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ON THE CUT-OFF IN TWO DIMENSIONAL GUIDING CENTER PLASMAS

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Numerous studies, both theoretical and computer simulations, have been devoted to the understanding of the negative temperature instability in the two dimensional hydrodynamic motion of interacting line vortices in ideal liquids and in the two dimensional (2-D) guiding center (g.c.) plasmas [1-9].

Onsager [1] first predicted the occurence of negative temperature states in 2-D turbulent liquids for interaction energies above a certain threshold value. Montgomery, Joyce, Taylor and others [2-9], theoretically described and numerically simulated such states in the 2-D g.c. plasma. They observed the basic feature of these systems in the negative temperature regime: small vortices clumping into larger ones. Similar behavior involving the filamentation of relativistic beams has been investigated by Lee and Lampe [10] in their recent nonlinear studies of the Weibel instability. Finally, a new analogous phenomenon in 2-D MHD, the tendency of current filaments to consolidate in high β turbulence, has been reported [11].

The primary purpose of this Letter is to present a calculation of the threshold energy for the onset of negative temperature states in a 2-D g.c. plasma. This will be done simply by equating the mean potential energy calculated from the pair correlation function to that calculated from the Debye value of the partition function containing the threshold energy as an unspecified parameter; the physical meaning of the threshold energy will then be elucidated in terms of a long wavelength cut-off.

Following the description of Vahala and Montgomery [12], we consider a

2-D g.c. plasma of volume $V = L^2$ containing N positive charges and N negative charges at the temperature $\beta^{-1} = kT$. The plasma is pervaded by a constant magnetic field $\underset{\sim}{B} = \hat{e}_z B$ so strong, that charged filaments are aligned with $\underset{\sim}{B}$. The plasma has therefore a 2-D character and the interaction potential between two charged rods is given approximately by $\phi_{ij} \approx -(2\, e_i e_j/\ell)\ \ln |\underset{\sim}{x}_i - \underset{\sim}{x}_j|$ where e_i is the total charge of rod i, $\underset{\sim}{x}_i$ is its position, and ℓ is its (very long) length. Its motion in the xy-plane is described by the microscopic g.c. equation $\underset{\sim}{v}_i = c\, \underset{\sim}{E}_i \times \underset{\sim}{B}/B^2$, $\underset{\sim}{E}_i$ being the electrostatic field at $\underset{\sim}{x}_i$ due to all the other rods in the system. The equation of state [12] has the same form as that for the two-component plasma in a strictly two dimensional world [13].

As a brief review of the cut-off problem, let us consider first Taylor's expression [4] for the energy from the microcanonical ensemble*,

$$(E)_{TAYLOR} = (Ne^2/\ell)\ln\left(\frac{k_1^2 + k_D^2}{k_0^2 + k_D^2}\right), \tag{1}$$

where $k_D^2 = 8\pi Ne^2\beta/(V\ell)$ and k_0 and k_1 are, respectively, minimum and maximum wavenumber values; k_0 is chosen to be $2\pi/L$ while k_1 is left unspecified. From (1), one obtains the threshold value

$$(E_t)_{TAYLOR} = \left[E(\beta=0)\right]_{TAYLOR} = (Ne^2/\ell)\ln(k_1/k_0)^2. \tag{2}$$

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*Note that Eq.(1) can be obtained from the Debye value of the canonical partition function Z, i.e., $E = \langle u \rangle = -(\partial/\partial\beta)\ \ln Z$ by use of the Edward's transformation [14] for the two dimensional case.

Montgomery [2], however, observed that the self energy of the particles,

$$u_{SELF} = 2\pi e \Sigma_{\underset{\sim}{k}} (1/4\pi\beta e)(k_D^2/k^2) = (Ne^2/\ell) \ln(k_1/k_0)^2, \quad (3)$$

should be subtracted out of Eq.(1) thus giving $u_t = 0$ for the value of the threshold energy above which negative temperature states can occur. In any case, we note that due to the peculiarity of the scaling property of the two dimensional logarithmic potential, neither the Taylor nor the Montgomery energy formulas are conveniently normalized so that they do not give rise to the recently established extensivity property of the energy in the thermodynamic limit [15,16], not even in the Debye approximation. This can be seen by noting from Eq.(1), that

$$E[k_0 = (2\pi/L) \ll k_D, k_1 \to \infty]_{TAYLOR} \approx (Ne^2/\ell) \ln k_1^2/k_D^2 \to \infty, ^* \quad (4)$$

while from the difference between Eqs.(1) and (3),

$$E[k_0 = (2\pi/L) \ll k_D, k_1 \to \infty]_{MONTG.} \approx -(Ne^2/\ell) \ln[(2\beta e^2/\pi\ell)N] \to \infty. \quad (5)$$

On the other hand, Seyler's [7] more recent rpa calculation results in an energy which is, in fact, extensive for $N \to \infty$. His method takes account of interactions with images and his threshold energy,

$$(E_t)_{SEYLER} = -2.62\,(Ne^2/\ell) + (Ne^2/\ell) \ln V, \quad (6)$$

calculated from the two-body noncentral Ewald potential, is the sum of the charged rod-image interaction energies. We too shall now derive an expres-

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*Note that if one adopts Taylor's assumption [4] that $k_1 = (8\pi/L)N$, the divergence of Eq.(4) has the same $N \ln N$ character as that of Eq.(5).

sion for the threshold energy such that in the thermodynamic limit, the extensivity property of the internal energy is preserved. We introduce at the outset an unspecified energy u_o (later shown to be u_t) into the Debye expression for the partition function to avoid the N $\ln$ N divergence (see Eq.(5)) encountered by Montgomery [2]. Our method then consists simply in equating the energies calculated from the pair correlation and partition functions in the thermodynamic limit. Unlike Seyler [7], we do not take account of image charges, and we therefore cannot interpret our threshold energy in terms of charged rod-image interactions, but rather in terms of a long wavelength cut-off intrinsic in the problem.

We start from the Debye expression for the partition function

$$Z \sim \exp\left\{-\beta u_o - (1/4\pi)\int_{k_o}^{k_1} k\,dk\left[\ln(1+k_D^2/k^2) - (k_D^2/k^2)\right]\right\}, \tag{7}$$

so that

$$u = -(\partial/\partial\beta)\ln Z = u_o + (Ne^2/\ell)\ln\left[\left(\frac{k_1^2+k_D^2}{k_o^2+k_D^2}\;\frac{k_o^2}{k_1^2}\right]. \tag{8}$$

Now, the mean energy u can also be calculated from the pair correlation formula,

$$u = (\bar{n}^2/2)\int d^2r\,\phi(r)p(r) = -2(Ne^2/\ell)\ln(k_D e^{\gamma}/2),^{*} \tag{9}$$

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*Alternatively, we note from the translational invariance property of the system and the definition of screening, that

$$u = \langle\phi\rangle = (\tfrac{1}{2})\left\langle \Sigma_{i\neq j}\, e_i e_j \phi(|\underline{r}_i - \underline{r}_j|)\right\rangle =$$
$$= \lim_{\underset{\sim}{r}\to 0} (2N)(\tfrac{1}{2}) \left\langle \Sigma_{j\neq 0}\, e e_j\, \phi(|\underline{r} - \underline{r}_j|)\right\rangle =$$
$$= \lim_{\underset{\sim}{r}\to 0} N \left\langle e \sum_{j=0} e_j\, \phi(|\underline{r} - \underline{r}_j|) - e e_o \phi(|\underline{r} - \underline{r}_0|)\right\rangle = \lim_{\underset{\sim}{r}\to 0} Ne\left[\Phi_{\text{screen}}(r) - \phi_{\text{coul}}(r)\right] = \lim_{\underset{\sim}{r}\to 0} Ne\left[2e\,K_o(k_D r) - (-\,2e\ln r)\right] = -2(Ne^2/\ell)\ln(k_D e^{\gamma}/2).$$

where $\bar{n} = (N_e + N_i)/V = 2\,N/V$, $\phi(r) = -(2e^2/\ell)\ln r$, $p(r) = 2\beta e^2 K_o(k_D r)$, $K_o(k_D r)$ being a Bessel function of imaginary argument [12], and $\gamma = 0.57...$ is the Euler constant. One can determine u_o by letting $k_o \ll k_D$ and $k_1 \to \infty$ in (8) and then equating the result to the r.h.s. of Eq.(9). One obtains:

$$u_o + (Ne^2/\ell)\ln k_o^2/k_D^2 = -2(Ne^2/\ell)\ln(k_D e^{\gamma}/2)$$

or (10)

$$u_o = -2(Ne^2/\ell)\ln(k_o e^{\gamma}/2).$$

The threshold energy $u_t = u_o$ is readily deduced from Eq.(8) by first letting $k_o \to o$ and then setting $k_D = k_o$. Adopting Seyler's value of $(4\pi/V)^{\frac{1}{2}}$ for k_o [7], this becomes:

$$u_t = -2.27(Ne^2/\ell) + (Ne^2/\ell)\ln V. \tag{11}$$

It is interesting to note that there is fair agreement between our threshold energy value and that of Seyler's. Eq.(10) reveals that the occurence of negative temperature states is connected with the physical cut-off k_o.

In conclusion, the negative temperature states of the two dimensional g.c. plasma in a finite container, as they are constructed and simulated numerically by Montgomery and Joyce [6], are possible if the energy shell is such that the energy per particle is at least $(u_o/2N) \approx -(e^2/\ell)\ln(2\pi/\lambda_{max}) = -(\frac{1}{2})\,\phi(\lambda_{max}/2\pi)$, where λ_{max} is the maximum wavelength in the system. This is connected with the appearance of macroscopic vortex structure and our result (10) is in agreement with the results of numerical experiments [5,6].

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