v > o\,, \\ -f_o(v)\; h(-t)\; g(-v,-t) \;, & v < o\,. \end{cases} \tag{7}$$

B. <u>The Excitation of the Acoustic Waves</u>

Compared to the scale of the acoustic wavelength the perturbation of the ion distribution in the vicinity of the grid can be considered localized at x = o. The relation (7) thus becomes a discontinuity condition for the distribution function at x = o. Let $F^+(x,v)$ and $F^-(x,v)$ be the distribution functions for $x > o$ and $x < o$ respectively, satisfying Vlasovs' equation in each half space, but not at x = o. By means of the step function

$$\eta(x) = \begin{cases} 1, & x > 0 \\ 0, & x > 0 \end{cases}$$

we express the distribution function for the entire space as

$$f(x,v) = \eta(x)\ F^{+}(x,v) + \left[1-\eta(x)\right] F^{-}(x,v)\ .$$

This distribution function obviously satisfies the equation

$$\begin{aligned} \frac{Df}{Dt} = \frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{e}{m_i} E \frac{\partial f}{\partial v} &= v\left[F^{+}(0,v) - F^{-}(0,v)\right]\delta(x) \\ &= v\left[f^{+}(v) - f^{-}(v)\right]\delta(x) \end{aligned} \tag{8}$$

where

$$f^{\pm}(v) = F^{\pm}(0,v) = f(\pm 0, v)\ .$$

Introducing the previously established discontinuity condition (7) into (8) one obtains

$$Df/Dt = \delta(x)\, f_0(v) \begin{cases} vq(v,t)\quad h(t), & v > 0 \\ -vq(-v,-t)\quad h(-t), & v < 0\ . \end{cases} \tag{9}$$

The source term to the right of equation (9) describes the excitation of the acoustic waves. It is the particular form of this term which characterizes the DP excitation mechanism. It must be supplemented by the equation $n_e = n_o \exp(eU/T_e)$ governing the electron density and by Maxwell's equation

$$\nabla^2 U = e\left[n_e - \int f d^3 v\right] .$$

The unknown functions shall be represented by their Fourier components, for instance

$$U = \sum_{\mu} \int \Phi_{\mu}(k) \exp(ikx - i\mu\omega t)\, dk/2\pi .$$

Since much of the necessary analyses is standard we may simply write down the Fourier transform of the potential

$$U_{\mu}(k) = \rho_{\mu}(k)\left[k^2\ \varepsilon(k,\mu\omega)\right]^{-1} . \qquad (10)$$

In this expression ε is the dielectric function and ρ_{μ} (k) the driving charge density due to the source term of equation (9). If we write

$$h(t)\ g(v,t) = \frac{1}{2\pi} \sum_{\mu} \psi_{\mu}(v) e^{-i\mu\omega t}$$

then

$$\rho_{\mu}(k) = -i(2\pi)^{-1/2} \frac{en_o}{k} \int_0^{\infty} \left[\frac{\psi_{\mu}(s)}{s-\mu z} - \frac{\psi_{\mu}^{x}(s)}{s+\mu z}\right] e^{-s^2/2}\, s ds . \qquad (11)$$

In this expression

$$z = (m_i/T_i)^{1/2}\ (\omega/k) \qquad (12)$$

and

$$s = (m_i/T_i)^{1/2}\ v\ . \qquad (13)$$

The Fourier components ψ_ν are obtained from the convolution

$$\psi_\nu = \sum_\mu h_{\nu-\mu} \, g_\mu \tag{14}$$

where

$$h(t) = \exp(\lambda \sin \omega t) - 1 = \sum_\mu h_\mu \exp(-i\mu\omega t) \tag{15}$$

with

$$h_0 = I_0(\lambda) - 1 , \tag{16}$$

$$h_\mu = i^\mu I_\mu(\lambda) , \; \mu \neq 0 . \tag{17}$$

The Fourier coefficients of g(v,t) are computed directly from

$$g_\mu(v) = (\omega/2\pi) \int_{-\pi/\omega}^{\pi/\omega} \exp(i\mu\omega t) \, g(v,t) \, dt$$

by simply remembering that

$$g(v,t) = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} , \; \frac{1}{2} m v_i^2 \gtrless e \Phi_0 \sin \omega t$$

Hence

$$g_0 = \frac{1}{2} + \frac{\tau}{\pi} \tag{18}$$

$$g_\mu = -i \left[e^{i\mu\tau} - (-)^\mu \, e^{-i\mu\tau} \right] / (2\pi\mu) \tag{19}$$

where

$$0 \leq \tau(v) = \arcsin(m_i\, v^2/2e\Phi_0) \leq \frac{\pi}{2}$$

or, expressed in terms of the variable $s = (m_i/T_i)^{\frac{1}{2}}v$,

$$\tau(s) = \arcsin(s^2/2\lambda) \,. \tag{20}$$

Introducing these results into (11) one obtains

$$\rho_\nu(k) = i\,\frac{en}{(2\pi)^{1/2}k}\sum_\mu \left[h_{\nu-\mu}\int_0^\infty \frac{q_\mu(s)\exp(-s^2/2)\,s\,ds}{s-\nu z}\right. \tag{21}$$

$$\left. - h^*_{\nu-\mu}\int_0^\infty \frac{q^*_\mu(s)\exp(-s^2/2)\,s\,ds}{s+\nu z}\right]$$

This is the driving charge density due to the source term of equation (9). Equation (10) must now be Fourier inverted using (21) to give the wave potential U(x,t). Before doing so we write $\rho_\nu(k)$ in a more convenient form.

We are not interested in the Fourier component $\nu = 0$ which describes a time independent perturbation of the sheaths. For $\nu \gg 1$ we may develop the denominators of the integrands of (21) in powers of s/z. This yields

$$\rho_\nu(k) = i\,\frac{en}{(2\pi)^{1/2}k}\sum_{\alpha=1}^\infty \left[\left(\frac{T_i}{m}\right)^{1/2}\frac{k}{\nu\omega}\right]^\alpha \sum_\mu \left(h_{\nu-\mu}\,g^\alpha_\mu - (-)^\alpha h^*_{\nu-\mu}\,g^{\alpha *}_\mu\right) \tag{22}$$

where

$$g^\alpha_\mu = \int_0^\infty q_\mu(s)\exp(-s^2/2)\,s^\alpha\,ds\,.$$

Using the expressions (18) and (19) for $g_m(s)$ we find

$$\mathcal{G}_0^{\alpha} = 2^{\alpha/2}\,\Gamma\left(\frac{1+\alpha}{2}\right) - \lambda^{(1+\alpha)/2} \int_0^{\pi/2} \left(\frac{1}{2} - \frac{\tau}{\pi}\right) \exp(-\lambda \sin\tau)(2\sin\tau)^{(\alpha-1)/2} \cos\tau\, d\tau \tag{23}$$

and for $\mu \neq 0$

$$\mathcal{G}_\mu^{\alpha} = -i\lambda^{(1+\alpha)/2} (2\pi\mu)^{-1} \int_0^{\pi/2} \left[e^{i\mu\tau} - (-)^{\mu} e^{-i\mu\tau}\right] \exp(-\lambda \sin\tau)(2\sin\tau)^{(\alpha-1)/2} \cos\tau\, d\tau . \tag{24}$$

When these expressions are introduced into the series (21) for the charge density one notices that only the even (odd) terms in α contribute to the odd (even) harmonics.

Provided that $z = (T_e/T_i)^{\frac{1}{2}} \gg 1$ the leading term in α suffices to obtain approximate expressions for the source density ς_ν. In this approximation the fundamental and second harmonic become

$$\varsigma_1(k) = -(2/\pi)^{1/2}\, e m_i^{-1} \omega^{-2}\, n_0 T_i\, k \sum_\mu \operatorname{Im}\left(h_{1-\mu}\, \mathcal{G}_\mu^{2}\right) \tag{25}$$

and

$$\varsigma_2(k) = i(2\pi)^{-1/2}\, e n_0 (T_i/m_i)^{1/2}\, \omega^{-1} \sum_\mu \operatorname{Re}\left(h_{2-\mu}\, \mathcal{G}_\mu^{1}\right). \tag{26}$$

These expressions further simplify in the limit of small excitation, in which it is possible to evaluate the integrals (23) and (24). In this limit, that is for $\lambda = e\,\Phi_0/T \ll 1$ we obtain for the source densities

$$\varsigma_1(k) = -2^{-1/2}\,(e^2 n_0/m_i)\,(k/\omega^2)\,\Phi_0 \tag{27}$$

and

$$\varsigma_2(k) = -i(\sqrt{2}-1)(8\pi)^{-1/2}(en_0/\omega)(T_i/m_i)^{1/2}(e\Phi_0/T_i)^2 . \qquad (28)$$

The expression (23) obtained for $\varsigma_\nu(k)$ or the approximations (25) and (26) must be substituted into Eq. 10. The Fourier inversion of this equation thus yields the wave potential

$$U_\nu(k,t) = \frac{1}{2\pi}\int_0^\infty \frac{\varsigma_\nu(k)\exp ikx}{k^2\,\varepsilon(k,\nu\omega)}\,dk$$

The dielectric function obtained from the Vlasov equation is

$$\varepsilon(\pm k,\omega) = 1 + (\delta_e k)^{-2} - \frac{\omega_{pi}^2}{k^2}\,\frac{m_i}{2T_i}\,Z'\left(\pm\sqrt{\frac{m}{2T_i}}\,\frac{\omega}{k}\right),\ k \gtrless 0$$

where Z(s) is the plasma dispersion function [11]. It is well known that $\varepsilon(k,\omega)$ cannot be continued into the complex plane as a single analytic function. Thus the inversion is not simply a sum of residues but contains a contribution of a branch cut integral. Nevertheless it has been shown [2] that the residue of $1/\varepsilon$ at the ion acoustic pole k, is the dominant term for x not too far from the source. Thus the wave potential of the ν^{th} harmonic becomes

$$U_\nu(x,t) = \left\{ i\varsigma(k)\left[k^2\,\partial\varepsilon/\partial k\right]^{-1}\exp(ikx - i\nu\omega t)\right\}_{k=k_\nu} + c.c.$$

$$= C_\nu \sin(k_\nu x - \nu\omega t + \alpha_\nu)\exp(-\beta_\nu x)$$

The amplitudes C_ν are independent of ω, to the extent that one may assume $k_\nu = c_s\nu\omega$, where c_s is the ion acoustic velocity.

Except for $\lambda \ll 1$ numerical computation was necessary to evaluate the amplitudes C_ν. The results are plotted in the graphs of Fig. 1 and Fig. 2 which give C_1 and C_2 as a function of λ for T_e/T_i = 10 and 20.

For small values of λ analytic expressions for C_1 and C_2 have been obtained using the dielectric function from the two fluid theory

$$\varepsilon = 1 + \frac{\omega_{pe}^2}{c_e^2 k^2 - \omega^2} + \frac{\omega_{pi}^2}{c_i^2 k^2 - \omega^2}$$

The amplitudes then become

$$e c_1 / T_i = i\, 2^{-1/2} (1 + 3 T_i/T_e)^{-2} \, (e \Phi_0 / T_i)$$

$$e c_2 / T_i = -(\sqrt{2} - 1)(32\pi)^{-1/2} \, (T_e/T_i)(1 + 3T_i/T_e)^{-3/2} (e\Phi_0/T_i)^2$$

It is surprising that these coefficients agree very well with the numerical results up to values of $\lambda \approx 1$ where a large number of terms are needed to evaluate the series.

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FIGURE CAPTION

Figure 1 : Variation of the eC_1/T_i and eC_2/T_i in function of $e\Phi_o/T_i$ for $T_e/T_i = 10$

Figure 2 : Variation of eC_1/T_i and eC_2/T_i in function of $e\Phi_o/T_i$ for $T_e/T_i = 20$

Figure 3a : Spatial potential in a DP device

Figure 3b : Ion orbits in phase space near the grid

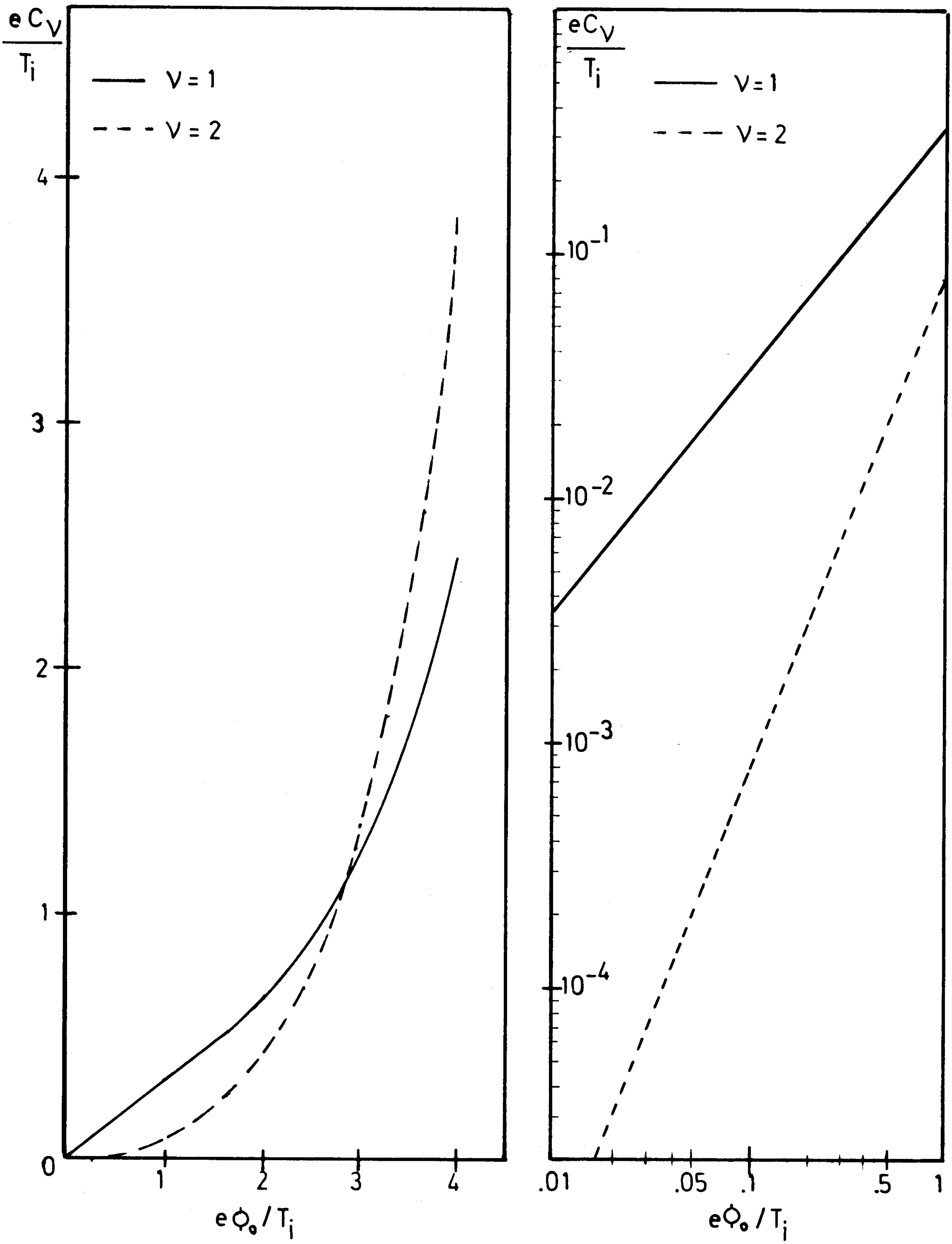
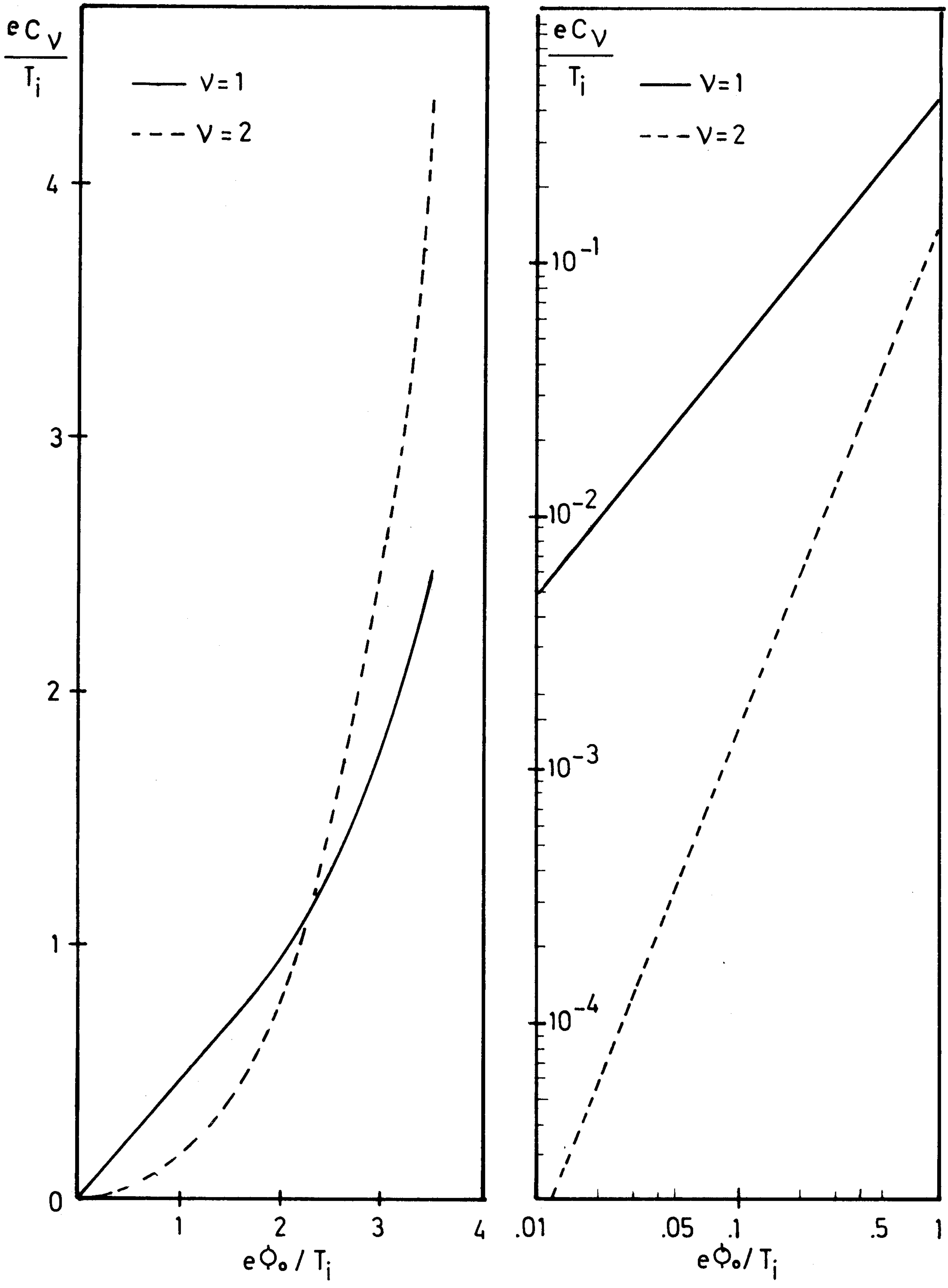


Figure 1



$\frac{eC_\nu}{T_i}$
$\nu = 1$
$\nu = 2$
4
3
2
1
0
1
2
3
4
$e\Phi_o / T_i$
$\frac{eC_\nu}{T_i}$
$\nu = 1$
$\nu = 2$
10^{-1}
10^{-2}
10^{-3}
10^{-4}
.01
.05
.1
.5
1
$e\Phi_o / T_i$

Figure 2

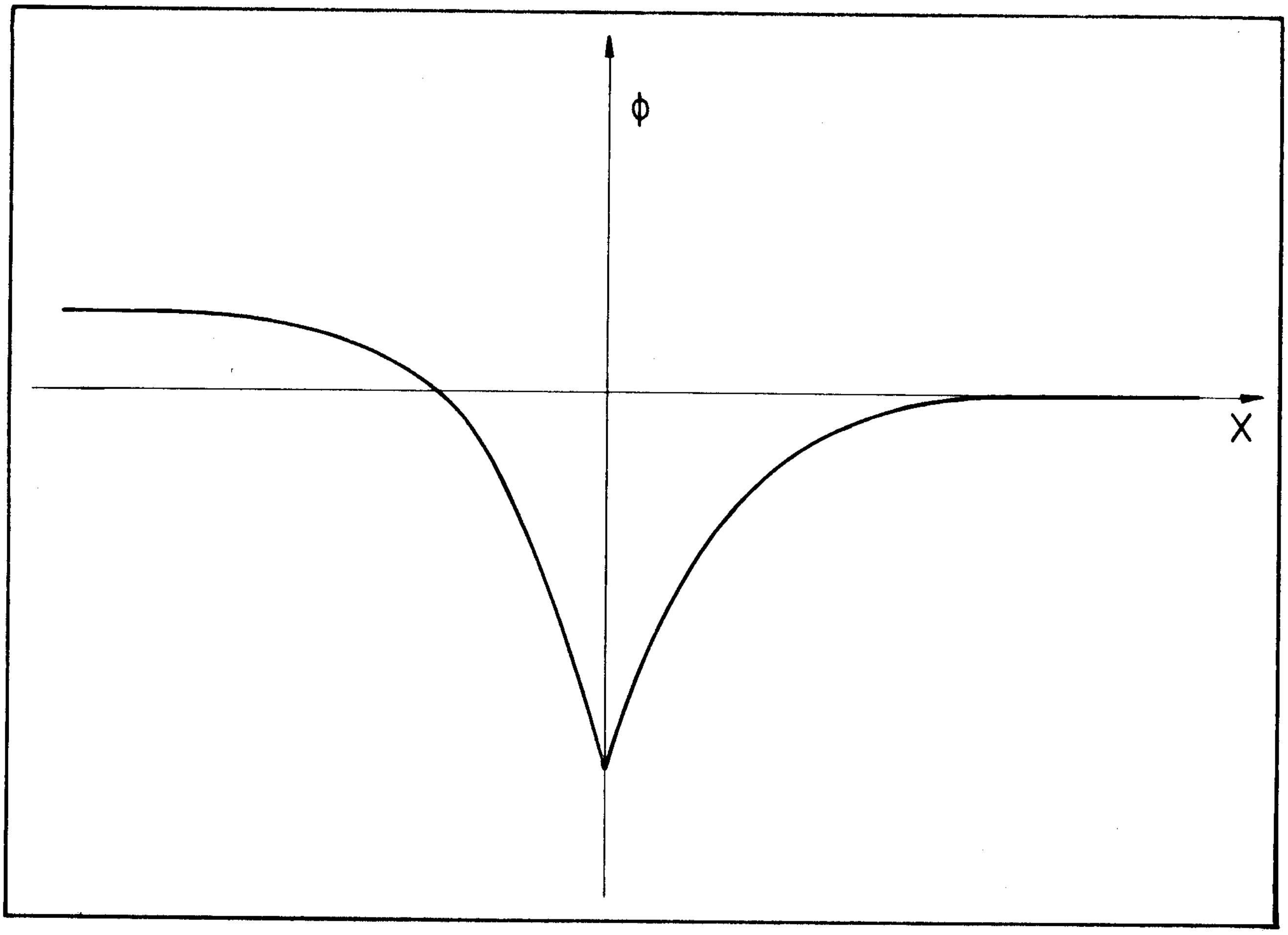


FIG. 3a

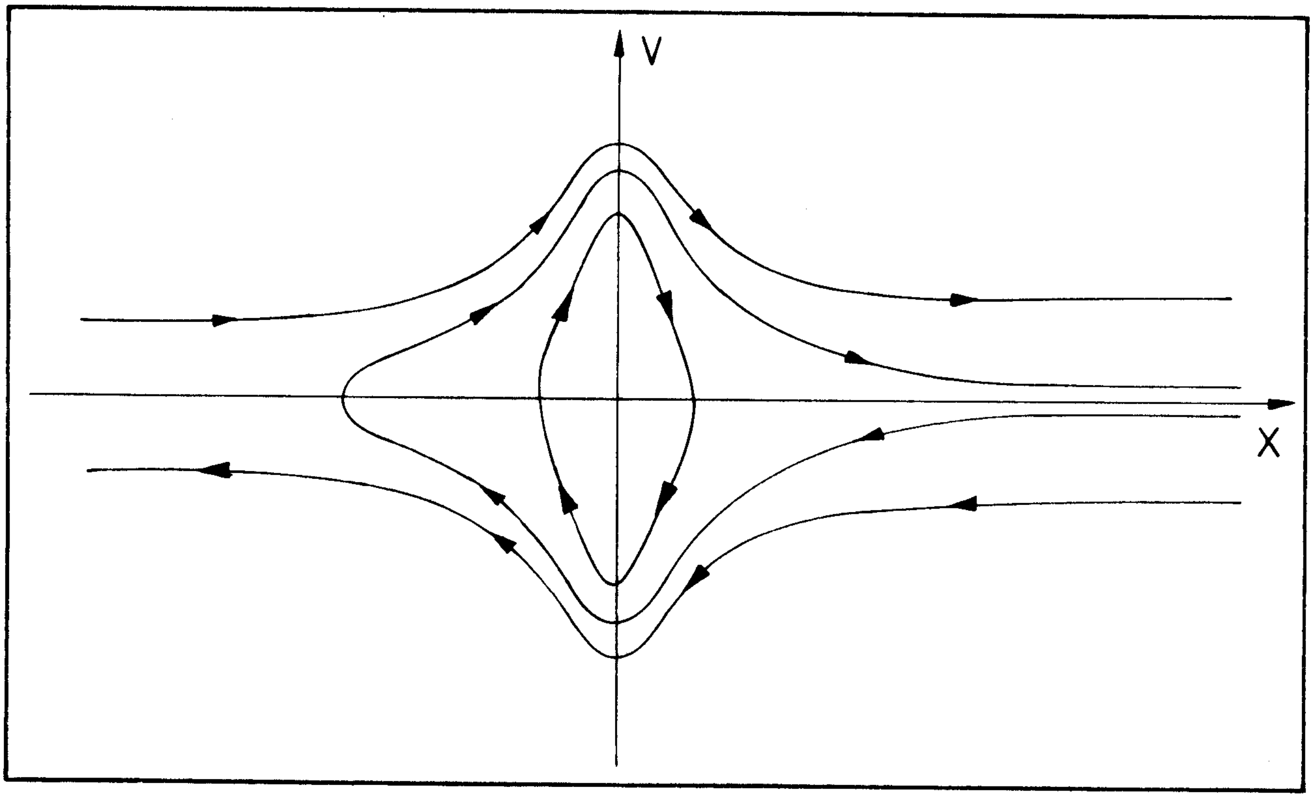


FIG. 3b