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INTRODUCTION TO RESISTIVE INSTABILITIES

J.M. Greene

Plasma Physics Laboratory, Princeton University

James Forrestal Campus*, Princeton, N.J. 08540, USA

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LECTURE I

In this set of lectures I will choose a few of the many things that could be discussed. The selection will be based on what I know best, in particular, analytic MHD theory. We all understand that MHD theory is incomplete, and that analysis is only one way to understand the theory. Analysis tends to emphasize certain phenomena simply because they are easy to calculate, and sometimes the methods of analysis become confused with the reality of the physical system. Nevertheless, it is useful. I hope to show the strengths and weaknesses of the theories, some of the results, and perhaps also indicate areas that are not well treated in analytic theory.

Analytic MHD theory is the study of singular perturbations. This is a slight exaggeration, but close to the truth. There are many kinds of singularities in MHD. This course could be made into a mathematical examination of singular perturbation theory. However, the physics is also interesting, so I hope to concentrate on it.

Before starting the discussion of the physics, I would like to devote this lecture to an examination of a basic mathematical technique that will be used throughout this course, a technique known as boundary layer theory and asymptotic matching. To motivate this discussion, consider a typical MHD problem:

$$
\begin{aligned}
D \frac{d}{d\eta} y(\eta) &= C_1\, y(\eta) - \eta\, C_2\, P(\eta) \\
D \frac{d}{d\eta} P(\eta) &= \frac{1}{\eta} C_3\, y(\eta) - C_1\, P(\eta)
\end{aligned}
\qquad (1)
$$

$$D \equiv \left[\rho\omega^2 - F^2(r)\right]\left[(\gamma P + B^2)\rho\omega^2 - \gamma P F^2(r)\right] \tag{2}$$

These equations should be well known here |1|, as describing the small amplitude oscillations of a circular cylinder plasma column. Today I only wish to point out that this is a typical MHD problem, and that D can vanish in the domain of interest, $0 \leq r \leq a$. When this happens, the equations are singular, and Y and P are badly behaved in a small nearby region. This bad behavior actually turns out to be good for analysis. Something like half the known results deal with this singularity. This bad behavior both requires and permits small effects, that are generally discarded in the MHD approximation, to play an important role. It requires other effects, because in nature nothing is infinite or exactly vanishes. It permits other effects because, for example, perturbations tend to be large there, so inertial effects can be exaggerated. Also, currents are frequently much larger in this region, so that the true effect of resistivity tends to be localized. Viscosity and dispersion effects, frequently dropped from MHD, may also be dominant in this vicinity.

In general, analysis likes the limit of an infinitessimal thickness. The technique for dealing with these thin layers is well developed, though not without some controversy |2,3,4| and is called boundary layer theory. Today I would like to do a couple of boundary layer problems to illustrate the method.

First, consider the equation

$$\epsilon L \frac{dI}{dt} + RI - Ve^{-\omega t} = 0 \tag{3}$$

with the boundary condition

$$I(0) = 0 \tag{4}$$

Physically, this equation represents the response of an inductive and resistive circuit to a voltage that is applied impulsively, and then decays away. We are particularly interested in the case where the inductance is small, and is important only for the initial transient response of the system. In this case we will want to treat the initial response and the steady behavior separately. This is the basic motivation for all boundary layer problems.

In Eq.(3) the inductance has been multiplied by a factor ϵ to indicate that it is small. There are two ways of handling small parameters. One can either form a dimensionless inductance and use this directly as the small parameter. Or one can multiply various expressions by functions of ϵ as a reminder that they should be small. These expressions can then be evaluated after the solution is found, with ϵ set equal to unity since it has no physical meaning, and the magnitudes confirmed. The latter convention has been adopted in these lectures.

Thus, we expect that a term multiplied by ϵ is small, and a term multiplied by ϵ^2 is somewhat smaller. It is more convenient, however, to consider ϵ to be, not a number or a magnitude, but the process of taking a limit. Thus a term that is multiplied by ϵ will be expected to vanish linearly in the limit as, in our example, the inductance vanishes. A term multiplied by ϵ^2 will be expected to vanish quadratically. Generally a quantity that vanishes quadratically will be smaller, for a given inductance, than one that vanishes linearly, but that is not the real point. In the rest of these lectures the work "small" will be more of a verb than a noun.

It follows that our interest is not in calculating the results for some particular value of the inductance, but in calculating the dependence of various quantities on inductance. This is true even when the small parameter is a known quantity, such as the ion/electron mass ratio, or the fine structure constant. It is an hypothesis that the resulting series are also particularly useful for obtaining results applicable to real physical systems*.

With this background, we set to work examining Eq.(3). We assume that the current can be expanded in powers of the small parameter ϵ:

$$I = \sum_{m=0}^{\infty} \epsilon^m I_m(t) \qquad (5)$$

and substitute this into Eq.(3). It is expected that this solution is valid for a range of inductances, therefore it is appropriate to equate the coefficient of each power of ϵ to zero. Thus we are led to

*See Sec. 1.2 of reference 2 for a good discussion of this.

$$RI_0 = Ve^{-\omega t}$$

$$RI_n = -L\frac{d}{dt}I_{n-1} \qquad n \geq 1 \tag{6}$$

Hence

$$I_n = \left(\frac{\omega L}{R}\right)^n \frac{V}{R} e^{-\omega t} \tag{7}$$

This shows that the natural dimensionless small number in our expansion is $\omega L/R$.

This can not be the whole solution because we are unable to apply the boundary condition at t=0, and indeed, our prescribed boundary condition, Eq.(4), is not satisfied. There is an initial transient that is not treated by this approximation. The time scale of the transient depends on the inductance, and vanishes as it vanishes. To treat this, we consider a new variable,

$$\tau \equiv \frac{R}{\epsilon\omega L} t \tag{8}$$

We will then consider limits with the inductance going to zero, but keeping τ fixed. Thus consideration of smaller and smaller inductances is coupled to consideration of earlier times. The behavior of the transient is nearly invariant in this limit, but the decay of the applied voltage appears to be slower and slower. In particular, substituting Eq.(8) into Eq.(3) we find

$$\frac{R}{\omega}\frac{d}{d\tau}I^* + RI^* - V\exp\left(-\frac{\epsilon\omega L}{R}\omega\tau\right) \tag{9}$$

where

$$I(t) \equiv I^*(\tau) \tag{10}$$

Again, we assume that the current can be expanded in powers of the inductance:

$$I^* = \sum_{n=0}^{\infty} \epsilon^n I_n^*(\tau) \tag{11}$$

and equate coefficients of power of ϵ to zero:

$$\begin{aligned} \frac{1}{\omega}\frac{d}{d\tau}I_0^* + I_0^* &= V/R \\ \frac{1}{\omega}\frac{d}{d\tau}I_1^* + I_1^* &= -\frac{V}{R}\left(\frac{\omega L}{R}\right)\omega\tau \end{aligned} \tag{12}$$

The solutions of these equations, with the prescribed boundary condition, Eq.(4), are

$$\begin{aligned} I_0^* &= \frac{V}{R}\left(1 - e^{-\omega\tau}\right) \\ I_1^* &= -\frac{V}{R}\left(\frac{\omega L}{R}\right)\left(\omega\tau - 1 + e^{-\omega\tau}\right) \\ I_2^* &= \frac{V}{R}\left(\frac{\omega L}{R}\right)^2\left[\frac{1}{2}(\omega\tau)^2 - \omega\tau + 1 - e^{-\omega\tau}\right] \end{aligned} \tag{13}$$

We now have two separate solutions, one describing the initial transient response, the other describing the effect of a slowly changing applied voltage. The theory of asymptotic matching says that these two solutions can agree somewhere, so that together they can describe the whole solution.

In our present problem, this would be a time span not far from the end of the transient response. Thus it would be a regime of large τ and small t. If we consider the first solution for small t, it is natural to expand each term in Eq.(7) in powers of t. This leads to a double series for the current in powers of ωt and $\omega L/R$. Thus:

$$\xrightarrow{\hspace{4em}} \text{Powers of } t$$

$$\frac{RI}{V} = \left\{ \begin{array}{llll} 1 & -\omega t & +\frac{1}{2}(\omega t)^2 & + \cdots \\ +\frac{\epsilon\omega L}{R} & -\left(\frac{\epsilon\omega L}{R}\right)\omega t & + \cdots & \\ +\left(\frac{\epsilon\omega L}{R}\right)^2 & -\left(\frac{\epsilon\omega L}{R}\right)^2\omega t & + \cdots & \\ \vdots & \vdots & \vdots & \end{array} \right. \qquad (14)$$

$$\Big\downarrow \ \text{Powers of } \epsilon$$

Similarly, the transient response can be evaluated for large τ in powers of $1/\omega\tau$. Here the exponential terms are smaller than any power of τ, and thus can be neglected. The resulting double series is

$$\xrightarrow{\hspace{4em}} 1/\tau^n$$

$$\frac{RI^*}{V} = \left\{ \begin{array}{llll} 0 & 0 & 1 & 0 + \cdots \\ +0 & -\left(\frac{\epsilon\omega L}{R}\right)\omega\tau & +\frac{\epsilon\omega L}{R} & + \cdots \\ +\frac{1}{2}\left(\frac{\epsilon\omega L}{R}\right)^2(\omega\tau)^2 & -\left(\frac{\epsilon\omega L}{R}\right)^2\omega\tau & +\left(\frac{\epsilon\omega L}{R}\right)^2 & 0 + \cdots \\ \vdots & \vdots & \vdots & \end{array} \right. \qquad (15)$$

$$\Big\downarrow \ \epsilon^n$$

Here the nonvanishing terms fall in a triangle opening to the left.

These two double series can be seen to be identical! The upper line of the first series is equal, term by term, to the upper, left trending diagonal of the lower series, on using Eq.(8). The other terms can be similarly identified.

This state of affairs is a bit like analytic continuation in the analysis of complex functions. We have two series representations of a given function, each valid in its own domain. The problem is to establish that these are both representations of the same function. The analogy should not be pressed to far. Here we only ask that these two representations asymptotically approximate the same function, a much weaker requirement.

With a little thought, we can see why these two series matched term by term. We can rearrange Eq.(3) in the form,

$$RI - V = V\left(e^{-\omega t} - 1\right) - \epsilon L \frac{dI}{dt} \tag{16}$$

We now apply the limiting procedure used to produce the double series of Eq.(14); both ϵ and t are small, but ϵ is smaller than any power of t. Then both terms on the right hand side are small, so that the equation can be solved recursively. A given approximation for I can be used on the right hand side and the solution for I from the left will then be a better approximation. Each term in Eq.(14) can be seen to be driven by a lower order term on the right. For example, the top line of Eq.(14) is produced by the expansion of the first term on the right hand side of Eq.(16). Equation (15) is also generated by the same kind of recursion on Eq.(16), differing only in the assumed relative magnitudes of the two terms on the right hand side.

Thus the two series agree because both are generated by essentially the same recursion. Terms generated by each part of the right hand side of Eq.(16) are independent of those generated by other parts, and so the resulting series are independent of the assumed relative magnitudes of the generators on the right. Thus it is not surprising that Eq.(14) and (15) agree.

It follows that any of a series of limits,

$$\epsilon \to 0, \quad t/\epsilon^n \text{ finite} \quad 0 < n < 1 \tag{17}$$

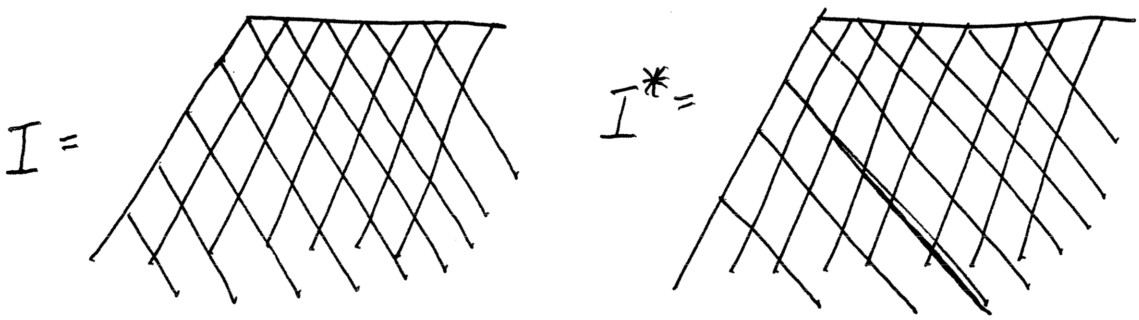
will again yield a series equivalent to Eq.(14) or (15), but with a different ordering of the terms. Thus in some sense the double series covers the entire space between the inner and outer regions.

How should one choose an ordering to obtain the most accurate results for a particular case when both ϵ and t are small ? It seems clear that one would want to pick up as many of the larger terms in the double series expansion as possible. It would appear to be best to pick some intermediate limit of the form of Eq.(17), and choose n to pick up these larger terms most efficiently.

What if the algebra is done correctly, and matching is not achieved ? Then it always happens that one of the intermediate limits, of the form of Eq.(17) is non trivial. This new region with its limiting equations can then be solved and matched separately to each of the two given regions. This happens in resistive instability theory much more often than appears in the published papers.

It should be noted that the outer region is determined completely by Eq.(3), according to Eq.(17), but the inner region depends on another constant, the boundary condition of Eq.(4). How can we get agreement when one solution has more independent constants than the other ? The answer is that this additional degree of freedom is lost in the decay of the exponentially small terms going from Eq.(13) to Eq.(15). This phenomenon occurs universally in boundary layer theory.

The model equation, Eq.(3), is a greatly simplified problem in many ways. Perhaps the most significant simplification is that the series in $1/\omega\tau$, occuring in Eq.(15), is truncated. In general each I_n^* will have an infinite series expansion in powers of $1/\omega\tau$. Then the nonvanishing terms of the double power series will fall in the hatched regions



These are pleasingly symmetric.

In order to demonstrate the usefullness of asymptotic matching for the solution of more complicated problems I will modify Eq.(3) by operating on it with

$$\frac{d}{dt} + 2\omega \tag{18}$$

This yields

$$\epsilon L \frac{d^2 I}{dt^2} + (2\epsilon\omega L + R)\frac{dI}{dt} + 2\omega R I - \omega V e^{-\omega t} = 0 \tag{19}$$

We now give two boundary conditions,

$$I(0) = 0 \qquad \left.\frac{dI}{dt}\right|_{t=0} = 0 \tag{20}$$

The physical meaning of this equation is not clear, but it provides a second order differential equation that is useful for the next point that I want to make.

We first consider the outer limit, ϵ small and t finite, assume a Taylor series expansion for the current I in powers of ϵ , and equate coefficients of ϵ^n to zero. The solutions of the resulting differential equations are,

$$I_n = A_n e^{-2\omega t} + \left(\frac{\omega L}{R}\right)^n \frac{V}{R} e^{-\omega t} \tag{21}$$

where the A_n's are constants of integration.

That is, Eq.(21) is Eq.(7) plus terms that are annihilated by Eq.(18).

Similarly, we take the inner limit, ϵ small and Υ , defined by Eq.(18), finite. To lowest order we find

$$\frac{1}{\omega^2}\frac{d^2}{d\Upsilon^2} I_0^* + \frac{1}{\omega}\frac{d}{d\Upsilon} I_0^* = 0 \tag{22}$$

and so

$$I_0^* = 0 \tag{23}$$

from Eq.(20). The next higher orders yield

$$\begin{aligned} I_1^* &= \frac{V}{R}\left(\frac{\omega L}{R}\right)\left(\omega\tau - 1 + e^{-\omega\tau}\right) \\ I_2^* &= -3\frac{V}{R}\left(\frac{\omega L}{R}\right)^2\left[\frac{1}{2}(\omega\tau)^2 - \omega\tau + 1 - e^{-\omega\tau}\right] \end{aligned} \tag{24}$$

Expansion of the outer region for small t yields

$$\begin{array}{l} \qquad\qquad\qquad\qquad \xrightarrow{\qquad\qquad\qquad} t^n \\ \dfrac{RI}{V} = \left\{ \begin{array}{ll} \dfrac{RA_0}{V} + 1 & -2\left(\dfrac{RA_0}{V} + 1\right)\omega t + \left(2\dfrac{RA_0}{V} + \dfrac{1}{2}\right)\omega^2 t^2 + \dots \\ + \dfrac{\epsilon RA_1}{V} + \dfrac{\epsilon\omega L}{R} & -\left(2\dfrac{\epsilon RA_1}{V} + \dfrac{\epsilon\omega L}{R}\right)\omega t + \dots \\ + \dfrac{\epsilon^2 RA_2}{V} + \left(\dfrac{\epsilon\omega L}{R}\right)^2 + \dots & \\ \quad \vdots & \end{array} \right. \\ \downarrow \\ \epsilon^n \end{array} \tag{25}$$

and expansion of the inner region for large τ yields,

$$\xrightarrow{\qquad\qquad\qquad\qquad} (\omega\tau)^{-n}$$

$$\frac{RI^*}{V} = \left\{ \begin{array}{rrrl} & & 0 & + \cdots \\ & +\left(\frac{\epsilon\omega L}{R}\right)\omega\tau & -\frac{\epsilon\omega L}{R} & + \cdots \\ -\frac{3}{2}\left(\frac{\epsilon\omega L}{R}\right)^2(\omega\tau)^2 & +3\left(\frac{\epsilon\omega L}{R}\right)^2\omega\tau & -3\left(\frac{\epsilon\omega L}{R}\right)^2 & + \cdots \\ \vdots & \vdots & \vdots & \end{array} \right. \qquad (26)$$

$$\big\downarrow \; \epsilon^n$$

If these two expressions are to represent the same solution, they must match term by term. This provides a series of conditions that must be satisfied by the undetermined constants in Eq.(25). First, the terms independent of both ϵ and t must match; the upper left corner of Eq.(25) and the zero on the top line of Eq.(26). Thus matching can only be achieved if

$$A_0 = -\frac{V}{R} \qquad (27)$$

With this choice each term in the top line of Eq.(25) matches the corresponding term along the appropriate diagonal of Eq.(26)! In the same way the leading terms containing higher order constants determine

$$\begin{aligned} A_1 &= -2\left(\frac{\omega L}{R}\right)\frac{V}{R} \\ A_2 &= -4\left(\frac{\omega L}{R}\right)^2\frac{V}{R} \end{aligned} \qquad (28)$$

It appears to be a miracle that many more terms are matched than there are free parameters. Again, as in the previous example, the reason in that each series is generated from the same recursion form,

$$R\frac{dI}{dt} = -\epsilon L\frac{d^2I}{dt^2} - 2\epsilon\omega L\frac{dI}{dt} - 2\omega RI - \omega V e^{-\omega t} \tag{29}$$

Higher order terms involving A_o are generated by inserting lower order terms that are proportional to A_o into the right hand side of Eq.(29). Thus higher order matching is automatic.

In this case matching determines boundary conditions to be applied to the outer region. Two boundary conditions are applied at the origin and determine the inner solution. Information from one boundary condition decays exponentially, while the information from the other condition, modified by its journey through the inner layer, is then ready to be applied to the outer solution. At this point it is much like the problem of matching across an internal boundary, such as a plasma-vacuum interface.

This is the most common, but not the only use of asymptotic matching. It can also be useful when exploring a complicated, multidimensional parameter space. It can provide a check that all interesting region have been covered.

The greatest difficulty with this theory is the very large impedance mismatch between the number and complexity of the terms in the double series, and the person generating these terms. Indeed, one is frequently doing well in practice to obtain matching of the first term in each series.

In these cases the theory is invoked to justify this single term matching, and place it in the context of more accurate, higher order results. In any case, one should pay some attention to the recursion form, so that obvious failures of matching are not overlooked.

The basic concepts of this lecture I have learned from Martin Kruskal. The broad outline of his ideas on asymptotics is contained in reference 5.

In the rest of these lectures we will find many more practical examples of this theory.

LECTURE II

In this lecture I will discuss the Mercier/Suydam instability. This instability is described by ideal MHD, but its growth rates can be calculated by the same methods as the resistive instabilities that will be discussed in subsequent lectures. Because the basic methods can be developed for an ideal system, with somewhat fewer parameters than the resistive system that will be studied later, it makes a good introduction to the more general theory. Furthermore, the results are of interest in themselves.

The ideal MHD equations are

$$\frac{\partial}{\partial t}\rho + \underline{\nabla}\cdot\rho\underline{v} = 0 \tag{30}$$

$$\frac{\partial}{\partial t}\rho\underline{v} + \underline{\nabla}\cdot\rho\underline{v}\,\underline{v} = \underline{j}\times\underline{B} - \underline{\nabla}P \tag{31}$$

$$\frac{\partial}{\partial t}\underline{B} = \underline{\nabla}\times\underline{v}\times\underline{B} \tag{32}$$

$$\frac{\partial}{\partial t}\left(P/\rho^{\gamma-1}\right) + \underline{\nabla}\cdot\left(P\underline{v}/\rho^{\gamma-1}\right) = 0 \tag{33}$$

$$\underline{\nabla}\cdot\underline{B} = 0 \qquad \underline{j} = \underline{\nabla}\times\underline{B} \tag{34}$$

The dependent variables are the density ρ, the velocity of the fluid $\underline{v}$, the magnetic field $\underline{B}$ and the pressure P. The quantity $\underline{j}$ need not be defined as the motion of a charge in these equations, but is another way of writing $\underline{\nabla}$ x $\underline{B}$. It is often convenient to substitute

$$\underline{J} \times \underline{B} \equiv \underline{\nabla} \cdot \left[\underline{B}\,\underline{B} - \frac{1}{2} B^2 \underline{\underline{I}} \right] \tag{35}$$

Then all the equations are written in conservation form, including Eq.(33) for the entropy. These equations can be changed from conservation to convective form by use of Eq.(30).

To analyze these equations we consider small motions about a steady state. Thus we first calculate a steady state and then evaluate the eigenfrequencies of small perturbations. Clearly, this is rather artificial. If we undertook to examine a pendulum in this way, we would first find two steady states, one down and one up. Then we would, giving equal weight to these two possibilities, calculate the small amplitude motion. If we understand the limitations, however, this can be a good way to start examining the problem.

For the present lecture, we take a steady state with $\partial/\partial t = 0$ and $\underline{v} = 0$. Then only Eq.(31) is not satisfied trivially. We now choose a particularly simple geometry, a circular cylinder in which the equilibrium depends only on the radius r. Then Eq.(30) is satisfied if

$$\frac{d}{dr}\left(p + \frac{1}{2} B^2 \right) + \frac{1}{r} B_\theta^2 = 0 \tag{36}$$

Next we consider perturbations around this equilibrium. We introduce a displacement $\underline{\xi}$ as the integrated velocity,

$$\underline{v} \equiv \partial \underline{\xi} / \partial t \tag{37}$$

Since the equilibrium is independent of θ, z, and t, the eigenfunctions depend on these quantities exponentially,

$$\underline{\xi} \propto \exp(im\Theta - inkz - i\omega t) \tag{38}$$

A couple of comments can be made on this exponential function. First, taking opposite signs for the θ and z terms makes lines of constant phase follow right handed helices. This helps intuition. Second, taking the coefficient of z to be nk is significant. Here $k \equiv 2\pi/L$ defines a length over which the system is strictly periodic, and thus has the topology of a torus. The factor n allows variations that are harmonics of this length. The straight system studied here has little or nothing in common with a real straight device because the boundary conditions are very different in the two cases.

After considerable algebra, the equations for the perturbed quantities can be written in the form familiar here |1|,

$$\begin{aligned} D\frac{d}{dr} r\xi_r &= C_1(r\xi_r) - rC_2 P \\ D\frac{d}{dr} P &= \frac{1}{r} C_3(r\xi_r) - C_1 P \end{aligned} \tag{39}$$

Here

$$D \equiv (\rho\omega^2 - F^2)G$$

$$C_1 \equiv \frac{B_\Theta^2}{r}\rho^2\omega^4 - \frac{1}{r}(F^2 - n^2k^2B^2)G$$

$$C_2 \equiv \rho^2\omega^4 - \frac{1}{r^2}(m^2 + n^2k^2r^2)G$$

$$
\begin{aligned}
C_3 &\equiv \frac{B_\theta^4}{r^2}\rho^2\omega^4 - \Big\{(\rho\omega^2 - F^2)^2 - 2\frac{B_\theta^2}{r^2}F^2 \\
&\qquad -2\frac{n^2k^2}{r^2}B_\theta^2 B^2 + (m^2 + n^2k^2r^2)\frac{B_\theta^4}{r^4}\Big\}G \\
F &\equiv \frac{mB_\theta}{r} - nkB_z = \frac{B_\theta}{r}(m - nq) \\
q &\equiv \frac{krB_z}{B_\theta} \\
G &\equiv (\gamma P + B^2)\rho\omega^2 - \gamma P F^2 \\
C_4 &\equiv (C_1^2 - C_2C_3)/D \\
&= 4\frac{n^2k^2}{r^2}B_\theta^2(\rho\omega^2B^2 - \gamma P F^2) - (\rho\omega^2 - F^2)C_2
\end{aligned}
\tag{40}
$$

and B_θ and B_z are components of the magnetic field $\underline{B}$. These equations differ slightly in detail from reference 1 , because the dependent variable P has been altered by adding $(B_\theta^2/r^2)\, r\,\xi_r$ to it. The chief effect of this transformation is to eliminate derivatives of equilibrium functions from all of Eq.(40). One of the nice things about this set of equations is that it is Hamiltonian, P is the conjugate momentum to $r\,\xi_r$. The other nice attributes of this equation have been given in reference 1.

Now we wish to look for the Mercier/Suydam instabilities. We will first sketch a line of reasoning used by Suydam to show that such instabilities exist. Then we will go through the argument again in more detail and calculate actual growth rates, using matched asymptotic expansions.

First consider Eq.(39) with $\omega^2 = 0$. This is reasonable because it has been shown that the eigenvalues ω^2 are always real. Thus any modes going into the unstable part of the complex ω plane must pass by $\omega^2 = 0$, making this a good place to watch. The resulting equations are singular where the quantity $F^2 = 0$ vanishes, that is, where

$$m - n\, q(r_s) = 0 \tag{41}$$

This equations selects a given m and n, and defines the singular point $r = r_s$. To understand the reduced Eq.(39), then, we must understand the nature of the solutions near the singular point. This introduces another theme in this set of lectures; understanding a set of differential equations means finding all the singular points, and analyzing the behavior of the solutions in the vicinity of each such point.

There is a slight algebraic problem in the present case because the right hand sides of the two parts of Eq.(39) are nearly proportional to each other as D vanishes, as can be seen from the last part of Eq.(40). One could beat through the algebra, but it is easier to eliminate P and find

$$\frac{d}{dr}\frac{D}{rC_2}\frac{d}{dr}(r\xi_r) - \left[\frac{d}{dr}\left(\frac{C_1}{rC_2}\right) + \frac{C_1^2 - C_2C_3}{rC_2D}\right](r\xi_r) = 0 \tag{42}$$

In the limit of vanishing ω^2,

$$\frac{D}{rC_2} = \frac{rF^2}{m^2 + n^2k^2r^2} \tag{43}$$

$$\frac{C_1}{rC_2} = \frac{F^2 - n^2k^2B^2}{m^2 + n^2k^2r^2}$$

Thus when both ω^2 and F^2 vanish, near the singular surface,

$$\frac{d}{dr}\left(\frac{C_1}{rC_2}\right) = -\frac{n^2k^2}{m^2+n^2k^2r^2}\left[\frac{d}{dr}B^2 - \frac{2r\,n^2k^2B^2}{m^2+n^2k^2r^2}\right]$$

$$\frac{C_1^2 - C_2C_3}{rC_2D} = -\frac{4n^2k^2}{m^2+n^2k^2r^2}\,\frac{B_\theta^2}{r} \tag{44}$$

When Eq.(41) is satisfied, the components of the equilibrium magnetic field are related, and

$$\frac{r\,n^2k^2B^2}{m^2+n^2k^2r^2} = \frac{B_\theta^2}{r} \tag{45}$$

To obtain the leading term in the coefficient of the derivative, F is expanded in a Taylor series around the singular surface,

$$F \simeq -\frac{B_\theta}{r}\,n(r-r_s)\,q' \tag{46}$$

where the prime denotes derivative. It has been assumed that q' does not vanish at the singular surface. When q has multiple roots the nature of the singularity changes, leading to a variety of different problems. Here we treat only the most important of these problems. Then the approximation for Eq.(42), or Eq.(39), valid near the singular point is

$$\frac{d}{dr}(r-r_s)^2\frac{d}{dr}(r\xi_r) + D_s(r\xi_r) = 0 \tag{47}$$

where

$$D_s \equiv -\left(\frac{2k^2 r}{B_\Theta^2 q'^2}\frac{dP}{dr}\right)\Bigg|_{r=r_s} \tag{48}$$

$$= -\frac{2q^2}{B_z^2 q'^2 r}\frac{dP}{dr}$$

and use has been made of Eq.(36).

This equation will occur over and over again in this set of lectures and it will be derived in many different ways. The parameter D_s is a measure of the destabilizing force associated with the pressure gradients. Its significance will be further elucidated as we go along.

Since Eq.(47) is homogeneous in the independent variable, it has a powerlike solution

$$r\xi_r = |r - r_s|^s \tag{49}$$

where

$$s(s+1) + D_s = 0 \tag{50}$$

or

$$s = \frac{1}{2}\left[-1 \pm (1-4D_s)\right]^{1/2} \tag{51}$$

When $D_s < \frac{1}{4}$, the two solutions for s are real, and $n\,\xi_n$ diverges near the singular point. This case will be treated in subsequent lectures. Here we will be concerned with the case $D_s > \frac{1}{4}$ so that s is complex. The meaning of a complex exponent is

$$X^s = X^{-\frac{1}{2}} \exp\left[\pm \frac{i}{2}(4D_s - 1)^{\frac{1}{2}} \ln X\right]$$

$$= X^{-\frac{1}{2}} \cos\left[\frac{1}{2}(4D_s - 1)^{\frac{1}{2}} \ln X\right] \tag{52}$$

$$\pm i X^{-\frac{1}{2}} \sin\left[\frac{1}{2}(4D_s - 1)^{\frac{1}{2}} \ln X\right]$$

That is, it oscillates infinitely rapidly as one approaches the singular point. This is really messy behavior. However, Suydam and others have shown that reducing the frequency a small amount reduces the number of oscillations. Thus this behavior implies the existance of instabilities. Rather than follow the formal procedures used previously in showing this, here we will derive formulas for calculating growth rates.

Now, having established the probable existance of localized, weakly unstable modes, we can consider how to calculate their growth rates. We wish to study cases where ω^2 is quite small. Over most of the range of r, ω^2 is much less than F^2 and can be neglected. However, near the singular point where $F^2 = 0$ this is not true. In a small layer near where ω^2 and F^2 are comparable, different equations must be used. Thus this problem is of the boundary layer type. Inertia is important only in this boundary layer, so both the physics and the equations are different in the two regions.

Now we consider the outer region in more detail. The solution here is determined by Eq.(39) and Eq.(40) with $\omega^2 = 0$. Near the singular point, these solutions have the form given in Eqs.(49-52). In particular, since we desire real solutions, we can set

$$r\xi_r \simeq G|r-r_s|^{-1/2} \exp i\left[\frac{u}{2}\ln\left|\frac{r-r_s}{a}\right| + \gamma\right] + \text{c.c.} \tag{53}$$

where

$$u \equiv (4D_s - 1)^{1/2} \tag{54}$$

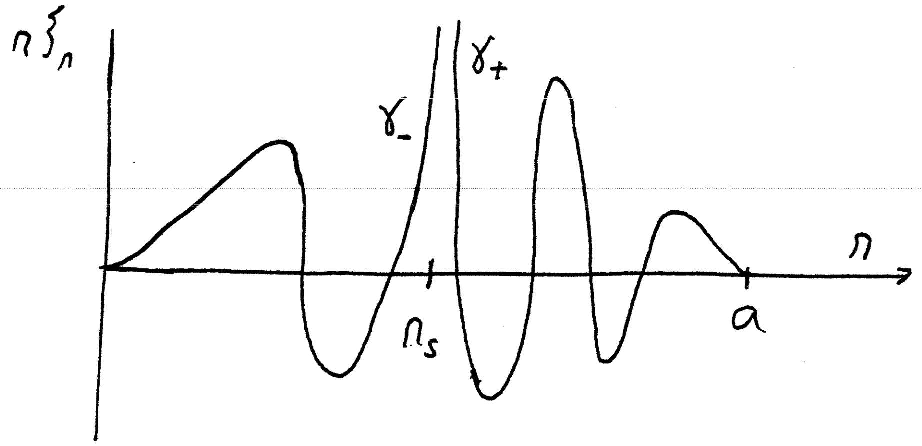
a is a length scale so that the argument of the log is dimensionless, and c.c. stands for the complex conjugate. This is another way of writing a solution proportional to two arbitrary constants $G\cos\gamma$ and $G\sin\gamma$, the two constants of integration of the second order differential equation,

$$\begin{aligned} r\xi_r \simeq 2|r-r_s|^{-1/2}\Big[&G\cos\gamma\cos\left(\frac{u}{2}\ln\left|\frac{r-r_s}{a}\right|\right) \\ &- G\sin\gamma\sin\left(\frac{u}{2}\ln\left|\frac{r-r_s}{a}\right|\right)\Big] \end{aligned} \tag{55}$$

The differential equation satisfies a regularity condition at r = 0, and a boundary condition at some radius r = a. For example, if there is a rigid boundary at the edge of the plasma, we have the conditions

$$\xi_r(0) = 0, \quad \xi_r(a) = 0 \tag{56}$$

These boundary conditions define the solution up to a multiplicative constant on either side of the singularity. That is, G is undetermined, but quantities γ_+ and γ_- are determined for solutions that approach the singular surface from $r > r_s$ and $r < r_s$



Note that

$$\gamma = \lim_{r \to r_s} \left\{ \operatorname{Im} \ln \left[-i \frac{|r-r_s|^{3/2}}{u} \left(\frac{d}{dr} r\xi_r + \frac{1+iu}{2|r-r_s|} r\xi_r \right) \right] - \frac{u}{2} \ln \left| \frac{r-r_s}{a} \right| \right\} \tag{57}$$

Equation (53) is the small $|r - r_s|$ limit of the outer solution that will be used for matching. This is just one term of the double series that was described in Lecture I. Other terms come from continuing the expansion of Eq.(53) in powers of $r - r_s$, and from higher order equations that calculate corrections in powers of ω^2. Thus the full double series would have the form:

$$\eta\xi_\eta = \cos\left(\frac{u}{2}\ln x\right)\begin{cases} 2G_0\cos\gamma\, x^{-\frac{1}{2}}\left[1+O(\eta-\eta_s)\right] \\ +\omega^2 G_1(\qquad) + \ldots \\ + \ldots \end{cases}$$

$$+\sin\left(\frac{u}{2}\ln x\right)\begin{cases} -2G_0\sin\gamma\, x^{-\frac{1}{2}}\left[1+O(\eta-\eta_s)\right] \\ +\omega^2 G_1(\qquad) + \ldots \\ + \ldots \end{cases} \tag{58}$$

This is an example of the very large impedance mismatch between the desires of matching theory for a reasonably large number of terms so that the structure of the matching can be seen, when it is coupled to the strength of the person generating these terms. In the present case we will only match the upper corner terms of the double series. Matching theory shows that this is a precise requirement when considered in a larger context.

Now consider the solution inside the boundary layer. First we must choose a scale for the expected radial dependence of ξ . This is chosen so that $|F^2| \sim \omega^2$. Thus we define a transformed independent variable t,

$$t \equiv \frac{n\xi'}{\Omega}(\eta-\eta_s) \qquad \Omega^2 \equiv -\omega^2\frac{\eta_s^2\rho}{B_\Theta^2}\bigg|_{\eta=\eta_s} \tag{59}$$

When ω^2 is small, t is finite when $r - r_s$ is small. When t is large, $\omega^2 << |F^2|$ and the solution should approach that of the outside region. The leading order terms in the limit as ω^2 goes to zero, but t is held constant, are:

$$\left(\frac{d}{dr}\right)^2 = n^2 q'^2 \frac{B_\Theta^2}{r_s \rho \omega^2}\left(\frac{d}{dt}\right)^2$$

$$\frac{D}{rC_2} \simeq -\frac{(\rho\omega^2 - F^2) r_s}{m^2 + n^2 k^2 r_s^2}$$

$$\frac{C_1}{rC_2} \simeq -\frac{n^2 k^2 B^2(r_s)}{m^2 + n^2 k^2 r_s^2}$$

$$\frac{C_1^2 - C_2 C_3}{r C_2 D} \simeq \frac{4n^2k^2}{r_s^3 C_2(r_s)} B_\Theta^2(r_s)\Big[(\gamma P$$

$$+ B^2)\rho\omega^2 - \gamma P F^2 - \rho\omega^2 \gamma P\Big] \tag{60}$$

$$\simeq -\frac{4n^2k^2B_\Theta^2}{r_s(m^2+n^2k^2r_s^2)}$$

$$+ \frac{4n^2k^2B_\Theta^2}{r_s(m^2+n^2k^2r_s^2)}\left[\frac{\rho\omega^2\gamma P}{(\gamma P + B^2)\rho\omega^2 - \gamma P F^2}\right]$$

$$F^2 \simeq -\rho\omega^2 t^2$$

$$r\xi_r \equiv y(t)$$

Then Eq.(42), or (39), reduces to

$$\frac{d}{dt}(1+t^2)\frac{d}{dt}y + \left[D_s - \frac{4k^2/q'^2}{1+(B^2/\gamma P)+t^2}\right]y = 0 \tag{61}$$

Note that for t large, this equation reduces to Eq.(47), the approximation in the outer region valid near the singular surface. This implies that matching can be done successfully.

Equation (61) appears fairly simple. However, it seems to have somewhat more singularities than can be treated with hypergeometric functions. Thus it appears that solutions must be calculated numerically. A given solution depends only on three numerical parameters, D_s, $B^2/\gamma P$, and $4k^2/q'^2$, and boundary conditions. The three parameters do not depend on t since their variation across the boundary layer is a higher order effect in the expansion.

Since Eq.(61) has the form of Eq.(47) for large t, the corresponding solutions will have the form of Eq.(53):

$$\begin{aligned} y &\simeq \left\{H t^{-1/2} \exp i\left[\frac{u}{2}\ln t + \eta\right] + c.c.\right\}\left[1 + O(1/t)\right] \\ &= H\left|\frac{\Omega}{nq'(\Omega-\Omega_s)}\right|^{1/2} \exp i\left[\frac{u}{2}\ln\left|\frac{\Omega-\Omega_s}{a}\right| - \frac{u}{2}\ln\left|\frac{\Omega}{nq'a}\right| + \eta\right] + c.c. \end{aligned} \tag{62}$$

Hence matching is accomplished by

$$G_+ = H_+ \left| \frac{\Omega}{n q' a} \right|^{1/2} \tag{63}$$

$$\gamma_+ = \eta_+ - \frac{u}{2} \ln \left| \frac{\Omega}{n q' a} \right| - j_1 \pi \tag{64}$$

for $\eta > \eta_s$, and by

$$G_- = H_- \left| \frac{\Omega}{n q' a} \right|^{1/2} \tag{65}$$

$$\gamma_- = \eta_- - \frac{u}{2} \ln \left| \frac{\Omega}{n q' a} \right| - j_2 \pi \tag{66}$$

for $\eta < \eta_s$. Here j_1 and j_2 are arbitrary integers. Since $\exp(ij\pi) = \pm 1$ matching is accomplished equally well for any $j_{1,2}$. The logarithm is formally large as ω^2 becomes small, but it diverges very weakly. It is generally more accurate to consider such logarithms as approximate constants in the limit of small ω^2, rather than to modify the asymptotic series.

We now count conditions and variables. There are two boundary solutions, one each at $\eta = 0$ and $\eta = a$. In addition there are the four matching conditions given above, yielding a total of six conditions to be satisfied. There are six independent constants of integration, coming from three second order differential equations. One of these equations describes the outside region $\eta < \eta_s$, one the boundary layer, and one the outside region $\eta > \eta_s$. The equations and boundary conditions are all homogeneous and linear, so we have properly defined an eigenvalue problem.

Two expressions for the eigenvalue are

$$\Omega = n q' a \exp \frac{2}{u}\left(\eta_+ - \gamma_+ - j_1 \pi\right) \tag{67}$$

from Eq.(64), and

$$\Omega = m q' a \exp \frac{2}{u}\left(\eta_- - \gamma_- - j_2 \pi\right) \tag{68}$$

from Eq.(66). Hence, equating these expressions, we find

$$\gamma_+ - \gamma_- = \eta_+ - \eta_- - \left(j_1 - j_2\right)\pi \tag{69}$$

which is a condition that must be satisfied by the various constants.

It remains to calculate the relation between η_+ and η_- implied by the differential equation, Eq.(61). We do this by introducing a technique taht will be used throughout the rest of these lectures, representing an arbitrary solution as a superposition of an even and odd solution. Since Eq.(61) is invariant under change of sign of t, it does have even and odd solutions. Each of these solutions has the form of Eq.(62) for large t, with phases denoted by η_e and η_o.

Now consider a mixture of the two solutions, with α the proportion of the odd solution. Then, for large positive t,

$$y \simeq t^{-1/2}\Big[H_e \exp i\left(\tfrac{u}{2}\ln t + \eta_e\right) + \alpha H_o \exp i\left(\tfrac{u}{2}\ln t + \eta_o\right)\Big] + \text{c.c.}$$

$$= t^{-1/2}\Big[(H_e \cos\eta_e + \alpha H_o \cos\eta_o)^2 + (H_e \sin\eta_e + \alpha H_o \sin\eta_o)^2\Big]^{1/2} \exp i\Big[\tfrac{u}{2}\ln t + \tan^{-1}\left(\frac{\sin\eta_e + z\sin\eta_o}{\cos\eta_e + z\cos\eta_o}\right)\Big] + \text{c.c.} \tag{70}$$

where

$$z \equiv \alpha \frac{H_o}{H_e} \tag{71}$$

Then η_+ is given in terms of η_e and η_o by

$$\eta_+ = \tan^{-1}\left(\frac{\sin\eta_e + z\sin\eta_o}{\cos\eta_e + z\cos\eta_o}\right) \tag{72}$$

and similarly, η_- is given by

$$\eta_- = \tan^{-1}\left(\frac{\sin\eta_e - z\sin\eta_o}{\cos\eta_e - z\cos\eta_o}\right) \tag{73}$$

These can be substituted into Eq.(69), and the resulting expression solved for Z, the mixture of odd and even solutions. This expression for Z can then be substituted back into Eqs.(72) and (73), yielding,

$$\eta_+ + \eta_- = \eta_e + \eta_o \pm \tan^{-1}\Big[\tan^2(\eta_e - \eta_o) + \frac{\tan^2(\gamma_+ - \gamma_-)}{\cos^2(\eta_e - \eta_o)}\Big]^{1/2} \tag{74}$$

The $\pm$ sign arises from the fact that the equation for Z is a quadratic, with two solutions. Multiplying Eq.(67) and (68) together, we find

$$\Omega^2 = n^2 q'^2 a^2 \exp \frac{2}{u} \left(\eta_+ + \eta_- - \gamma_+ - \gamma_- - j\pi \right) \tag{75}$$

$$j \equiv j_1 + j_2$$

Now we have an expression for the growth rates, we must determine its meaning.

First, j is an arbitrary integer, so Eq.(75) describes an infinite sequence of unstable modes, accumulating to zero frequency for j approaching infinity. The frequency of each of these modes is a constant multiple of the frequency of the preceding mode. In fact, in general, these localized stability criteria of the Mercier/Suydam type are conditions for the existence of an infinite sequence of unstable modes.

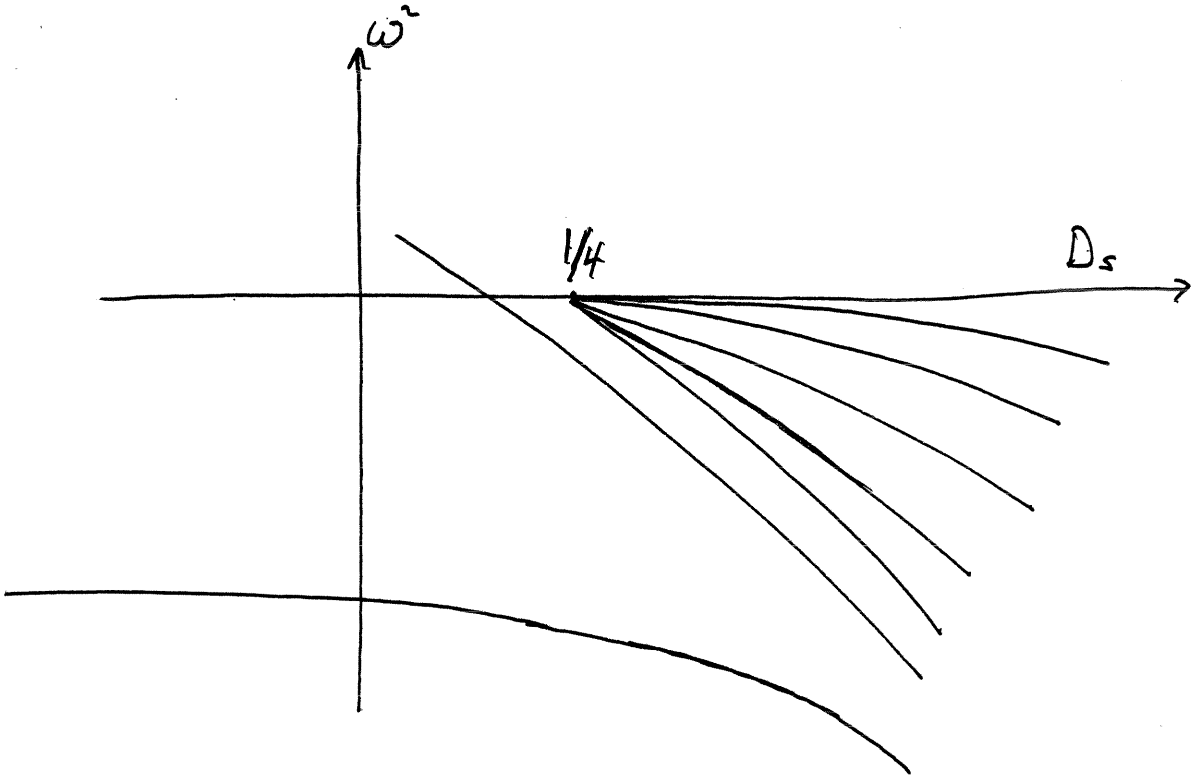
Next, j cannot be completely arbitrary. The basic approximation used in deriving Eq.(75) was that $\Omega/nq'a$ is small in Eq.(59). Then the variation of the equilibrium over the range of finite t is small, and can be neglected. This reduces to the requirement that the argument of the exponential in Eq.(75) be negative, thus determining the range of validity in terms of j.

Actually, there are two sets of unstable modes, corresponding to the two solutions of Eq.(74). These are something like a set of even solutions, and a set of odd solutions, but they can be mixed if the boundary conditions are not symmetric.

It is too bad that we could not calculate the quantities η_e and η_o analytically. Thus the problem of finding actual growth rates still requires a fair amount of numerical calculation. However, we can calculate an infinite number of frequencies from the four numbers γ_+, γ_-, η_o and η_e. Further, the latter two parameters depend only on the three constants D_s, $B^2/\gamma p$, and $4k^2/q'^2$

Analytic results for growth rates of Suydam modes were first given by Kulsrud |6|. The discussion here follows more closely Appendix D of |7|.

Now we should consider where these results fit into a larger context. Let us consider a one parameter family of equilibria, with differing values of D_s, the instability driving parameter defined in Eq.(48). Then the growth rates of the unstable modes might have the following dependence on D_s.



In this figure there are an infinite number of unstable modes for $D_s > \frac{1}{4}$ and all but two converge to zero frequency at $D_s = \frac{1}{4}$. In this figure the lowest mode is a kink mode, driven by currents flowing parallel to the magnetic field, and is more or less independent of pressure gradients and D_s. The remaining mode, as drawn here, seems related to pressure gradients. However, for small values of D_s at least, it appears to associated with a mode for which the argument of the exponential is positive in Eq.(75). This figure is speculative, and not based on a specific calculation. Nevertheless, it reflects the results of a considerable amount of thought that has gone into speculations into the nature of localized instabilities. The trend of this line of thought will now be discussed. Considerablely more detail is given in reference 8.

In that paper the energy driving ideal instabilities has been written,

$$\begin{aligned}\delta W = \int d\tau \Big\{ & \Big[\underline{\nabla}\times(\underline{\xi}\times\underline{B}) - \frac{\underline{B}}{B^2}\,\underline{\xi}\cdot\underline{\nabla}P\Big]^2 \\ & + \gamma P(\underline{\nabla}\cdot\underline{\xi})^2 - \frac{\underline{j}\cdot\underline{B}}{B^2}\,\underline{\xi}\times\underline{B}\cdot\underline{\nabla}\times(\underline{\xi}\times\underline{B}) \\ & - 2(\underline{\xi}\cdot\underline{\nabla}P)(\underline{\xi}\cdot\underline{\kappa})\Big\}\end{aligned} \tag{76}$$

where

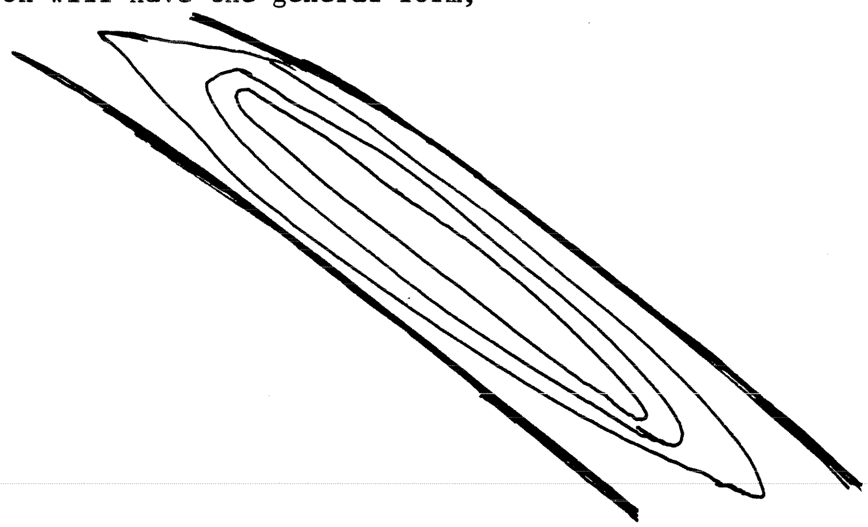
$$\underline{\kappa} \equiv \frac{1}{2B^4}\Big[\underline{B}\times\underline{\nabla}(2P+B^2)\Big]\times\underline{B} \tag{77}$$

is the curvature of the magnetic lines of force. The first two terms in the energy are stabilizing. The third term is related to the destabilizing force associated with parallel current flow, and is not treated here. The last term is related to destabilization due to the curvature of the field lines, and is what this lecture is all about.

When the curvature points in the same direction as the pressure gradient, it is destabilizing. However, whenever the pressure gradient and curvature are not exactly antiparallel, there is a range of directions of $\underline{\xi}$ for which the curvature term can be destabilizing. Two facts prevent this term from always producing instabilities. First, the average of the component of the curvature lying in the surfaces always vanishes. Thus, if the average of the curvature normal to the surfaces is stabilizing, $\underline{\xi}$ must wind and twist to produce an instability. Then the second fact becomes important, that variation of along lines of force produces stabilizing energy from the first term of Eq.(76). Thus, instability of this type results from competition between the first and last terms of the energy.

The quantity D_s in the Suydam criterion is an average of the normal curvature over a field line. Since in the straight case the curvature is constant along a field line, the average is equal to the local value. In more general systems, however, there will be deviations from the local value. It is worthwhile to discuss these, though they are slightly outside the scope of the present lecture.

These slowly growing modes are nearly divergence-free, so they can be represented by a stream function. For localized modes, this stream function will have the general form,



where the solid outer lines represent magnetic surface. That is, the convection cells are aligned along magnetic surfaces. Small changes in the perturbation lying in the surface, resonating with changes in the curvature lying in the surface, can produce destabilizing energy. This can be comparable to that produced by the much smaller normal perturbation working against the normal curvature, without significantly increasing magnetic field energy, the first term of Eq.(76). This effect is already included in the general Mercier stability criterion.

On the other hand, variations in the normal perturbations are coupled strongly to variations in the surface perturbations, through the divergence condition, for modes of the form of the figure above. This gives strong stabilization in the magnetic energy term of Eq.(76). Thus for modes of that shape, it is difficult or impossible to obtain additional destabilization by varying the normal perturbation.

However, it is also possible to arrange for convection cells aligned perpendicular to the magnetic surfaces. Cells of this type can vary along field lines with a diminished, but not negligible, effect on the magnetic field energy. These cells can be formed by considering perturbations with large values of n. That is, we have been considering integers m and n, and a magnetic surface where

$$m - n\,q(r_s) = 0$$

We can equally well multiply m and n by a large integer and the above equation is satisfied at the same singular surface.

The result of this discussion, then, is that larger deviations from the localized Mercier/Suydam stability criteria are to be expected when n is large. These deviations should have the form of a finite number of unstable modes when the localized criterion does not predict instability. This deviation should be enhanced when there is considerable variation of normal curvature following a magnetic field line.

It is also possible that there can be an infinite number of unstable modes as n approaches infinity even when the localized criterion predicts stability. Since these localized criteria more exactly are criteria for an accumulation point of unstable modes, this implies that the criteria do not work when n is coupled to the integer j of Eq.(75). In that case other accumulation points of the spectra may arise. Both Mikhailovsky |9| and several French groups |10| have attempted such calculations, but the results have not been entirely satisfactory.

One case is known where large n is required to obtain a satisfactory localized stability criterion, and this was the first realistic MHD stability calculation ever done |11|. They treated an axisymmetric example without shear, corresponding to a general tokamak with $q = 0$. In that case, the curvature of the lines of force is strictly normal to the magnetic surfaces.

Other cases that have been treated are those in which the average of the normal curvature is very small compared with local values. Then non-constant perturbations, with large n, must certainly be more unstable than constant perturbations. Such calculations were the original motivation for calculating ballooning modes.

Alltogether, this discussion illustrates a weak point in analytic theory. There appear to be significant deviations from Suydam/Mercier criteria that are difficult to calculate in an orderly way. This means that practically there may be curvature driven modes that are unstable over a wider ranger of parameters than indicated in the localized theories, and there may even be different accumulation points if somewhat different orderings are taken. To me, all these effects can be called ballooning. The analogy is with elastic membranes that tend to suffer their largest perturbations where they are weakest. They thus tend to be unstable pressure containers.

It appears that these ideas will be most fruitful when they can be coupled to the results of numerical calculations. These will be difficult because they involve the most marginal of instabilities. Some idea of the way numerical results can complement analytic results can be seen from the following table, taken from reference 7.

$-\omega^2$

Finite element	Eq.(75)
$2.28 \cdot 10^{-1}$	$3.62 \cdot 10^{-2}$
$1.78 \cdot 10^{-3}$	$1.78 \cdot 10^{-3}$
$4.92 \cdot 10^{-5}$	$6.56 \cdot 10^{-5}$

Finite elements lose accuracy for very localized modes, and Eq.(75) loses accuracy for larger growth rates. Thus the ranges of accuracy tend to be complementary. In this case it is fortunate that there is a region where both are sufficiently accurate.

Returning to Eq.(61), we have discussed the parameter D_s at some length. A few words can be said about the last term in that equation. This has been derived and discussed in more detail in reference 12. Most important, it represents the inertia of fluid driven along field lines by inhomogeneities of the perturbed pressure. It is larger when the shear, q' is small, and also when the fluid is nearly incompressible, or has a large pressure.

Finally, some attention should be paid to the validity of the matching used in this lecture, since it has not been carried beyond the lowest order. However, inspection of Eq.(42) shows that the matching region,

$$1 \gg |F^2| \gg |\omega^2|$$

can equally well be approached from either side, where F^2 is near the top or bottom of the above range. Thus matching can be obtained to as many terms as one can manage to calculate.

LECTURE III

Now, in the third lecture, we are ready to consider resistivity. Basically, the story here has been forcast by the previous lectures. We will be considering unstable modes with slow growth rates, and the region of significant behavior will be localized, as in the Mercier/Suydam modes treated in the previous lecture. This localized region will be centered about the same singular surface that was treated there. The small terms that are important here include inertia, as before, but will include resistivity also. The chief difference between this lecture and last will be a modification of Eq.(61), the equation determining the behavior in the boundary layer, to include the effects of resistivity.

A number of other small terms could also be important in the boundary layer. Among these terms are viscosity, heat conduction, and the dispersion associated with gyration effects that is often called FLR terms. However, I find it easier to do one thing at a time. The simple addition of resistivity will give new insight into the nature of the boundary layer. Then other terms can be fit into the scheme according to taste and necessity.

Before considering the boundary layer, I will first set the stage by considering the problem from the beginning.

The full fluid equations Eqs.(30-34), are modified by the addition of resistivity,

$$
\begin{aligned}
&\frac{\partial}{\partial t}\rho + \underline{\nabla}\cdot\rho\underline{v} = S_m \\
&\frac{\partial}{\partial t}\rho\underline{v} + \underline{\nabla}\cdot\rho\underline{v}\,\underline{v} + \underline{\nabla}P - \underline{j}\times\underline{B} = S_P \\
&\frac{\partial}{\partial t}\underline{B} = \underline{\nabla}\times(\underline{v}\times\underline{B} - \eta\underline{j}) \\
&\frac{\partial}{\partial t}(P/\rho^{\gamma-1}) + \underline{\nabla}\cdot(\underline{v}P/\rho^{\gamma-1}) \\
&\qquad = \frac{\gamma-1}{\rho^{\gamma-1}}\left(\eta j^2 + S_E\right) \\
&\underline{j} = \underline{\nabla}\times\underline{B} \qquad \underline{\nabla}\cdot\underline{B} = 0
\end{aligned}
\tag{78}
$$

This differs from the previous equations only in two terms involving resistivity; one involving the induction equation, the other the heating of the fluid. They also differ in the addition of source terms for mass, momentum, and energy. The sources have been introduced so that it is possible to have a steady state, analogous to that used in the previous lecture. The problem is that resistivity allows fluid diffusion, and there are only a few special configuration that do not diffuse. One such configuration has a constant current in the z direction, with the associated pressure distribution, so that the curl of the current vanishes. We wish to consider perturbations around more general configurations. Then the steady state will have a nonvanishing velocity. This velocity, and the associated sources, will vanish linearly with the resistivity when it is small.

Now we consider the limit of small resistivity. Always in doing such things you have to use intuition or make some guesses about the nature of the interesting behavior. Now I am going to tell you that we want to consider time variations such that $\partial/\partial t \sim \eta^{1/3}$ in the limit of small resistivity. In fact, an MHD fluid wiggles and oozes in many ways. Ideal modes are independent of resistivity when it is small. Diffusion modes will vanish linearly in this limit. So the statement that $\partial/\partial t \sim \eta^{1/3}$ is a way of focusing on a particular type of behavior that may be interest at a particular time and place, perhaps here and now. We will cut the cloth of the theory to calculate this type of mode.

First we calculate the equilibrium in powers of the resistivity. To lowest order the results are the same as those obtained in the previous lecture. To next order there are terms that are linear in resistivity, but they will not affect the much faster growing perturbations that will be calculated in this lecture, so they can be ignored.

In calculating the perturbations around this equilibrium, we again introduce the integrated velocity $\underline{\xi}$ as an independent variable, and also the perturbed magnetic field

$$\underline{B} = \underline{B} + \underline{b} \tag{79}$$

Henceforth we will not dear with the total magnetic field, so we will not distinguish it from the equilibrium field. Again, the equilibrium is independent of t, θ, and z, so we can introduce

$$\underline{\xi} = \underline{\xi}(r)\exp(\gamma t + im\theta - inkz) \tag{80}$$

The factor γ is introduced because we are interested primarily in calculating real growth rates. This γ should not be confused with the ratio of specific heats, since the latter is always associated with the pressure P.

The equations for the perturbed quantities can then be written

$$\rho\gamma^2\underline{\xi} = (\underline{\nabla}\times\underline{b})\times\underline{B} + \underline{j}\times\underline{b} + \underline{\nabla}\left[\gamma P(\underline{\nabla}\cdot\underline{\xi}) + \underline{\xi}\cdot\underline{\nabla}P\right] \tag{81}$$

$$\underline{b} - \frac{\eta}{\gamma}\underline{\nabla}\times\underline{\nabla}\times\underline{b} = \underline{\nabla}\times(\underline{\xi}\times\underline{B}) \tag{82}$$

When the resistivity vanishes, this reduces to the equations for ideal modes.

We now consider the limit in which the resistivity and growth rate γ both vanish. The limit takes us right back to the equations for the outer region in the previous lecture, particularly Eq.(47). In that lecture we were primarily interested in the case $D_s > \frac{1}{4}$. Here we are mostly interested in the case $D_s < \frac{1}{4}$. Resistivity may have a strong effect on weakly unstable Suydam modes, but we will first want to show the existence of unstable modes in regimes that are Suydam stable. Thus, near the singular point we find in the region $r < r_s$

$$\Omega \xi_{\Omega} \sim A_{I} |\Omega - \Omega_s|^{s} + B_{I} |\Omega - \Omega_s|^{-1-s} \tag{83}$$

where

$$s \equiv -\frac{1}{2} + \frac{1}{2}\left(1 - 4D_s\right)^{1/2} \tag{84}$$

and A_I and B_I are constants determines by Eq.(39) or (42) with $\omega^2 = 0$, and the boundary condition of Eq.(56). In fact, only the ratio of the constants is determines since the equations and boundary conditions are homogeneous. Similarly, for the region $\Omega > \Omega_s$,

$$\Omega \xi_{\Omega} \sim A_{III} |\Omega - \Omega_s|^{s} + B_{III} |\Omega - \Omega_s|^{-1-s} \tag{85}$$

This is very similar to the outside region discussed previously, but perhaps simpler because there are no complex quantities. The ratio of the constants here is equivalent to the phase parameter γ that was introduced in Eq.(53).

We now consider the inner layer, where the length scale will be ordered to the resistivity. First, we will want to order the length scale and growth rate so that all three terms of the induction equation, Eq.(82), are comparable,

$$\gamma (\Omega - \Omega_s)^2 \sim \eta \tag{86}$$

This says that the boundary layer thickness is about a resistive skin depth on the growth rate time scale. We now need another relation to determine the length and times scales in terms of the resistivity. It is convenient to use the relation of the previous lecture,

$$\rho\omega^2 \sim \underline{B}\cdot\underline{\nabla}(\underline{B}\cdot\underline{\nabla}) \sim (F')^2(\Omega-\Omega_s)^2 \tag{87}$$

so that the present boundary layer is a modification of the previous one. We then find

$$\Omega-\Omega_s \sim \gamma \sim \eta^{1/3} \sim \epsilon \tag{88}$$

The ϵ has been introduced as a tag to keep track of the size of various things.

We now consider the projections and magnitudes of the perturbation quantities,

$$\begin{aligned}\underline{\xi} = {} & (\epsilon\xi_n^{(1)} + \ldots)\,\underline{e}_n \\ & + (\xi_\perp^{(0)} + \ldots)\,\frac{\underline{B}\times\underline{e}_n}{B^2} \\ & + (\xi_B^{(0)} + \ldots)\,\underline{B}/B^2\end{aligned} \tag{89}$$

$$\begin{aligned}\underline{b} = {} & (\epsilon^2 b_n^{(2)} + \ldots)\,\underline{e}_n \\ & + (\epsilon b_\perp^{(1)} + \ldots)\,\frac{\underline{B}\times\underline{e}_n}{B^2} \\ & + (\epsilon b_B^{(1)} + \ldots)\,\underline{B}/B^2\end{aligned} \tag{90}$$

Furthermore, we will find,

$$\underline{\nabla}\cdot\underline{\xi} = \epsilon(\underline{\nabla}\cdot\underline{\xi})^{(1)} + \cdots \tag{91}$$

The reason for taking the above ordering for the components of the perturbation will be discussed below.

First consider the divergence. According to Eq.(91),

$$(\underline{\nabla}\cdot\underline{\xi})^{(0)} = \frac{\partial \xi_r^{(1)}}{\partial r} + \frac{i}{r_s B^2}(mB_z + nkr_sB_\theta)\,\xi_\perp^{(0)} = 0 \tag{92}$$

since $\partial/\partial r \sim \epsilon^{-1}$. This relation explains the relative ordering between ξ_r and $\xi_\perp$ in Eq.(89). The condition that $\underline{\nabla}\cdot\underline{b} = 0$ yields a relation similar to Eq.(92).

Next consider the radial component of the induction equation, Eq.(82). To lowest order, the coefficient of ϵ^2 is,

$$\begin{aligned} b_r^{(2)} - \frac{\eta}{\gamma}\frac{\partial^2}{\partial r^2} b_r^{(2)} &= \underline{B}\cdot\underline{\nabla}\xi_r^{(1)} \\ &= -i\,\frac{B_\theta}{r_s}\,nq'(r-r_s)\,\xi_r^{(1)} \end{aligned} \tag{93}$$

This tells us why b_r is one order smaller than ξ_r.

We now consider the momentum equation, and take γ P very small so that this term is negligible. Part of the reason for doing this is that the pressure is small in tokamaks. As we will see later, however, dropping this term is not completely justified. In any event, the effect of the γP term will be saved for Lecture V.

First consider the radial component of the momentum equation, Eq.(81). We can see immediately that the term $\partial(\underset{\sim}{\xi}\cdot\underset{\sim}{\nabla}P)/\partial\eta$ is much larger than the inertial term. The other lowest order term can be found by writing

$$(\underset{\sim}{\nabla}\times\underset{\sim}{b})\times\underset{\sim}{B}+\underset{\sim}{j}\times\underset{\sim}{b}=\underset{\sim}{\nabla}\cdot\left(\underset{\sim}{B}\,\underset{\sim}{b}+\underset{\sim}{b}\,\underset{\sim}{B}-\underset{\sim}{B}\cdot\underset{\sim}{b}\,\underset{\sim}{\mathbb{I}}\right)$$
$$\simeq\epsilon\underset{\sim}{\nabla}\,b_B^{(1)} \tag{94}$$

Thus, to lowest order, the momentum equation tells us

$$b_B^{(1)}=\xi_\eta^{(1)}\frac{dP}{d\eta} \tag{95}$$

This says that the perturbed fluid and magnetic pressures approximately balance perpendicular to the field lines. If it were not so, fast magnetosonic waves would be excited and would dominate the fluid motion. Since that is not what we wish to study today, we assume by Eq.(95) that they are absent, or have damped out.

The perpendicular component of the momentum equation, $\underset{\sim}{e}_\eta\times\underset{\sim}{B}$, yields the same information as Eq.(95). That is, the two components of the momentum equation become proportional to each other in the limit as ϵ vanishes. To obtain additional information from this equation we must go to higher order, and subtract the two components from each other. A convenient way to do this is to find an operator that annihilates the lowest order information, apply it to the full equation before ordering, and then take the lowest order terms. This is the concept of annihilation introduced by Kruskal, and will be used in many different ways in these lectures.

The most convenient form of the annihilator for these equations seems to be

$$\underline{\nabla} \cdot \frac{\underline{B}}{B^2} \times \qquad (96)$$

By vector identities, this is closely related to the parallel component of the curl, which is the obvious annihilator of a gradient. The largest term inertial is then

$$\underline{\nabla} \cdot \frac{1}{B^2} \underline{B} \times (\rho \gamma^2 \underline{\xi}) \simeq -\frac{1}{r} \frac{\partial}{\partial r} [\rho \gamma^2 r \xi_\perp^{(0)} / B^2]$$

$$\simeq -\frac{\epsilon \rho \gamma^2}{B^2} \frac{\partial}{\partial r} \xi_\perp^{(0)} \qquad (97)$$

$$\simeq -\frac{i \epsilon r_s}{m B_z + n k r_s B_\theta} \rho \gamma^2 \frac{\partial^2}{\partial r^2} \xi_n^{(1)}$$

on using Eq.(92).

The remaining terms of Eq.(81) reduce to

$$\underline{\nabla} \cdot \frac{1}{B^2} \underline{B} \times [(\underline{\nabla} \times \underline{b}) \times \underline{B} + \underline{j} \times \underline{b}]$$

$$= \underline{j} \cdot \underline{\nabla} \frac{b_B}{B^2} - \underline{b} \cdot \underline{\nabla} \frac{\underline{j} \cdot \underline{B}}{B^2} - \underline{B} \cdot \underline{\nabla} \frac{(\underline{\nabla} \times \underline{b}) \cdot \underline{B}}{B^2} \qquad (98)$$

and

$$\underline{\nabla} \cdot \left[\frac{1}{B^2} \underline{B} \times \underline{\nabla}(\underline{\xi} \cdot \underline{\nabla} P)\right] = \frac{1}{B^2} \underline{j} \cdot \underline{\nabla}(\underline{\xi} \cdot \underline{\nabla} P)$$

$$+ \frac{\underline{B} \times \underline{\nabla} B^2}{B^4} \cdot \underline{\nabla}(\underline{\xi} \cdot \underline{\nabla} P) \qquad (99)$$

The lowest order of the first term of Eq.(98) combines with Eq.(99) to yield,

$$\frac{1}{B^2}\left(2\underline{j} + \frac{\underline{B}\times\underline{\nabla}B^2}{B^2}\right)\cdot\underline{\nabla}b_B^{(1)}$$
$$\simeq \frac{1}{B^4}\,\underline{B}\times\underline{\nabla}(2P+B^2)\cdot\underline{\nabla}b_B^{(1)} \tag{100}$$

In deriving this we have used Eq.(95) to express everything in terms of $b_B^{(1)}$, expressed the current $\underline{j}$ as

$$\underline{j} = \frac{\underline{B}\times\underline{\nabla}P}{B^2} + \frac{\underline{j}\cdot\underline{B}}{B^2}\,\underline{B}$$

and dropped a gradient parallel to the field lines since it is small. The significant thing about Eq.(100) is that the quantity $\underline{\nabla}(2P + B^2)$ is proportional to the curvature of the field lines by Eq.(77). It will thus represent the destabilizing force in terms of field line curvature.

The second term of Eq.(98) is small in the expansion, while the third term yields, to lowest order,

$$\underline{B}\cdot\underline{\nabla}\left(\frac{\underline{B}\cdot\underline{\nabla}\times\underline{b}}{B^2}\right) \simeq \underline{B}\cdot\underline{\nabla}\left[\frac{\epsilon}{B^2}\,\underline{B}\cdot\left(\underline{e}_r\times\frac{\underline{B}\times\underline{e}_r}{B^2}\right)\frac{\partial b_\perp^{(1)}}{\partial r}\right]$$
$$\simeq \underline{B}\cdot\underline{\nabla}\left[-\frac{i\epsilon r_s}{mB_z + nkr_sB_\theta}\,\frac{\partial^2 b_r^{(2)}}{\partial r^2}\right] \tag{101}$$

since only the term with the radial derivative will be sufficiently large to make a significant contribution. The expression for $\underline{\nabla}\cdot\underline{b} = 0$ has been used in the derivation.

Combining Eqs.(97), (100), and (101) yields,

$$\rho\gamma^2 \frac{\partial^2}{\partial r^2} \xi_r^{(1)} = 2\frac{B_\theta^2}{r_s^3 B^4}\left(mB_z + nkr_sB_\theta\right)^2 b_B^{(1)} - i\frac{B_\theta}{r_s} mq'(r-r_s)\frac{\partial^2 b_r^{(2)}}{\partial r^2}$$
$$= \frac{2m^2B_\theta^2}{r_s^3 B_z^2} b_B^{(1)} - i\frac{B_\theta}{r_s} mq'(r-r_s)\frac{\partial^2 b_r^{(2)}}{\partial r^2} \qquad (102)$$

where we have used the relation between B_θ and B_z at the singular surface.

Now Eqs.(93), (95), and (102) are closed, so we do not need to find any more equations. In some way, we found a closed set of equations without finding a full solution of the problem. A number of quantities have not been calculated, and a full check of the ordering has not been made. This will be corrected when we return to the problem of the effect of the γP term in the fifth lecture. Until then, we will concentrate on the closed set that has been derived here.

Before analyzing these equations, we will scale out a lot of parameters. First we define a dimensionless growth rate Q.

$$\gamma \equiv \left(\frac{\eta n^2 q'^2 B_\theta^2}{\rho r_s^2}\right)^{1/3} Q \qquad (103)$$

and a dimensionless length x,

$$r - r_s = L_R x \qquad L_R \equiv \left(\frac{\eta^2 \rho r_s^2}{n^2 q'^2 B_\theta^2}\right)^{1/6} \qquad (104)$$

Further, it is convenient to scale the perturbed magnetic field,

$$b_n^{(2)} = -i \frac{L_R n q' B_\Theta}{n_s} \Psi \tag{105}$$

Then Eqs.(93) and (102) become,

$$\Psi'' = Q(\Psi - x\xi) \tag{106}$$

$$\begin{aligned} Q^2 \xi'' &= -D_s \xi - x \Psi'' \\ &= -D_s \xi + Q x^2 \xi - Q x \Psi \end{aligned} \tag{107}$$

where prime denotes derivative, and D_s is exactly the same quantity defined in Eq.(48) and used extensively in the previous lecture. Examination of the algebra shows that it comes from Eq.(100), and is a measure of field line curvature.

Several things could be noted about these equations.

Sometimes in fluid dynamics one constructs dimensionless variables and small parameters before starting the problem. The method used here of waiting until after the equations have been put in final form makes it clear that all the factors have been put in the length scales. For example, the dependence of L_R on the shear might not be obvious until q' had been scaled out of the equations. Here everything has been scaled out except the growth rate and the instability parameter D_s.

The most important approximation in this lecture has been that the inner layer is localized, that is, $L_R \ll a$, where a is the diameter of the plasma. In fact, in tokamaks L_R turns out to be of the order of millimeters. This is indeed very localized. There is more worry that the layer is so localized that fluid equations are not valid than that localization is a poor approximation.

Finally, these equations are a straightforward generalization of the inner layer of the previous lecture, Eq.(61). Indeed, the only term that arises from resistivity is the left hand side of Eq.(106). When this is dropped we recover the first two terms of Eq.(61). In recovering this result we must remember that the last two terms in the second form of Eq.(107) is inversely proportional to resistivity, so the first form must be used to recover the correct result. The approximate form of Eq.(106) is then used together with

$$X(X\xi)'' \equiv (X^2\xi')' \tag{108}$$

The remaining term in Eq.(61) is small when γP is very small, as has been assumed here.

Now, what can we learn about the solutions of this set of equations ? It is a fourth order ordinary differential equation whose only singularity is at infinity. In the vicinity of this singularity these are solutions that behave algebraically, and for which the resistive term is small. These solutions then behave as those in the ideal boundary layer,

$$\xi \simeq x^{s}, \; x^{-1-s}$$

$$\Psi \simeq x^{s+1}, \; x^{-s}$$

where s is defined in Eq.(84). Then indeed, $\Psi'' \sim \Psi / x^2 << \Psi$. These solutions can then be matched without difficulty to Eqs.(83) and (85).

The other two solutions for large x are exponential,

$$\xi \simeq \exp\left(\pm x^2/2Q^{1/2}\right)$$

The terms that decay exponentially have no affect on the outer region, so they are no problem. The exponentially large terms are linked to resistivity for all x. They can not contribute to the behavior we are interested in here. This leads to additional conditions that these exponentially large terms vanish.

Now we count conditions and constants to see if the problem is well posed. There are now eight constants of integration, four coming from the fourth order inner layer equations, and four coming from the outer regions as in the previous lecture. Again there are two boundary conditions, as those in Eq.(56), and four matching conditions from the algebraic solutions on either side of the inner layer. The remaining two conditions are that the boundary layer solution not diverge exponentially on either side, for $x \rightarrow \pm \infty$. This gives eight homogeneous conditions for the eight constants of integration, and thus yields as eigenvalue problem for the growth rate.

To conclude this lecture, we will get an estimate of these eigenvalues.

A very useful technique in resistive theory has been the use of Fourier transform. Note that the equations are fourth order in d/dx, but only second order in x. Thus, introducing,

$$\xi = \int_{-\infty}^{\infty} d\mu \, J(\mu) \exp(i\mu x) \tag{109}$$

Eqs.(106) and (107) reduce to,

$$\frac{d}{d\mu} \frac{\mu^2}{\mu^2+Q} \frac{d}{d\mu} J - Q\left(\mu^2 - \frac{D_s}{Q^2}\right) J = 0 \tag{110}$$

This equation is singular at small μ , where it behaves as

$$J \simeq \mu^s, \ \mu^{-1-s}$$

with s given by Eq.(84), again, and also singular for large μ with behavior

$$J \simeq \exp\left(\pm \frac{1}{2} Q^{1/2} \mu^2\right)$$

The condition that J not diverge leads to a well posed eigenvalue problem. The singularity at μ = 0 is a bit of an annoyance. There is a relation between this behavior and the matching condition for the real problem in configuration space, but we do not need to go into that here.

For now, we can just point out that there is a sequence of exact solutions of Eq.(110),

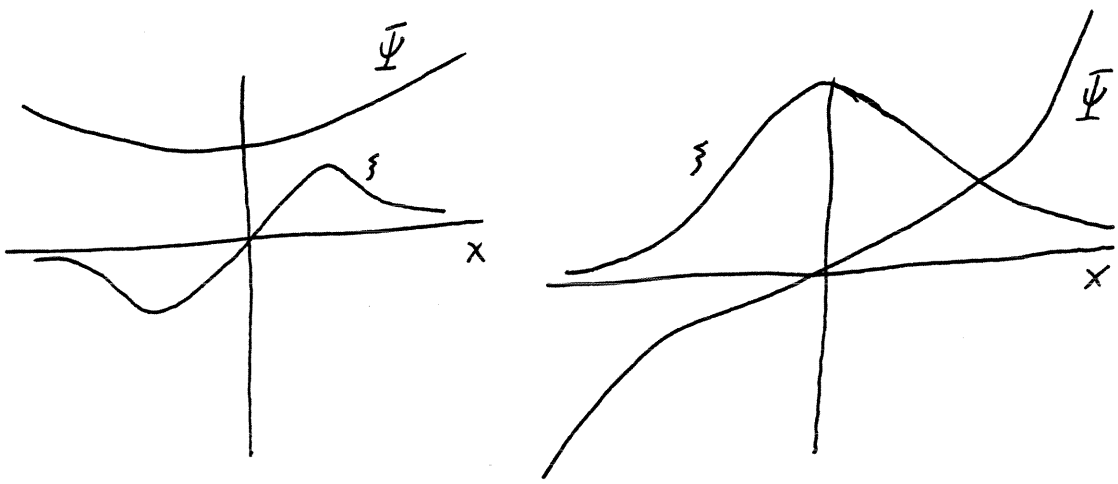
$$\xi = \mu^{s} \exp\left(-\tfrac{1}{2} Q^{1/2} \mu^{2}\right) \sum_{j=0}^{n} a_j \mu^{2j} \tag{111}$$

where the summation is truncated. The resulting eigenvalues are given by

$$Q^{3/2} = \frac{D_s}{s + 2n + \frac{1}{2} + \left[4sn + \left(2n + \frac{1}{2}\right)^2\right]^{1/2}} \tag{112}$$

Here n is an integer that depends on the number of terms in the truncated series. The procedure only works when D_s is positive so $Q^{\frac{1}{2}}$ is positive and the solution decays at infinity. This again is an infinite sequence of unstable modes as was obtained in the previous lecture for the ideal Mercier/Suydam modes. Here, however, the instability criterion has been extended down to $D_s > 0$. Further, higher modes decay less rapidly to zero with increasing n.

Direct numerical integration of Eqs.(106) and (107) reveals that there are even and odd solutions,



Further, the eigenvalues of Eq.(112) are degenerate, that is, they are valid for both even and odd solutions.

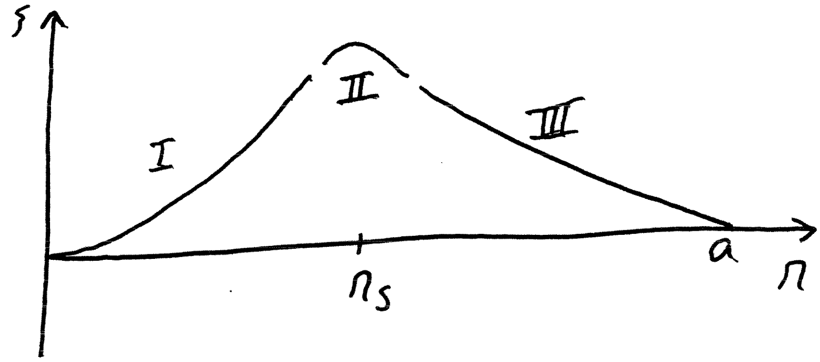
In this lecture we have given short shrift to the boundary conditions. That will be the main subject for next time.

The results of this lecture were all given, in some form or other in the original paper on resistive instabilities by Furth, Killeen, and Rosenbluth |12|. We found that paper extremely difficult to understand in our own terms, and after working it all over, wrote a paper applying the theory to a stellarator, the configuration of interest at the time |13|. A stellarator is a difficult configuration to use, so we then wrote a paper using the straight circular pinch configuration of these lectures |14|. Most of this lecture has been taken from the latter paper.

LECTURE IV

In this lecture the effects of the matching conditions on resistive modes will be considered. In the previous lecture, equations for the resistive layer were derived, and the existance of instabilities was demonstrated, but the matching conditions were hardly treated at all. By treating the matching conditions, we will gain a more precise understanding of the stability criterion.

The overall form of the perturbation can be sketched as follows,



Region I is the outer region closest to the magnetic axis. Generally, when treating boundary layers, the boundary layer is called the inner region, and the rest is called the outer region. Here, however, we have two outer regions, one of which is in some sense inside the boundary layer. We can just call it region I. Region II is then the inner, resistive boundary layer, and Region III the other outer region.

The solution in region I near the boundary layer approaches the form

$$\xi \simeq A_I \left|\frac{r-r_s}{a}\right|^{s} + B_I \left|\frac{r-r_s}{a}\right|^{-1-s} \qquad (113)$$

where the length scale a has been introduced to make the constants A_I and B_I dimensionless and s is given in Eq.(84). Similarly, in region II the solution approaches, for large negative x,

$$\xi \simeq A_{II\,L} |X|^{s} + B_{II\,L} |X|^{-1-s} \tag{114}$$

where

$$X \equiv (\Omega - \Omega_s)/L_R$$

and L_R is defined in Eq.(104). The matching conditions then are

$$\begin{aligned} A_I &= (a/L_R)^{s} A_{II\,L} \\ B_I &= (a/L_R)^{-1-s} B_{II\,L} \end{aligned} \tag{115}$$

Since the equations and boundary conditions are all homogeneous, the magnitude of ξ can be scaled in each region, and complete matching can be accomplished if

$$\frac{A_I}{B_I} = \left(\frac{a}{L_R}\right)^{2s+1} \frac{A_{II\,L}}{B_{II\,L}} \tag{116}$$

where

$$2s+1 = (1-4D_s)^{1/2} > 1$$

This is equivalent to the situation in the previous lecture where only the matching of phases was important.

We expect that for most cases A_I/B_I will be some finite number. It depends on the parameters of the outside region, and not on the growth rate. To obtain matching, the growth rate must be adjusted in the resistive layer so that $A_{IIL} \approx 0$. The exceptional case occurs when B_I vanishes. According to Newcomb's stability criterion |15|, this is the condition for marginal ideal stability. It is not surprising that a system that is close to being unstable in the ideal theory will have a strongly modified resistive instability.

Similarly, there is a matching condition between regions II and III of the form,

$$\frac{A_{III}}{B_{III}} = \left(\frac{a}{L_R}\right)^{2s+1} \frac{A_{IIR}}{B_{IIR}} \tag{117}$$

These matching constants can be calculated numerically from the resistive layer equations as the limit

$$\frac{A_{IIL}}{B_{IIL}} = \lim_{X \to -\infty} \left[-X^{-4s-2} \frac{\frac{d}{dX}\left(X^{1+s}\xi\right)}{\frac{d}{dX}\left(X^{-s}\xi\right)} \right]$$

$$= \lim_{X \to -\infty} \left[-X^{-4s-2} \frac{\frac{d}{dX}\left(X^{s}\Psi\right)}{\frac{d}{dX}\left(X^{-1-s}\Psi\right)} \right]$$

These forms are something like a logarithmic derivative, particularly when s is small, but it has its own character.

We are now prepared to tackle the main subject of this lecture. First, we wish to consider Q and D_s both small, to see how they depend on each other in the limit. The motivation here is to examine the stability criterion more closely. It turns out that in this limit, the growth rates are equally affected by the matching conditions. Thus it is possible to derive stability criteria that involve the matching conditions also.

The limit of small D_s and Q is singular. One can see this by looking at the behavior of the exponential solutions of the resistive layer equations, Eqs.(106) and (107). In the previous lecture it was shown that for large x there are solutions that behave as

$$\xi \sim \exp\left(\pm x^2/2Q^{1/2}\right)$$

Thus as Q becomes small, we must also consider the length scale becoming small as $Q^{1/4}$. Otherwise, the exponential would try to grow infinitely rapidly, and the equations would become impossible. That is, there is an inner layer of significant behavior that is even thinner than the resistive skin depth, by another factor of $Q^{1/4}$. Further, it is clear from Eq.(112) that we wish to assume that D_s becomes small as $Q^{3/2}$ in the limit of small Q.

There are two kinds of solutions in this limit, one with

$$\Psi \sim x\xi$$

and one with

$$\Psi \sim Q^{3/2}x\xi$$

These two solutions corresponds to the even and odd modes discussed at the end of the previous lecture. In the present theory these are not degenerate. It turns out that the fastest growing modes that determine marginal stability are of the first class, so we will concentrate on them.

With these approximations the leading term of Eq.(106) becomes

$$\Psi^{(0)''} = 0 \tag{119}$$

with solution

$$\begin{aligned}\Psi^{(0)} &= B_{II}^{(0)} + A_{II}^{(0)} X \\ &= B_{II}^{(0)} + A_{II}^{(0)} \left(\frac{a}{L_R}\right)\left(\frac{\Omega - \Omega_s}{a}\right)\end{aligned} \tag{120}$$

We have taken D_s small, so that s approximately vanishes, and there is agreement with Eq.(114). To obtain matching to arbitrary conditions in the outer regions, $A_I^{(o)}$ must be very small. In fact, the method here works by calculating A_{II} from higher orders in the expansion in Q. It is then small, and can be used to match to the outside solutions.

To lowest order, then, Ψ is a constant, $B_{II}^{(o)}$. This is the origin of the so-called constant Ψ approximation, that has become something of a cliché in the field.

This is a case where it not really obvious that matching between the inner and outer regions should work. Here the inner layer is completely dominated by the resistive term. Thus, it will not necessarily merge with a region where the resistive term is negligible. In fact, one should include an intermediate layer, scaled to the resistive skin depth, between the solution calculated here and that in the outer regions. However, it turns out that everything is constant in this intermediate layer. Thus, its existance is necessary, particularly in higher orders, but it has no affect on the present calculation. One does need to watch this sort of thing to make sure that the calculations are correct.

Using the approximation for Ψ, Eq.(107) can be solved for lowest order approximation to ζ,

$$Q^2 \zeta^{(0)\prime\prime} - Qx^2 \zeta^{(0)} + D_s \zeta^{(0)} = -Qx\Psi^{(0)} = -Qx B_{II}^{(0)} \tag{121}$$

With the present ordering, each term on the left is the same order in Q. The solution of this equation that does not diverge exponentially for $x \to \pm \infty$ is odd, and approaches $\zeta \to B_{II}^{(o)}/x$ in the limit.

Next, we examine the higher approximation for Ψ. The equation for the first correction is

$$\Psi^{(1)\prime\prime} = Q\left(\Psi^{(0)} - x\zeta^{(0)}\right) \tag{122}$$

where $\Psi^{(1)}$ is smaller than $\Psi^{(o)}$ by a factor of $Q^{3/2}$ in the limit of small Q. For large x, the important terms are the corrections to Eq.(120),

$$\Psi^{(1)} \approx B_{II\,L}^{(1)} + A_{II\,L}^{(1)} X \qquad X \to -\infty$$

$$\approx B_{II\,R}^{(1)} + A_{II\,R}^{(1)} X \qquad X \to +\infty$$

Thus

$$\Psi^{(1)\prime}\Big|_{X\to-\infty} = A_{II\,L}^{(1)} \tag{123}$$

and similarly on the right. Equation (122) can be integrated to yield

$$\Psi^{(1)\prime}\Big|_{X=-X}^{X=+X} = Q\int_{-X}^{+X}\left(\Psi^{(0)} - X\,\xi^{(0)}\right)dX \tag{124}$$

$$\sim Q^{5/4}\,B_{II}^{(0)}$$

when Q is small, since the integration variable also scales with Q in this limit.

It follows that the constant A_{II} of Eq.(120) will be of order $Q^{5/4}$. Thus, if

$$Q^{5/4} \sim L_R/a$$

the right hand side of Eq.(116) and (117) will be of order unity. Then changes in the matching condition will affect the modes significantly. That is the object of the present lecture. With this ordering we find

$$\Omega - \Omega_s \sim L_R Q^{1/4}$$
$$\sim L_R (L_R/a)^{1/5} \qquad (125)$$
$$\sim \eta^{2/5}$$

The growth rates are also altered so that

$$\gamma \sim \eta^{3/5} \qquad (126)$$

Thus these the modes considered in this lecture grow much more slowly than those considered in the previous lecture, when the resistivity is small. When we find a limiting process that is distinctly different from those considered before, as in the present case, it is necessary to go back to the original equations and see if the new limit will give rise to new terms. This has been done, for example in reference 14, and nothing significant shows up. Thus we can continue to push on from our present position.

It turns out that Eqs.(121) and (123) can be evaluated explicitly. This was done in reference 12, and is also sketched in references 13 and 14. The process will only be outlined here.

First consider Eq.(121). The left hand side of this equation is a Hermite operator. Thus if we expand

$$\xi^{(0)} = \exp\left(-X^2/2Q^{1/2}\right) \sum_0^\infty c_n H_n\left(X/Q^{1/4}\right) \qquad (127)$$

where the H_n's are Hermite polynomials, then

$$Q^2 \frac{d^2\xi^{(o)}}{dX^2} - QX^2\xi^{(o)} + D_s\xi^{(o)}$$

$$= Q^{3/2} \exp\left(-X^2/2Q^{1/2}\right) \tag{128}$$

$$\times \sum_0^{\infty} \left(\frac{D_s}{Q^{3/2}} - 2n - 1\right) c_n H_n$$

Further, it can be shown

$$z = 2^{1/2} \exp\left(-z^2/2\right) \sum_0^{\infty} \frac{H_{2n+1}(z)}{4^n \Gamma(n+1)} \tag{129}$$

Equating the coefficients of H_n to zero yields equations for the c_n, and hence a formal solution for $\xi^{(o)}$. This solution can be inserted into Eq.(123), and the integrals evaluated, so that $\Psi^{(1)\prime}(X) - \Psi^{(1)\prime}(-X)$ is given in terms of an infinite series. This can be identified as a hypergeometric series, and thus evaluated, yielding,

$$\delta = \frac{1}{B_{II}^{(o)}}\left[\Psi^{(1)\prime}(X) - \Psi^{(1)\prime}(-X)\right]$$

$$= 2\pi Q^{5/4} \frac{\Gamma\left[\frac{1}{4}\left(3 - D_s/Q^{3/2}\right)\right]}{\Gamma\left[\frac{1}{4}\left(1 - D_s/Q^{3/2}\right)\right]} \tag{130}$$

Now we come to a complicated argument to show that δ is the only quantity needed from the inner layer to evaluate the dispersion relation. First we note from Eq.(123) that to lowest order

$$\delta = \frac{A_{II R}}{B_{II R}} - \frac{A_{II L}}{B_{II L}} \tag{131}$$

According to Eqs.(116) and (117), the individual terms of the right hand side of the equation above must be matched to the outside regions. However, solutions of the homogeneous left hand side of Eq.(124) can be added in, in particular a solution $A^{(1)}_{II\ H}x$. Then

$$\frac{A_{II R}}{B_{II R}} = \frac{1}{2}\delta + \frac{A^{(1)}_{II\ H}}{B^{(0)}_{II}} = \frac{A_{III}}{B_{III}}\left(\frac{L_R}{a}\right)^{2s+1}$$

$$\frac{A_{II L}}{B_{II L}} = -\frac{1}{2}\delta + \frac{A^{(1)}_{II\ H}}{B^{(0)}_{II}} = \frac{A_{I}}{B_{I}}\left(\frac{L_R}{a}\right)^{2s+1} \tag{132}$$

Eliminating $A^{(1)}_{II\ H}$ yields the dispersion relation

$$\Delta = \Delta' \tag{133}$$

where

$$\Delta = \left(\frac{a}{L_R}\right)^{2s+1}\delta = \left(\frac{a}{L_R}\right)2\pi Q^{5/4}\,\frac{\Gamma\left[\frac{1}{4}\left(3 - D_s/Q^{3/2}\right)\right]}{\Gamma\left[\frac{1}{4}\left(1 - D_s/Q^{3/2}\right)\right]} \tag{134}$$

since s is approximately zero, and

$$\Delta' = \frac{A_{III}}{B_{III}} - \frac{A_{I}}{B_{I}} \tag{135}$$

Thus Δ depends on parameters of the inner region, and Δ' depends on the outer region. The result is an equation for the growth rate Q. We now analyze this to obtain information on the dependence of the growth rate on D_s and Δ'.

When D_s is positive, the gamma functions have an infinite sequence of poles, accumulating at Q = 0. Thus Δ alternately vanishes and diverges, so it passes through all values many times. That is, for a given Δ' there is an infinite sequence of growth rates that satisfy the dispersion relation. When Δ' vanishes, these growth rates are given by

$$\frac{1}{4}\left(1 - D_s/Q^{3/2}\right) = -n \tag{136}$$

This agrees with the result obtained in the preivous lecture, Eq.(112), when s vanishes.

The result is, that when D_s is positive, there is always an instability.

When D_s is negative, Δ is always positive. Thus, there are no unstable solutions when Δ' is negative. A more detailed examination shows that Δ diverges as $Q^{5/4}$ for large Q, vanishes as $(-DQ)^{\frac{1}{2}}$ for small Q, and is monotonic. Thus the result here is there is an instability only when $\Delta' > 0$.

The two kinds of instability, for D_s positive or negative, have quite different character. The first kind depends only on the parameters within the singular surface. Furhter, there is an infinite sequence of unstable modes, as in the Mercier/Suydam instabilities discussed in the second lecture. Both of these modes depend only on the local average field line curvature. It seems reasonable to call them by the same name, so they are known as ideal and resistive interchanges.

It should be noted that the quantity D_s is strongly affected by toroidicity in a tokamak. While it is nearly always positive in a straight system, it is multiplied by a factor $1-q^2$ in the most straightforward tokamak approximation. Thus in the latter case D_s is nearly always negative and there are no unstable interchanges. This conceivably is an important reason for the successful operation of tokamaks, though there is no experimental confirmation.

In the unstable modes with D_s negative, matching conditions from the outer regions play a crucial role. Thus these modes appear to be driven by forces in the outer regions. The role of the inner layer is to permit motions that would be excluded in ideal theory. For this reason, such instabilities are called tearing modes. In ideal theory, $\underline{\xi}$ must be bounded. As a result, Newcomb obtained a condition for marginal stability that $(\Delta')^{-1}$ vanish. In resistive theory this has been relaxed to the condition that $\Delta' = 0$.

There is an instructive calculation along this line that can be made concerning the region outside a plasma column. The region between the main plasma column and the conducting wall, where currents and pressure gradients vanish, can be considered to be a vacuum, a perfectly conducting fluid, or a resistive fluid. In the former case, $\underline{\xi}$ has no meaning, and the solution is given in terms of the perturbed magnetic field. This is equivalent to Ψ of the present lecture. The quantity Ψ is regular at the singular point. Thus the condition for marginal stability is continuity of Ψ and its derivative, or $\Delta' = 0$, exactly the same as the condition for a resistive plasma.

Thus there is a close relation between tearing modes and the kink modes that can be calculated for ideal systems that have a surrounding vacuum region. The approximation that the entire column is filled with a perfectly conducting fluid yields much more optimistic results, since then there is an additional constraint that ξ be finite at the singular surface.

In this lecture, and the previous lecture, we have assumed that the pressure, γP is very small. This approximation will be considered in the next lecture. It will turn out that the chief effect of this term is to modify the tearing criterion $\Delta' > 0$. In some ways this correction is small, but it can be quite significant.

LECTURE V

In the previous lecture, we found the stability criterion

$$D_s < 0 \quad , \quad \Delta' < 0 \tag{137}$$

Here D_s is a measure of the average curvature, that drives interchange instabilities, and

$$\Delta' \equiv \frac{A_{III}}{B_{III}} - \frac{A_I}{B_I}$$

is a measure of the destabilization arising outside the resistive layer. In that lecture, we dropped the γ P term, assuming that the plasma pressure was small. According to further calculations that have been made, this neglect of the plasma pressure can be misleading. Inclusion of these terms can lead to significant corrections to the stability criterion given above. In particular, some range of positive Δ' can be stabilized. This will be discussed in this lecture.

In addition, the calculation here helps to fill out that of the previous lecture. Then we found a set of equations that could be solved, without calculating every quantity. Here we will get approximations for every component of the perturbation. It will then be possible to show that the solution is completely consistent.

We now review the development of lecture III, and add the new terms where necessary. The orderings assumed there are valid here also. Thus the divergence condition, Eq.(92) is unchanged.

The radial induction equation, Eq.(93), is not affected by the pressure.

The balance of total pressure, Eq.(95), now has a term representing pressure changes induced by the perturbation,

$$b_B^{(1)} = \xi_r^{(1)} \frac{dP}{dr} + \gamma P (\nabla \cdot \xi)^{(1)} \tag{138}$$

The annihilated momentum equation, Eq.(102) is unchanged. Fortunately we do not have to rederive it.

The new term that has been added to the total pressure balance equation prevents us from closing the equations. We now have to evaluate the rest of the equations in order to obtain a complete set.

The quantity $\underline{\nabla} \cdot \underline{\xi}$ is related to pressure balance along the lines of force. The parallel component of the momentum equation yields

$$\rho \gamma^2 \xi_B^{(0)} = - \frac{dP}{dr} b_r^{(2)} - i \frac{B_\Theta}{r_s} n q'(r - r_s) b_B^{(1)} \tag{139}$$

Thus the parallel perturbation is zero order in the expansion, as assumed. Finally, we must evaluate the parallel component of the induction equation. From reference 8 we can evaluate

$$\underline{B} \cdot \underline{\nabla} \times (\underline{\xi} \times \underline{B}) = B^2 (\underline{B} \cdot \underline{\nabla} \xi_B - \underline{\nabla} \cdot \underline{\xi}) - \underline{\xi} \cdot \underline{\nabla} (P + B^2) \tag{140}$$

Thus the induction equation becomes,

$$b_B^{(1)} - \frac{\eta}{\gamma} \frac{\partial^2}{\partial r^2} b_B^{(1)} = - i \frac{B_\Theta}{r_s} n q'(r - r_s) \xi_B^{(0)}$$

$$- B^2 (\nabla \cdot \xi)^{(1)} - \frac{d}{dr} (P + B^2) \xi_r^{(1)} \tag{141}$$

This establishes the divergence of the perturbation as a first order quantity. This completes the demonstration that the assumed ordering gives a consistent set of equations. We now have a closed set of equations for $b_r^{(2)}$, $\xi_r^{(1)}$, $b_B^{(1)}$, $\xi_B^{(o)}$, and $(\underline{\nabla}\cdot\underline{\xi})^{(1)}$. Note that $(\underline{\nabla}\cdot\underline{\xi})^{(1)}$ is independent of $\xi_r^{(1)}$ since it depends on a higher order correction, $\xi_r^{(2)}$. We again introduce the same scaling as before, Eqs.(103-105). We also define a scaled parallel perturbed magnetic field,

$$b_{\beta}^{(1)} = \frac{dP}{d\eta}\Upsilon \tag{142}$$

Then,

$$\Psi'' = Q(\Psi - X\xi) \tag{143}$$

$$Q^2\xi'' = QX^2\xi - QX\Psi - D_s\Upsilon \tag{144}$$

$$\begin{aligned}\Upsilon'' = {} & Q\left(1 + \frac{X^2}{Q^2} + \frac{B^2}{\gamma P}\right)\Upsilon \\ & + Q\left(\frac{S}{D_s} - 1 - \frac{B^2}{\gamma P}\right)\xi - \frac{1}{Q}X\Psi\end{aligned} \tag{145}$$

where

$$S \equiv 4K^2/q'^2 \tag{146}$$

We can obtain the results of lecture III by making γP very small. Then

$$\Upsilon = \xi$$

from Eq.(145), and Eq.(144) reduces to Eq.(107).

Further, we can obtain the results of lecture II for the equation of the inner layer by making the resistivity vanish. The resistive terms are Ψ'' and Υ''. We further remember from lecture III that Eq.(143) must be used together with Eq.(144) in this limit. When the proper terms are dropped, and Ψ and Υ eliminated using the resulting algebraic equations, Eq.(61) is obtained, exactly. The pressure terms that have been added in this lecture are just those that were treated in the second lecture on ideal interchanges. The method used here is thus an alternate derivation of Eq.(61).

Now we have a new set of equations. We need to look them over, understand their character, and determine the type of boundary conditions they require.

As before, the only singularity of these equations is at infinity. Near this singular point, as before, there is a pair of algebraic solutions that asymptotically approach ideal solutions. For these, again,

$$\xi \sim X^{s}, X^{-1-s} \qquad s \equiv \frac{1}{2}\left[-1+\left(1-4D_s\right)^{\frac{1}{2}}\right]$$

There are also four exponential solutions,

$$\xi \sim \exp\left(\pm X^2/2Q^{1/2}\right) X^{n}\left(1+\cdots\right)$$

and each solution has a different value of the exponent r. This provides conditions that two exponentially large terms vanish on either side of the resistive layer.

Now we can count conditions and variables: Compared to the counting that was done in the third lecture, there are two more free constants of integration, because the resistive layer equations are now sixth order. There are also two more conditions, because there are more exponentially large solutions. Thus again we have a well posed eigenvalue problem.

It is difficult to calculate these equations numerically. The exponentially large solutions dominate everything. One can choose initial conditions to minimize the exponential terms, but, because of roundoff, they can never be eliminated.

We can, however, carry through the tearing ordering again. The calculation is very similar to that of the preceding lecture, but somewhat more tedious. The ordering for small Q is the same as before. Now we also assume that

$$\frac{\gamma P}{B^2} \sim Q^{3/2}$$

so that it remains in the equations. To lowest order

$$\Psi^{(0)} = B_{II}^{(0)}$$

again. The approximate equations for $\zeta^{(0)}$ and $\Upsilon^{(0)}$ are coupled,

$$Q^2 \zeta^{(0)''} - Q x^2 \zeta^{(0)} + D_s \Upsilon^{(0)} = -Q x B_{II}^{(0)}$$

$$Q^2 \Upsilon^{(0)''} - Q x^2 \Upsilon^{(0)} - Q^3 \frac{B^2}{\gamma P} \Upsilon^{(0)}$$
$$- Q^3 \left(\frac{S}{D_s} - \frac{B^2}{\gamma P} \right) \zeta^{(0)} = -Q x B_{II}^{(0)} \tag{147}$$

but the two equations have the same hermite operator. Thus $\xi^{(0)}$ and $\eta^{(0)}$ can be expanded in Hermite functions, as in Eq.(127), and the constants evaluated. The dispersion relation again is

$$\Delta = \Delta'$$

with

$$\Delta' \equiv \frac{A_{III}}{B_{III}} - \frac{A_I}{B_I}$$

and Δ is evaluated as an integral, as in Eq.(130), with the result,

$$\Delta = \frac{a}{L_R} \frac{\pi D_s}{4Q^{1/4}} F(\alpha, \delta, S) \tag{148}$$

$$\begin{aligned} F \equiv & \left[\alpha - 1 + \frac{1 + \alpha^2\delta - \alpha\delta + \alpha S}{\tau}\right] \\ & \times \frac{\Gamma\left[\frac{1}{4}(3 + \alpha\delta - \tau)\right]}{\Gamma\left[\frac{1}{4}(5 + \alpha\delta - \tau)\right]} \\ & + \left[\alpha - 1 - \frac{1 + \alpha^2\delta - \alpha\delta + \alpha S}{\tau}\right] \\ & \times \frac{\Gamma\left[\frac{1}{4}(3 + \alpha\delta + \tau)\right]}{\Gamma\left[\frac{1}{4}(5 + \alpha\delta + \tau)\right]} \end{aligned} \tag{149}$$

where

$$\alpha \equiv Q^{3/2}/D_s$$

$$\delta \equiv D_s B^2/\gamma P$$

$$\Upsilon^2 \equiv \frac{Q^3 B^4}{(\gamma P)^2} + 2\,\frac{D_s B^2}{\gamma P} - S \tag{150}$$

$$= \alpha^2 \delta^2 + 2\delta - S$$

With patience, a lot of information can be gathered from this formula. For example, it can be shown that when D_s is positive, Δ takes on all values for real, positive Q. Thus the growth rates of the interchange instability are altered here, but the mode is not affected in a fundamental way. The tearing mode is more interesting and will be considered further.

The results of the previous lecture are obtained when δ is large. In that limit the second term in F becomes very small because the denominator diverges faster than the numerator, while the first term approaches

$$F = 2\left(\frac{Q^{3/2}}{D_s} - 1\right) \frac{\Gamma\left[\frac{1}{4}\left(3 - D_s/Q^{3/2}\right)\right]}{\Gamma\left[\frac{1}{4}\left(5 - D_s/Q^{3/2}\right)\right]} \tag{151}$$

This is equivalent to Eq.(134). This again yields the stability criterion $\Delta' < 0$.

The other limit, γP large so that δ is small, can be evaluated. Then,

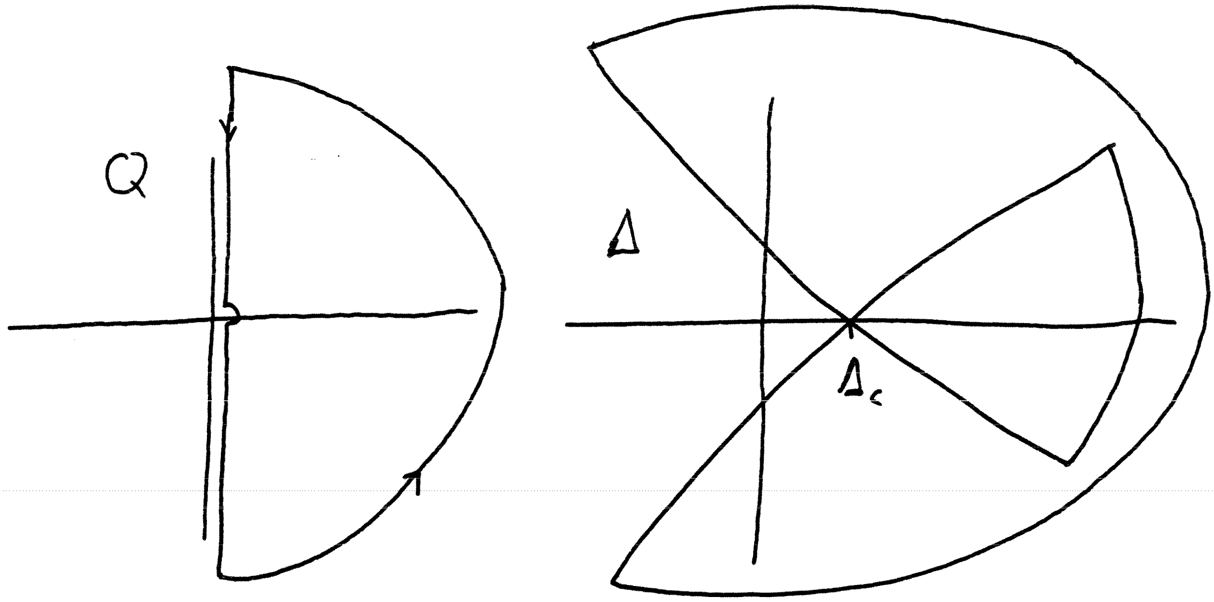
$$F = \frac{\pi}{2^{1/2}\Delta} \frac{(1+\Delta)^2}{\left|\Gamma\left[\frac{1}{4}(5+i\Delta)\right]\right|^2} \times \left[\frac{\alpha\Delta\cosh\pi\Delta/4 - \sinh\pi\Delta/4}{\cosh^2\pi\Delta/4 + \sinh^2\pi\Delta/4}\right] \tag{152}$$

where

$$\Delta = S^{1/2}$$

The resulting Δ is positive when D_s is negative. Further, it diverges as $Q^{-1/4}$ for small Q, and as $Q^{5/4}$ for large Q. It has some minimum, nonvanishing value for some finite Q.

Up to this point we have always looked for real growth rates, and they have always turned out to be real. One day we were discussing this particular dispersion relation with Paul Rutherford, and he suggested that perhaps this was one place in resistive theory that the growth rates might be complex. Somehow that possibility had been overlooked here, after having been examined for all the other cases. So we need to examine the dispersion relation in the whole complex plane. This is most easily done by employing a Nyquist, or Cauchy plot. One takes the following contour in the Q plane,



and calculates its image in the Δ plane, as above.

The unstable Q half plane is mapped into the inside of the contour in the Δ plane. Real values of Δ satisfying

$$\Delta < \Delta_c \tag{153}$$

can not be achieved by unstable values of Q. This then provides a stability criterion.

The quantity Δ_c can be evaluated for large pressure using Eq.(152). At marginal stability, Q will be purely imaginary, and Δ real. The frequency of oscillation can be calculated from

$$Im\,\Delta = 0 = \frac{a}{L_R}\,\frac{\pi^2 D_s}{4\,2^{1/2}\Delta}$$

$$\times \frac{(1+\Delta)^2}{\left|\Gamma\left[\frac{1}{4}(1+i\Delta)\right]\right|^2}\,\frac{\cosh \pi\Delta/4}{\cosh^2 \pi\Delta/4 + \sinh^2 \pi\Delta/4} \tag{154}$$

$$\times\left[\frac{\Delta}{D_s}\left|Q_c\right|^{5/4}\sin 5\pi/8 + \left|Q_c\right|^{-1/4}\tanh(\pi\Delta/4)\sin\pi/8\right]$$

or

$$Q_c^{3/2} = -\frac{D_s}{\Delta}\tanh(\pi\Delta/4)\tan\pi/8 \tag{155}$$

where $Q = Q_c \exp(i\pi/2)$

This is positive, since we are taking D_s negative here.

The real part of Δ can then be evaluated,

$$\Delta_c = \mathrm{Re}\,\Delta(Q_c) = \frac{a}{L_R}\,\frac{\pi^2}{8}\,\frac{(1+\Delta)^2}{\left|\Gamma\left[\frac{1}{4}(1+is)\right]\right|^2} \times \frac{\cosh\pi\Delta/4}{\cosh^2\pi\Delta/4+\sinh^2\pi\Delta/4}\left(\frac{Q_c^{5/4}}{\sin\pi/8}\right) > 0 \tag{156}$$

The conclusion is that for very small pressure, the results of the previous lecture are obtained, and

$$\Delta_c = 0$$

while for large pressure Δ_c is given by Eq.(156) above. Then the question is, which is more accurate for some particular pressure ? As a partial answer to this question, we return to consider the small pressure case.

Another theme in this set of lectures is that nothing physical is infinite or zero. We have met a number of cases where quantities diverged in some simple approximation, and we have found improved approximations that have allowed us to find how large it is. One example of this is the size of the perturbation near the singular surface. It is much the same with small quantities. Ultimately we want to know, not just whether something is large or small, but how big it is. Now we want to know the magnitude of Δ_c when the pressure is small.

The previous calculation for large δ must be modified along the lines of the calculation for large pressure. First, we must determine the way the critical oscillatory frequency scales with pressure when they are small. It turns out that we need

$$Q^3 \sim \gamma P/B^2 << 1 \tag{157}$$

In this limit, Δ can be expressed in terms of

$$\eta \equiv Q^3 B^2/\gamma P D_s \tag{158}$$

as

$$\Delta = -\frac{a}{L_R}\frac{\pi}{4} D_s^{2/3}\left(\frac{\gamma P}{B^2}\right)^{1/6} \times \frac{1}{\eta^{1/12}}\left\{\left[1+\left(\frac{\eta}{\eta+2}\right)^{\frac{1}{2}}\right]\left[\eta^{\frac{1}{2}}-(\eta+2)^{\frac{1}{2}}\right]^{-\frac{1}{2}} + \left[1-\left(\frac{\eta}{\eta+2}\right)^{\frac{1}{2}}\right]\left[\eta^{\frac{1}{2}}+(\eta+2)^{\frac{1}{2}}\right]^{-\frac{1}{2}}\right\} \tag{159}$$

Again, we find the pure imaginary value of η for which the imaginary part of Δ vanishes. The quantity Δ_c is given by the corresponding real value of Δ .

The important point is that Δ_c scales as the one sixth power of the pressure, for small pressure. It goes to zero in the limit, but very slowly.

Here is a case where the large pressure limit may be more accurate than the small pressure limit, even when the pressure is small. When $\gamma P/B^2$ is of the order of 1% the high pressure limit may be off by a factor of two or so, while the low pressure limit is in error by an infinite factor.

The quantity Δ_c can be significant because it is multiplied by the large factor a/L_R. This is a case where it is important to estimate the size of quantities, rather than dismiss them simply as large or small. Recent calculations indicate that this effect could stabilize Mirnov oscillations in future devices |16|.

Finally, I should make a comment on the validity of these equations. In general, MHD works best for motions perpendicular to the magnetic field. The long mean free paths of typical experimental devices have a strong affect on parallel motions. Viscosity and ion Landau damping have a dominant effect on the slow magnetosonic modes that are calculated in ideal theory. Here is a case where the equations for parallel motion play an important role in a mechanism for stabilization. We need to check to see that the results are reasonable.

It appears that ideal equations may actually underestimate this stabilization. Raising the pressure allows ξ and Υ to differ in Eq.(145). The term in this equation that comes from the parallel motion of the fluid is

$$\frac{1}{Q}\left(x^2\Upsilon - x\Psi\right)$$

These terms, when combined with Eq.(143) also force $\xi = \Upsilon$ when x is large. Thus if they are reduced, by forces impeding parallel motion, ξ and Υ are less constrained to equality. Thus there is no reason to expect this stabilization to disappear when nonideal effects are included. In any event, this is being looked into.

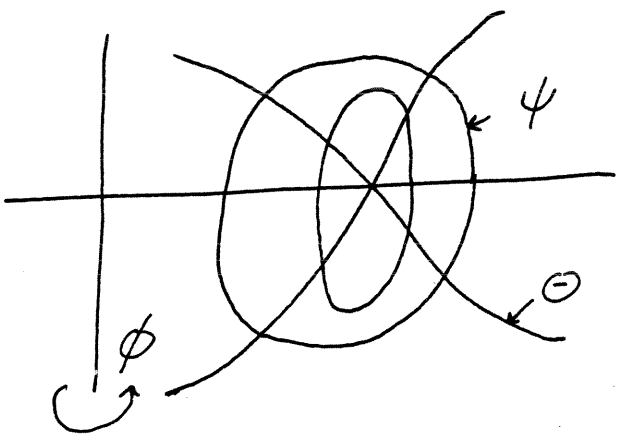
Most of the basic results of this lecture were obtained in reference 14. The specific results involving complex growth rates were given in references 16 and 17.

LECTURE VI

In the previous three lectures we have covered resistive instabilities in a straight system. Here we examine the modifications that arise in toroidal systems. Now there is only one ignorable coordinate, so we will have to deal with nontrivial partial differential equations. In spite of this complication, with the more complicated algebra that it produces, the character of the results is not strongly affected by toroidicity. A good part of this lecture will be spent deriving the toroidal form of the various constants, particularly the D_s that played the decisive role in interchange instabilities.

The calculation is much simplified in a convenient coordinate system. The most convenient of such coordinate systems are the class known as Hamada systems. Our first task will be to construct an axisymmetric Hamada system.

We start with a coordinate system in which magnetic surfaces are used as a coordinate. Then ψ is an arbitrary labeling of the magnetic surfaces; θ is an angle like variable that has a period of 2π. Surfaces of constant θ are axisymmetric surfaces that radiate from the magnetic axis. Finally, ϕ is the angle around the major axis of the torus.



In terms of these quantities the magnetic field is given by

$$\underline{B} = B_o f(\psi)\, \underline{\nabla}\phi \times \underline{\nabla}\psi + R_o B_o \, g(\psi)\, \underline{\nabla}\phi \tag{160}$$

where B_o is the toroidal magnetic field at a distance R_o from the major axis. The quantity f is related to the flux the short way by

$$\chi = 2\pi B_o \int f(\psi)\, d\psi \tag{161}$$

so that if one changes ones mind about the arbitrary way ψ labels the magnetic surfaces, the change is absorbed in f.

It follows directly that the Jacobian of the coordinate system is given by,

$$\mathcal{J} = \left(\underline{\nabla}\psi \times \underline{\nabla}\Theta \cdot \underline{\nabla}\phi\right)^{-1} = \frac{B_o f}{\underline{B}\cdot\underline{\nabla}\Theta} \tag{162}$$

Then the flux the long way is given by

$$\begin{aligned} \Phi &= \frac{1}{2\pi}\int \underline{B}\cdot\underline{\nabla}\phi \, d\tau \\ &= \frac{B_o}{2\pi}\int f \, \frac{\underline{B}\cdot\underline{\nabla}\phi}{\underline{B}\cdot\underline{\nabla}\Theta}\, d\psi\, d\Theta\, d\phi \end{aligned} \tag{163}$$

so the safety factor q is given by

$$q \equiv \frac{d\Phi}{d\chi} = \frac{1}{2\pi}\oint d\Theta \, \frac{\underline{B}\cdot\underline{\nabla}\phi}{\underline{B}\cdot\underline{\nabla}\Theta} \tag{164}$$

The first requirement of an Hamada system is that the Jacobian be a function of ψ only. We can choose the coordinate θ to accomplish this. If, for some reason, an angle like variable has been constructed for which the Jacobian is not constant, a change of variables can be constructed so that

$$\hat{\Theta} = 2\pi \frac{\int d\Theta / \underline{B}\cdot\underline{\nabla}\Theta}{\oint d\Theta / \underline{B}\cdot\underline{\nabla}\Theta} \tag{165}$$

Then

$$(\nabla\psi \times \nabla\hat{\Theta}\cdot\nabla\phi)^{-1} = \frac{B_0 f}{2\pi} \oint \frac{d\Theta}{\underline{B}\cdot\underline{\nabla}\Theta} \tag{166}$$

We now drop the hat on θ and assume that the Jacobian is constant on magnetic surfaces.

The final requirement for an Hamada system is that the contravariant components of the magnetic field be constant on magnetic surfaces. Then, in some sense they can be represented as straight lines. To accomplish this we have to wrinkle the constant ∅ surfaces. Now these are planes radiating from the major axis. If we introduce a new coordinate $\mathcal{J}$,

$$\mathcal{J} \equiv \phi + \int d\Theta \left[q(\psi) - \frac{\underline{B}\cdot\underline{\nabla}\phi}{\underline{B}\cdot\underline{\nabla}\Theta} \right] \tag{167}$$

surfaces of constant $\mathcal{J}$ will not be planes, but each such surface will have the same shape as all the others. Thus they still nest, and radiate from the major axis. Axisymmetric quantities will depend on ψ and θ but will be independent of $\mathcal{J}$. Thus this is a good axisymmetric coordinate system, and the Jacobian is unchanged by the transformation.

To evaluate the magnetic field in this coordinate system, we note that the gradient is given by,

$$\underline{\nabla} \zeta = \underline{\nabla}\phi + \left(q - \frac{\underline{B}\cdot\underline{\nabla}\phi}{\underline{B}\cdot\underline{\nabla}\theta} \right)\underline{\nabla}\theta + (\quad)\,\underline{\nabla}\psi \tag{168}$$

Fortunately, we do not need to evaluate the component in the $\underline{\nabla}\psi$ direction. It is then straightforward to show that the magnetic field is given by,

$$\underline{B} = B_0 f(\psi)\left[\underline{\nabla}\zeta - q(\psi)\,\underline{\nabla}\theta\right] \times \underline{\nabla}\psi \tag{169}$$

Since the coefficients of this equation are functions of ψ only, we have an Hamada coordinate system. Then the operator $\underline{B}\cdot\underline{\nabla}$, that occurs so often in MHD, is given by,

$$\underline{B}\cdot\underline{\nabla} = \frac{f}{J}\left(\frac{\partial}{\partial\theta} + q(\psi)\frac{\partial}{\partial\zeta} \right) \tag{170}$$

Now we make another coordinate change, from ψ, θ, ζ to ψ, θ, u with

$$u \equiv n\zeta - m\theta \tag{171}$$

Axisymmetric quantities are independent of u. Other physical quantities are periodic in u with period $2n\pi$. In fact, we will assume that perturbed quantities are periodic over the shorter period 2π, just as we have used n in the previous lectures to indicate periodicity in a harmonic of the fundamental period. Also, the Jacobian of Eq.(166) is divided by a factor of n in the new coordinate system. Now

$$\int d\tau = \int d\psi \int_0^{2\pi} d\theta \int_0^{2n\pi} du\, \mathcal{J}$$
$$= n \int d\psi \int_0^{2\pi} d\theta \int_0^{2\pi} du\, \mathcal{J} \tag{172}$$

when all quantities have the harmonic period. Thus we use period 2π for u, and the Jacobian of Eq.(166).

In the new coordinate system

$$\underline{B} = \frac{f}{n}\left[\underline{\nabla} u + (m - nq)\,\underline{\nabla}\theta\right] \times \underline{\nabla}\psi \tag{173}$$

As before, we will be working near the magnetic surface where $m - nq(\psi_s)$ vanishes. Then we can approximate

$$\underline{B}\cdot\underline{\nabla} = \frac{f}{\mathcal{J}}\left[\frac{\partial}{\partial\theta} - (m - nq)\frac{\partial}{\partial u}\right]$$
$$\sim \frac{f}{\mathcal{J}}\left[\frac{\partial}{\partial\theta} + nq'(\psi - \psi_s)\frac{\partial}{\partial u}\right] \tag{174}$$

Quantities that are constant on the closed field lines of the singular surface are independent of θ.

The present coordinate system seems a bit complicated in structure, but it can easily represent the two types of functions we will be most interested in. Axisymmetric quantities are independent of u, and perturbations constant along closed field lines on the singular surface of interest are independent of θ. My experience is that it is much better to have a complicated coordinate system and simpler equations, than the other way around. It should also be noted that the coordinate system is completely nonorthogonal, there are no vanishing elements of the metric tensor. That is no worry here, because we need only a few components of that tensor in these lectures.

The general considerations regarding resistive instabilities in this geometry are almost identical with those given in Lecture III, for a straight system. We start from the general equation, Eq.(78). The equilibrium, to lowest order in resistivity, must satisfy

$$R\frac{\partial}{\partial R}\frac{1}{R}\frac{\partial}{\partial R}\chi + \frac{\partial^2}{\partial z^2}\chi = -4\pi^2\left(R^2\frac{dP}{d\chi} + \frac{R_0^2 B_0^2}{2}\frac{dg^2}{d\chi}\right)$$

rather than Eq.(36). Perturbations around this equilibrium must satisfy Eqs.(81) and (82). Now, however, only t and u are ignorable, so we set

$$\underline{\xi} = \xi(\psi,\theta)\exp(\gamma t - iu) \tag{175}$$

The vector perturbed quantities are projected along three directions, in a straightforward generalization of Eqs.(89) and (90),

$$\underline{\xi} = \frac{\underline{\nabla}\psi}{|\underline{\nabla}\psi|^2}\xi_\psi + \frac{\underline{B}\times\underline{\nabla}\psi}{B^2}\xi_\perp + \frac{\underline{B}}{B^2}\xi_B \tag{176}$$

$$\underline{b} = \frac{\underline{\nabla}\psi}{|\underline{\nabla}\psi|^2}b_\psi + \frac{\underline{B}\times\underline{\nabla}\psi}{B^2}b_\perp + \frac{\underline{B}}{B^2}b_B$$

In this lecture we will only consider the inner, resistive layer. Most of the physics is contained here, and most of the progress has been made here. Further discussion of the outside regions will be given in the next lecture.

The size of the various terms in the inner layer in the expansion in resistivity is exactly the same as that given in Eqs.(88) to (91), with ψ replacing r. For now, we will assume that this is true, and then show that it leads to a consistent set of equations.

Now we consider the induction equation, Eq.(82), to lowest order. According to the assumed magnitudes of the various quantities, the only first order term in the radial component is

$$\underline{B}\cdot\underline{\nabla}\,\xi_\psi^{(1)} \simeq \frac{f}{J}\frac{\partial}{\partial\theta}\xi_\psi^{(1)} = 0 \tag{177}$$

Thus ξ_ψ must be approximately constant along the closed field lines of the singular surface. Returning to the ψ, θ, ζ coordinate system, this says that one Fourier component in θ dominates all the others. In the straight case we could exclude all the other components by symmetry. Here we can only argue that certain components will dominate.

A similar argument from the perpendicular component of the induction equations yields,

$$\frac{\partial}{\partial\theta}\xi_\perp^{(0)} = 0 \tag{178}$$

and from the parallel component we find, using Eq.(140),

$$\left(\underline{\nabla} \cdot \left[\underline{B} \times (\underline{\xi} \times \underline{B}) \right] \right)^{(0)} = 0 \tag{179}$$

The parallel component of the momentum equation yields

$$\left(\underline{B} \cdot \underline{\nabla} \left[\gamma P (\underline{\nabla} \cdot \underline{\xi}) \right] \right)^{(0)} = 0 \tag{180}$$

when the pressure is taken to be finite.

These last two equations are two separate relations for parts of $(\underline{\nabla} \cdot \underline{\xi})^{(0)}$. Since the magnetic field is not constant on a surface, they can only be consistent for a particular parallel motion. We will now solve for that quantity.

To lowest order

$$\left(\underline{\nabla} \cdot \left[\underline{B} \times (\underline{\xi} \times \underline{B}) \right] \right)^{(0)} = B^2 \frac{\partial}{\partial \psi} \xi_\psi^{(1)} + \underline{B} \times \underline{\nabla} \psi \cdot \underline{\nabla} \xi_\perp^{(0)} \tag{181}$$

since $\underline{\nabla} \cdot (\underline{B} \times \underline{\nabla} \psi) = 0$ from the equilibrium, and

$$\begin{aligned} (\underline{\nabla} \cdot \underline{\xi})^{(0)} &= \frac{\partial}{\partial \psi} \xi_\psi^{(1)} + \frac{\underline{B} \times \underline{\nabla} \psi}{B^2} \xi_\perp^{(0)} \\ &\quad + \xi_\perp^{(0)} \underline{\nabla} \cdot \frac{\underline{B} \times \underline{\nabla} \psi}{B^2} + \underline{B} \cdot \underline{\nabla} \left(\xi_B^{(0)} / B^2 \right) \end{aligned} \tag{182}$$

so that

$$\underline{B} \cdot \underline{\nabla} \left(\xi_B^{(0)} / B^2 \right) + \xi_\perp^{(0)} \underline{\nabla} \cdot \frac{\underline{B} \times \underline{\nabla} \psi}{B^2} = 0 \tag{183}$$

We can evaluate the second term by returning to the equilibrium. Starting with

$$\underline{J} \times \underline{B} = \underline{\nabla} P$$

we can write

$$\underline{J} = \frac{\underline{B} \times \underline{\nabla} P}{B^2} + \sigma \underline{B} \tag{184}$$

where

$$\sigma \equiv \frac{\underline{J} \cdot \underline{B}}{B^2} \tag{185}$$

Then the condition that $\underline{\nabla} \cdot \underline{j} = 0$ yields

$$\underline{B} \cdot \underline{\nabla}\, \sigma = - P'(\psi)\, \underline{\nabla} \cdot \frac{\underline{B} \times \underline{\nabla} \psi}{B^2} \tag{186}$$

Hence,

$$\underline{B} \cdot \underline{\nabla} \left(\xi_B^{(0)} / B^2 \right) = \left(\xi_\perp^{(0)} / P' \right) \underline{B} \cdot \underline{\nabla} \sigma$$

$$= \underline{B} \cdot \underline{\nabla} \left(\xi_\perp^{(0)}\, \sigma / P' \right) \tag{187}$$

from Eq.(178). This can be integrated to yield

$$\xi_B^{(0)} = B^2 \left(\frac{\sigma}{P'} \xi_\perp^{(0)} + \text{Const} \right)$$

$$= \frac{B^2}{P'} \left(\sigma - \frac{\oint \sigma B^2 d\theta}{\oint B^2 d\theta} \right) \xi_\perp^{(0)} + \frac{2\pi B^2}{\oint B^2 d\theta} \bar{\xi}_B^{(0)} \tag{188}$$

where

$$\bar{\xi}_B^{(0)} = \frac{1}{2\pi} \oint \xi_B^{(0)} d\theta \tag{189}$$

Thus we have solved for the variation of the parallel motion over the magnetic surfaces in terms of two functions of ψ. It is not constant, but can be evaluated for each equilibrium. The only approximation that has been made is that the modes have low frequency.

We have now calculated the θ dependence of $\underline{\xi}$ to lowest order, using the components of the induction equation. We now wish to do the same thing for the perturbed magnetic field $\underline{b}$, using the momentum equation. The first order parallel momentum equation yields

$$\underline{B} \cdot \underline{\nabla} \left[\xi^{(1)} \cdot \underline{\nabla} P + \gamma P (\nabla \cdot \xi)^{(1)} \right] = 0 \tag{190}$$

and the radial component again yields

$$b_B^{(1)} = \xi_\psi^{(1)} P'(\psi) + \gamma P (\nabla \cdot \xi)^{(1)} \tag{191}$$

as in Eqs.(95) and (138). Combining these yields

$$\partial b_B^{(1)} / \partial \theta = 0 \tag{192}$$

To evaluate the θ dependence of $b_\perp^{(1)}$ we must annihilate the momentum equation with $\underline{\nabla} \cdot B^{-2}\, \underline{B}x$, and keep terms of zero order in ϵ. This goes as in Eqs.(97-99). The inertial term of Eq.(97) does not contribute in this order. Only the last term of Eq.(98) is sufficiently large to survive. To leading order it is

$$-\underline{B}\cdot\underline{\nabla}\left[\frac{\underline{B}\cdot\underline{\nabla}\times\underline{b}}{B^2}\right] = \underline{B}\cdot\underline{\nabla}\left\{\frac{\underline{B}}{B^2}\cdot\left[\frac{\underline{B}\times\underline{\nabla}\psi}{B^2}\times\underline{\nabla}\psi\,\frac{\partial b_\perp^{(1)}}{\partial\psi}\right]\right\}$$

$$= -\underline{B}\cdot\underline{\nabla}\left[\frac{|\underline{\nabla}\psi|^2}{B^2}\frac{\partial}{\partial\psi}b_\perp^{(1)}\right] \tag{193}$$

The term in Eq.(99) is more conveniently expressed as

$$\nabla\cdot\left[\frac{\underline{B}}{B^2}\times\underline{\nabla}\,b_B^{(1)}\right] \simeq \left(\underline{\nabla}\cdot\frac{\underline{B}\times\underline{\nabla}\psi}{B^2}\right)\frac{\partial b_B^{(1)}}{\partial\psi}$$

$$= -\underline{B}\cdot\underline{\nabla}\left(\frac{\sigma}{p'}\frac{\partial b_B^{(1)}}{\partial\psi}\right) \tag{194}$$

where Eq.(191) has been used to eliminate the perturbed pressure, higher order terms have been dropped, and Eqs.(186) and (192) have been used to achieve the final form. Remember that in lecture III, and Eq.(99), the $\gamma P(\underline{\nabla}\cdot\underline{\xi})$ term has been dropped. We thus obtain

$$\underline{B}\cdot\underline{\nabla}\left(\frac{|\underline{\nabla}\psi|^2}{B^2}\frac{\partial b_\perp^{(1)}}{\partial\psi}\right) + \underline{B}\cdot\underline{\nabla}\left(\frac{\sigma}{p'}\frac{\partial b_B^{(1)}}{\partial\psi}\right) = 0 \tag{195}$$

or

$$\frac{\partial b_\perp^{(1)}}{\partial\psi} = -\frac{1}{p'}\left[\frac{\sigma B^2}{|\underline{\nabla}\psi|^2} - \frac{B^2}{|\nabla\psi|^2}\frac{\oint\frac{\sigma B^2}{|\underline{\nabla}\psi|^2}d\theta}{\oint\frac{B^2}{|\underline{\nabla}\psi|^2}d\theta}\right]\frac{\partial b_B^{(1)}}{\partial\psi}$$

$$+\frac{B^2}{|\underline{\nabla}\psi|^2}\frac{2\pi}{\oint\frac{B^2}{|\underline{\nabla}\psi|^2}d\theta}\frac{\partial\bar{b}_\perp^{(1)}}{\partial\psi} \tag{196}$$

where

$$\bar{b}_\perp^{(1)} = \frac{1}{2\pi}\oint b_\perp^{(1)} d\theta \tag{197}$$

Again, $b_\perp^{(1)}$ is not constant along field lines, but its variation can be evaluated.

In previous lectures the important quantity was the radial perturbed magnetic field. We can find an expression for it in terms of $\bar{b}_\perp^{(1)}$ by using $\underline{\nabla} \cdot \underline{b} = 0$. We use

$$\underline{\nabla}\cdot\underline{b} = \frac{1}{\mathcal{J}}\left[\frac{\partial}{\partial\psi}\left(\mathcal{J}\,\underline{b}\cdot\underline{\nabla}\psi\right) + \frac{\partial}{\partial\theta}\left(\mathcal{J}\,\underline{b}\cdot\underline{\nabla}\theta\right) + \frac{\partial}{\partial u}\left(\mathcal{J}\,\underline{b}\cdot\underline{\nabla}u\right)\right] \tag{198}$$

To lowest order the Jacobians drop out of this expression, since it is nearly constant in the resistive layer. Then

$$\oint d\theta\, \underline{\nabla}\cdot\underline{b} = 0$$

yields

$$\frac{\partial}{\partial\psi}\bar{b}_\psi^{(2)} + \frac{1}{2\pi}\oint d\theta\, \frac{\underline{B}\times\underline{\nabla}\psi}{B^2}\cdot\underline{\nabla}u\,\frac{\partial b_\perp^{(1)}}{\partial u} = 0 \tag{199}$$

where $\bar{b}_\psi^{(2)}$ is again the average value, and the invariance of the equilibrium with respect to u has been used. Now to lowest order, Eq.(173) yields

$$\frac{\underline{B}\cdot\underline{\nabla}\psi\times\underline{\nabla}u}{B^2} = -\frac{n}{f} \tag{200}$$

so that finally,

$$\frac{\partial}{\partial\psi}b_\psi^{(2)} + \frac{in}{f}\,b_\perp^{(1)} = 0 \tag{201}$$

Similarly,

$$\frac{\partial}{\partial\psi}\xi_\psi^{(1)} + \frac{in}{f}\xi_\perp^{(0)} = 0 \tag{202}$$

We have now examined the leading order of each component of the equations for the perturbed quantities, and have evaluated the θ dependence of everything. Still, we do not have equations for the average values $\bar{\xi}_\psi^{(1)}$, $\bar{\xi}_B^{(0)}$, $\bar{b}_\psi^{(2)}$, $\bar{b}_B^{(1)}$. That is, we do not have the analog of any of the equations that have been used in the other lectures. Examination of the equations shows that we have not used all the leading information they contain. For example, the portion of the parallel momentum equation we have used is given by Eq.(190), but this does not have a component independent of θ. To obtain the rest of the information in this equation we need to project out the component independent of θ, in other words, average it over θ. This again is an annihilation, eliminating terms containing $\partial/\partial\theta$.

Averaging the parallel momentum equation yields to second order

$$\rho\gamma^2\bar{\xi}_B^{(0)} = -\frac{dP}{d\psi}\bar{b}_\psi^{(2)} - \frac{inf}{\mathcal{J}}q'(\psi-\psi_s)\bar{b}_B^{(1)} \tag{203}$$

where Eq.(191) has been used to eliminate the perturbed pressure, and Eq.(174) to expand the operator $\underline{B}\cdot\underline{\nabla}$ to next order. This is essentially identical to Eq.(139). The density has been assumed to be constant on a magnetic surface.

The average of the radial component of the induction equation yields, to second order,

$$\bar{b}_\psi^{(2)} + \frac{inf}{\gamma} q'(\psi - \psi_s)\xi_\psi^{(1)}$$

$$= -\frac{n}{2\pi\gamma} \oint d\theta \, \underline{\nabla}\psi \cdot \underline{\nabla} \times \underline{\nabla} \times \underline{b} \tag{204}$$

$$= \frac{n}{2\pi\gamma} \oint d\theta \, \underline{\nabla} \cdot (\underline{\nabla}\psi \times \underline{\nabla} \times \underline{b})$$

We use Eq.(198) to evaluate the average of a derivative, and keep only the leading order terms to finds,

$$\oint d\theta \, \underline{\nabla} \cdot (\underline{\nabla}\psi \times \underline{\nabla} \times \underline{b})$$

$$= -\oint d\theta \left(\underline{\nabla} \times \frac{\partial \underline{b}}{\partial u} \right) \cdot \underline{\nabla}\psi \times \underline{\nabla} u$$

$$\simeq -\oint d\theta \frac{\partial^2 b_\perp^{(1)}}{\partial \psi \partial u} \left(\underline{\nabla}\psi \times \frac{\underline{B} \times \underline{\nabla}\psi}{B^2} \right) \cdot \underline{\nabla}\psi \times \underline{\nabla} u$$

$$= -\frac{in}{f} \oint d\theta \, |\underline{\nabla}\psi|^2 \frac{\partial}{\partial \psi} b_\perp^{(1)} \tag{205}$$

$$= \frac{in}{fP'} \left[\oint d\theta \, \sigma B^2 - \oint B^2 d\theta \frac{\oint \frac{\sigma B^2}{|\underline{\nabla}\psi|^2} d\theta}{\oint \frac{B^2}{|\underline{\nabla}\psi|^2} d\theta} \right] \frac{\partial b_B^{(1)}}{\partial \psi}$$

$$+ 2\pi \frac{\oint B^2 d\theta}{\oint \frac{B^2}{|\underline{\nabla}\psi|^2} d\theta} \frac{\partial^2}{\partial \psi^2} \bar{b}_\psi^{(2)}$$

In the last step Eqs.(196) and (201) have been used.

After substituting Eq.(205) back into (204), we can compare it with the radial induction equation for the straight case, Eq.(93). Aside from the inevitable change of constants appropriate for the new equilibrium, the important difference between these two equations is the first term in Eq.(205), proportional to $b_B^{(1)}$. This has no counterpart in the straight system. A good part of the next lecture will be devoted to a discussion of the effect of this term.

Note from Eq.(204) and the first form of Eq.(205) that the resistive term is proportional to the parallel perturbed current, and thus to the parallel electric field. It is the existance of a parallel electric field, forbidden by the ideal Ohm's law,

$$\underline{E} + \underline{v} \times \underline{B} = 0$$

that permits these instabilities to form. Other nonideal effects that give rise to parallel electric fields also cause instabilities of the same general type.

We still need two more equations to complete the full set. The derivation of these equations is more tedious than the derivations done so far, without having any intrinsic interest. One is formed from

$$\oint d\theta \left(\frac{1}{B^2} \underline{B} + \frac{\sigma}{p'} \frac{\underline{B} \times \underline{\nabla}\psi}{|\underline{\nabla}\psi|^2} \right) \cdot (\text{Induction Eq.}) = 0 \tag{206}$$

This involves two components of the induction equation because the parallel component, from the argument between Eqs.(181) and (187), yields a term

$$\xi_\perp \; \underline{B}\cdot\underline{\nabla}(\sigma/p')$$

to annihilate this cleanly by averaging over θ we need to add a term

$$(\sigma/p')\,\underline{B}\cdot\underline{\nabla}\xi_\perp$$

This is accomplished by the second term of Eq.(206).

Finally, by the same kind of argument, we take for the next equation

$$\oint d\theta \left\{ \underline{\nabla}\cdot\frac{\underline{B}}{B^2}\times(\text{Momentum}) - \frac{\sigma}{p'}\frac{\partial}{\partial\psi}\,\underline{B}\cdot(\text{Momentum})\right] = 0 \tag{207}$$

The last two equations involve the average

$$\oint \sigma\, b_\perp^{(1)}\, d\theta \tag{208}$$

However, a differential equation can be obtained for this quantity by an appropriate average over Eq.(196).

This lecture has been devoted to consideration of how a complete set of equations could be obtained for resistive instabilities in toroidal systems. In the next lecture the equations will be collected and discussed.

The calculation given in this lecture was started in reference 18. In that paper we barely got further than this lecture. Some years later Glasser took up the problem from that point, and put it in good form in reference 17. That latter paper will form the basis for the next lecture. I would like to thank Alan Glasser for help in preparing these two lectures.

LECTURE VII

In the previous lecture equations were derived describing the inner layer behavior of resistive instabilities in toroidal, axisymmetric systems. In fact, only the method of derivation was given. Here we pull together the results of that lecture, and discuss the resulting equations. Because there are many factors, we will do this slowly, and discuss them one at a time.

It turns out that the appropriate scaling for these equations, equivalent to Eqs.(103) and (104), is

$$\gamma = \left[\frac{\eta\, n^2 f^2 q'^2 \oint B^2 d\theta}{J^2 \rho M \oint \frac{B^2}{|\nabla\psi|^2} d\theta} \right]^{1/3}_{\psi=\psi_s} Q \tag{209}$$

$$\psi - \psi_s = \left[\frac{J^2 \rho M\, \eta^2 \left(\oint B^2 d\theta\right)^2}{n^2 f^2 q'^2 \left(\oint \frac{B^2}{|\nabla\psi|^2} d\theta\right)^2} \right]^{1/6}_{\psi=\psi_s} X = L_R X \tag{210}$$

where

$$M \equiv \frac{1}{4\pi^2} \oint \frac{B^2}{|\nabla\psi|^2} d\theta \left[\oint \frac{|\nabla\psi|^2}{B^2} d\theta + \frac{1}{p'^2} \left(\oint \sigma^2 B^2 d\theta - \frac{\left(\oint \sigma B^2 d\theta\right)^2}{\oint B^2 d\theta} \right) \right] \tag{211}$$

The quantity $\mathcal{J}$ is the Jacobian, defined in Eq.(162). It can also be expressed as

$$\mathcal{J} = \frac{1}{4\pi^2}\frac{dV}{d\psi} \tag{212}$$

where V is the volume inside the surface labeled by ψ, since we have taken $\mathcal{J}$ to be constant over the magnetic surfaces. The quantities f, q, and σ have been defined in Eqs.(160), (164), and (185).

This reduces to the scaling for the straight case if we take $\psi = \Omega$ so that $|\underline{\nabla}\psi|^2 = 1$, $\mathcal{J} = \Omega/K$ and $f = B_\theta/k$ where k was introduced in Eq.(80). Note that in this limit σ is a constant, so M reduces to unity.

The dependent variables are also scaled as in lecture III and V,

$$\begin{aligned} \bar{\xi}_\psi^{(1)} &\equiv \xi \\ \bar{b}_\psi^{(2)} &\equiv -i\,\frac{nfq'}{\mathcal{J}}\,L_R\,\Psi \\ \bar{b}_B^{(1)} &= (dP/d\psi)\,\Upsilon \end{aligned} \tag{213}$$

and also

$$\Gamma \equiv \frac{\mathcal{J}^2}{4\pi^2 f^2 q'^2}\oint\frac{B^2}{|\underline{\nabla}\psi|^2}d\theta\Big[P'^2\Big(\oint\frac{1}{B^2}d\theta\Big)\Upsilon + \frac{2\pi f^2 q'}{\mathcal{J}}\frac{\oint\sigma B^2 d\theta}{\oint B^2 d\theta}\Psi_x - \oint\sigma b_\perp^{(1)}d\theta\Big] \tag{214}$$

The latter quantity contains the average of $\sigma b_\perp^{(1)}$ in the last term.

Then the radial induction equation becomes, collecting Eqs.(204) and (205),

$$\Psi_{xx} = Q\Psi - Qx\xi + H\Upsilon_x \tag{215}$$

with

$$H \equiv \sigma \frac{\oint \frac{B^2}{|\nabla\psi|^2} d\theta}{2\pi f^2 q'} \left[\frac{\oint \sigma B^2 d\theta}{\oint B^2 d\theta} - \frac{\oint \sigma \frac{B^2}{|\nabla\psi|^2} d\theta}{\oint \frac{B^2}{|\nabla\psi|^2} d\theta} \right] \tag{216}$$

This is the same as Eq.(106) for the straight case, with the addition of the term containing H.

This constant is the only completely new term arises from axisymmetry. It appears to be a rather strange term, whose physical meaning is not apparent. Note that it vanishes if $|\nabla\psi|^2$ is constant over a magnetic surface. Further, if the pressure gradient vanishes, σ is a constant by Eq.(186), and H vanishes. In reference 16 it has been evaluated for a large aspect ratio circular cross section tokamak, with the result,

$$H \simeq -\frac{2q^5}{B_0^2 r^4 q'} \frac{dP}{dr} \int_0^r \frac{r^3}{q^2} \left(1 - \frac{2R_0^2}{B_0^2 r} \frac{dP}{dr} \right) \tag{217}$$

Since generally, $dP/dr < 0$, it appears that H will tend to be small and positive.

Pushing on to examine the other equations, the annihilated momentum equation is given by, after evaluating Eq.(207),

$$Q^2 \xi_{xx} - Qx^2 \xi + E\Upsilon + Qx\bar{\Psi} + \Gamma = 0 \tag{218}$$

where

$$E \equiv \frac{\mathcal{J}}{8\pi^3 f^4 q'^2} \oint \frac{B^2}{|\nabla\psi|^2} d\theta \left[-p'V'' + J'\bar{\Phi}'' - I'\chi'' - 4\pi^2 f^2 q'^2 \frac{\oint \sigma B^2 d\theta}{\oint B^2 d\theta} \right] \tag{219}$$

If the Γ term were absent, this equation would be the same as Eq.(144) for the straight case, with E taking the place of D_s. Thus E is a measure of the average normal curvature. We will see later that the Γ term introduces some corrections. Of the various quantities entering E, V" is the second derivative of the volume with respect to ψ, and χ and $\bar{\Phi}$ are the fluxes introduced in Eqs.(161) and (163). The quantities I and J are the current fluxes, given by

$$I \equiv \mathcal{J} f \oint \frac{|\nabla\psi|^2}{R^2} d\theta$$

$$J \equiv -2\pi R_0 B_0 \, g(\psi) \tag{220}$$

Together they satisfy the equilibrium condition |19|

$$p'V' = J'\phi' - I'\chi' \tag{221}$$

Very early the CTR program the quantity V" was recognized as important in determining stability, for example in reference 11. Thus the quantity call E here is often referred to as V" .

The parallel induction equation can be represented, after carrying out the operations indicated in Eq.(206),

$$\frac{1}{Q}\Upsilon_{xx} - \frac{X^2}{Q^2}\Upsilon - G\Upsilon + (G-kE)\zeta + \frac{X}{Q^2}\Psi - k\Gamma = 0 \tag{222}$$

The constant

$$G \equiv \frac{\oint B^2 d\Theta}{2\pi M \gamma P} \tag{223}$$

is the straightforward generalization of the factor $B^2/\gamma P$ that appeared in earlier lectures. The constant

$$K \equiv \frac{f^4 q'^2}{M P'^2 g^2} \frac{\oint B^2 d\Theta}{\oint \frac{B^2}{|\nabla\psi|^2} d\Theta} \tag{224}$$

absorbs some other factors.

Finally, the equation for Γ is obtained by averaging Eq.(106),

$$\Gamma_x = H\Psi_{xx} + F\Upsilon_x \tag{225}$$

where

$$F \equiv \frac{g^2}{4\pi^2 f^4 q'^2}\oint\frac{B^2}{|\nabla\psi|^2}d\Theta\left[P'^2\oint\frac{1}{B^2}d\Theta + \oint\frac{\sigma^2 B^2}{|\nabla\psi|^2}d\Theta - \frac{\left(\oint\frac{\sigma B^2}{|\nabla\psi|^2}d\Theta\right)^2}{\oint B^2/|\nabla\psi|^2 d\Theta}\right] \tag{226}$$

The set of equations, Eqs.(215), (218), (222), and (225), has the exact solution

$$\bar{\Psi} = x \qquad \xi = \Upsilon = 1 \qquad \Gamma = -E \tag{227}$$

It appears that this does not match to the outer regions. The condition that it not appear is that the integrated form of Eq.(225) vanish,

$$\Gamma = H\bar{\Psi}_x + F\Upsilon \tag{228}$$

This seems like a reasonable thing to do. Then Γ can be eliminated and we find,

$$\bar{\Psi}_{xx} - Q\bar{\Psi} + Qx\xi - H\Upsilon_x = 0 \tag{229}$$

$$Q^2\xi_{xx} - Qx^2\xi + (E+F)\Upsilon + Qx\bar{\Psi} + H\bar{\Psi}_x = 0 \tag{230}$$

$$\begin{aligned} Q^2\Upsilon_{xx} - Qx^2\Upsilon - Q^3(G+kF)(\Upsilon-\xi) \\ - Q^3k(E+F)\xi + Qx\bar{\Psi} \\ - Q^3kH\bar{\Psi}_x = 0 \end{aligned} \tag{231}$$

Comparing this set with Eqs.(143) to (145) of lecture V, each equation of the new set has a term proportional to H that did not appear in the previous set. The other factors can be identified; E + F replaces D_s, G replaces $B^2/\gamma P$ as mentioned above, KF replaces unity, and K(E+F) replaces S/D_s. Each factor reduces to the proper limit when evaluated for the straight case. The effect of H is the only new and interesting thing in these equations.

Following previous procedure, we now consider the solutions of these equations near the singular point at infinity. The exponential solutions are not changed from lecture V. The leading terms for each equation, in the approximation that yields the algebraic solutions, is

$$\begin{aligned} &\Psi - x\xi = 0 \\ &x\Psi_{xx} - xH\Upsilon_x + (E+F)\Upsilon + H\Psi_x = 0 \\ &\Psi - x\Upsilon = 0 \end{aligned} \tag{232}$$

As before, the middle equation is derived from a combination of Eqs.(229) and (230). Eliminating yields

$$x\Psi_{xx} + (E+F+H)\Psi/x = 0$$

or

$$\Psi = x^{s+1},\ x^{-s} \qquad s \equiv -\tfrac{1}{2} + \tfrac{1}{2}\left[1 - 4(E+F+H)\right]^{1/2} \tag{233}$$

Now the generalized Mercier stability criterion |20| for ideal interchanges in axisymmetric systems can be written

$$D_I < 0 \qquad D_I \equiv E+F+H-1/4 \tag{234}$$

This is also the criterion that s is real. By the arguments of lecture II it is clear that we have rederived this general stability criterion here, by a different method than that used previously. This give confidence that it will be possible to obtain matching to outside solutions. Thus D_I is the ideal generalization of the quantity D_s. The $1/4$ has been added here for reasons that will be discussed later.

Now we need to find the equivalent resistive stability criterion. We proceed as in the straight case, taking the approximation that Q is small. The ordering is as before, except that we will not take G to be large. Otherwise, the equations become too complicated to treat analytically. This is not an unreasonable approximation, as discussed in lecture V, where the small G approximation turned out to be more realistic than the infinite G approximation.

To lowest order, when Q is small, Eqs.(229) and (230) yield

$$\begin{aligned} &\bar{\Psi}_{xx} - H\Upsilon_x = 0 \\ &(E+F)\Upsilon + H\bar{\Psi}_x = 0 \end{aligned} \tag{235}$$

We have assumed $\bar{\Psi} \sim X\xi$, following the argument above Eq.(119) in lecture IV. This pair of equations has a nontrivial solution only when

$$E+F+H^2=0$$

In other words, there can be no solutions with small Q unless this quantity is small. One could correctly guess that this will turn out to be the resistive stability criterion. To prove it requires more work. We define

$$D_R \equiv E + F + H^2 \tag{236}$$

and assume that it is of order $Q^{3/2}$ in the small Q limit, following previous procedure. Then the leading order equations are

$$\begin{aligned} &\bar{\Psi}_{xx} - H\Upsilon_x = 0 \\ &\left(Q^2 \xi_{xx} - Qx^2 \xi\right)_x - QHx\xi \\ &\qquad + Q(x\bar{\Psi})_x + QH\bar{\Psi} + D_R \Upsilon_x = 0 \\ &Q^2 \Upsilon_{xx} - Qx^2 \Upsilon + Qx\bar{\Psi} = 0 \end{aligned} \tag{237}$$

As with most of our approximate equations, Eq.(229) and (230) have been combined to derive the middle equation.

As with the equations in lecture IV, it is not clear that they should match to solutions in the outside region. The first term above is entirely resistive, and never approaches the ideal regime. Again, an intermediate region of order of the skin depth is needed to carry the solution from dominantly resistive to nearly ideal. This solution is essentially trivial, again as before.

The first and last equations above can be solved together. When combined they yield

$$Q\Psi_{xxx} - x^2\Psi_x + Hx\Psi = 0 \tag{238}$$

This equation no longer yields the constant Ψ approximation, but it is not hopeless.

This equation has two exponential and one algebraic solution when x is large. The algebraic solution goes as

$$\Psi \sim x^H$$

that is

$$-s = \frac{1}{2} - \frac{1}{2}\left[1 - 4(E+F+H)\right]^{1/2}$$

$$\simeq \frac{1}{2} - \frac{1}{2}\left[1 + 4(H^2 - H)\right]^{1/2}$$

$$= H$$

since D_R has been taken to be small. Requiring the solution to be odd fixes two conditions at x = 0, and thus, barring a miracle, the odd solution must diverge exponentially. The even solution has more free parameters, so the useful solution here must be even.

We can solve Eq.(238) by Fourier transforms,

$$\Psi = \int_{-\infty}^{\infty} \exp(i\mu x)\,\hat{\Psi}(\mu)\,d\mu$$

The singularity that arises at $\mu = 0$, as in the similar treatment in lecture III, is an unecessary annoyance. There is no particular reason that μ should be real. When the contour is taken to lie in the complex plane, this method of solution is known as Laplace's method. It has been carried through for this equation in reference 17.

Having decided that algebra involved in Laplace's method is too detailed to include in these lectures, we are cut off from following the calculation. The general trend is very similar to that in lecture IV. The middle of Eqs.(237) can be solved for ξ , knowing Ψ and Υ , again by Laplace's method. From the next order corrections to Ψ and Υ, a dispersion relation can be derived in the same form as Eq.(133).

$$\Delta = \Delta' \tag{239}$$

with Δ' coming from coefficients of solutions in the outer region

$$\Delta' \equiv \frac{A_{III}}{B_{III}} - \frac{A_I}{B_I} \tag{240}$$

and Δ evaluated from the inner, resistive region solution as

$$\Delta = \pi \left(\frac{2 x_s}{L_R}\right)^{1-2H} \frac{\Gamma(\frac{1}{4})\,\Gamma^2(1-\frac{H}{4})\,\Gamma(\frac{3}{4}-\frac{H}{2})}{(1-2H)\cos^2 \pi H/2\,\Gamma(1-\frac{H}{2})\,\Gamma^2(\frac{1+H}{4})\,\Gamma^2(1-H)}$$

$$\times Q^{(2H+5)/4}\left[1 - \frac{\Gamma(\frac{3}{4})\,\Gamma^2(\frac{1}{2}-\frac{H}{4})\,\Gamma(\frac{1}{4}-\frac{H}{2})}{\Gamma(\frac{1}{4})\,\Gamma^2(1-\frac{H}{4})\,\Gamma(\frac{3}{4}-\frac{H}{2})}\;\frac{D_R}{4Q^{3/2}}\right] \tag{241}$$

There are several things that one can learn from this messy formula.

First, when D_R is positive, Δ takes on all values for positive Q. Thus there is always an instability. When D_R is negative, Δ is positive, at least when H is small. Thus, as in lecture V, there is a stability criterion of the form,

$$D_R < 0 \qquad \Delta' < \Delta_c \tag{242}$$

The stability criterion on D_R is the most immediately useful result of this whole exercise. As we have seen in lecture II, interchange stability criteria are the most useful results of analytic theory because they explore regions that are almost impossible to calculate numerically. Further, the resistive interchange stability criterion complements the ideal criterion of Eq.(234).

Next, consider extending the range of H from consideration of small H. Really terrible things happen at H = ½. First off, note that

$$D_I = D_R - \left(H - \frac{1}{2}\right)^2 \tag{243}$$

so that the two stability criteria are equal here. Incidently, the ¼ was put in the expression for D_I to simplify this relation between D_I and D_R. Furthermore, the large and small solutions, that are to be matched in the outer region, degenerate and exchange sizes at this point. Thus the factor

$$\left(\psi_s / L_R\right)^{1-2H}$$

becomes small instead of large. There must be many interesting things happen when H is $\frac{1}{2}$ and larger, but this has not been thoroughly explored.

As H becomes more negative, the first difficulty seems to appear at $H = -5/2$. At this point, the term

$$Q^{(2H+5)/4}$$

no longer diverges for large Q. I believe that here the approximation that Q is small breaks down. There are weakly unstable modes with finite oscillatory frequencies that escape from the ordering we have used. Again, this is a region that has not been thoroughly explored.

Understanding the behavior for large H does not seem to be terrible pressing, because H is nearly always small.

There are several other problems that should be done, that could be important.

First, it is probably important to evaluate the tearing mode stability parameter, more accurately. Other nonideal effects, such as heat conductivity and viscosity, might well make a significant difference. Some work is being done on this.

More fundamentally, we need a way to calculate Δ' from the outer regions. In these lectures we have put off discussion of this parameter until now. In fact, we have no good way to calculate it at the present time. It would be ideal to express it as an integral, so that the result is weighted over the entire outer region. Such an expression could make this theory very tidy and would make numerical calculation of tearing modes straightforward.

I close with this somewhat uncertain note. In a way it is satisfying to know that there are still open problems that must be faced before the subject becomes nature. We are working with an interesting, growing theory.

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