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LONG WAVELENGTH SUM RULES FOR TWO DIMENSIONAL PLASMAS

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ABSTRACT

Using the exact equation of state for the two dimensional ocp, asymptotic static long wavelength limits and compressibility sum rules are derived for the linear and quadratic polarizabilities.

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Recent progress in the formulation of an exact equation of state for the two dimensional (2D) one component plasma (ocp)[1,2,3,4] now makes it possible to establish the 2D ocp long wavelength compressibility sum rules. These model independent sum rules (valid for arbitrary values of the plasma parameter $\gamma = \beta q^2$ (q = charge, β^{-1} = temperature in energy units) up to the threshold of thermodynamic instability[4]) can be especially useful for verifying the correctness of dynamical ($\omega \neq 0$) response functions derived from model kinetic equations for strongly correlated 2D ocp's.

In this Comment, we derive the long wavelength $1/\omega$ frequency moment sum rules for (i) the <u>internal</u> linear and quadratic polarizability functions $\alpha(\underset{\sim}{k}\omega)$, $\alpha(\underset{\sim}{k}_1\omega_1;\ \underset{\sim}{k}_2\ \omega_2)$ defined by the relations[5]

$$\tilde{E}^{(1)}(\underset{\sim}{k}\omega) = -\alpha(\underset{\sim}{k}\omega)E^{(1)}(\underset{\sim}{k}\omega), \tag{1}$$

$$\tilde{E}^{(2)}(\underset{\sim}{k}\omega) = \frac{-1}{\epsilon(\underset{\sim}{k}\omega)} \sum\!\!\!\!\!\!\int_{\underset{\sim}{q}\mu} \alpha(\underset{\sim}{q}\mu;\underset{\sim}{k}-\underset{\sim}{q},\omega-\mu)E^{(1)}(\underset{\sim}{q}\mu)E^{(1)}(\underset{\sim}{k}-\underset{\sim}{q},\omega-\mu), \tag{2}$$

and (ii) the corresponding <u>external</u> polarizabilities $\hat{\alpha}(\underset{\sim}{k}\omega)$, $\hat{\alpha}(\underset{\sim}{k}_1\omega_1;\ \underset{\sim}{k}_2\omega_2)$ defined by[6]

$$\tilde{E}^{(1)}(\underset{\sim}{k}\omega) = -\hat{\alpha}(\underset{\sim}{k}\omega)\hat{E}(\underset{\sim}{k}\omega), \tag{3}$$

$$\tilde{E}^{(2)}(\underset{\sim}{k}\omega) = -\sum\!\!\!\!\!\!\int_{\underset{\sim}{q}\mu} \hat{\alpha}(\underset{\sim}{q}\mu;\underset{\sim}{k}-\underset{\sim}{q},\omega-\mu)\hat{E}(\underset{\sim}{q}\mu)\hat{E}(\underset{\sim}{k}-\underset{\sim}{q},\omega-\mu). \tag{4}$$

We first establish the static (zero frequency) long wavelength limit of and sum rules for the internal polarizabilities. In this limit, the isothermal fluid motion of the 2D ocp is appropriately described by hydrodynamic equations. From first and second order perturbation expansions (in $\hat{E}$) of these equations, one obtains polarizabilities whose $\underset{\sim}{k}$ space structures are the same as those for the 3D case[7]:

$$\lim_{\underset{\sim}{k}\to 0} \alpha'(\underset{\sim}{k}0) = \alpha_0'(\underset{\sim}{k}0)(\beta mc^2)^{-1}, \tag{5}$$

$$\lim_{\underset{\sim}{k}_1 \to 0} \lim_{\underset{\sim}{k}_2 \to 0} \alpha''(\underset{\sim}{k}_1 0; \underset{\sim}{k}_2 0) = \alpha_0''(\underset{\sim}{k}_1 0; \underset{\sim}{k}_2 0)(\beta mc^2)^{-2} \left[1 - na/(mc^2) \right], \tag{6}$$

where: prime and double primes refer, respectively, to the real and imaginary part,

$a = (\partial^2 p/\partial n^2)_\beta$, $n = N/\Lambda$, p = pressure

$c = \sqrt{(1/m)(\partial p/\partial n)_\beta}$ is the isothermal sound speed,

and

$$\alpha_0'(\underset{\sim}{k} 0) = 4\pi n\beta q^2/k^2 \tag{7}$$

$$\alpha_0''(\underset{\sim}{k}_1 0; \underset{\sim}{k}_2 0) = 2\pi n\beta^2 q^3/(k_1 k_2 |\underset{\sim}{k}_1 + \underset{\sim}{k}_2|) \tag{8}$$

are the Vlasov values of $\alpha'(\underset{\sim}{k}0)$, $\alpha''(\underset{\sim}{k}_1 0; \underset{\sim}{k}_2 0)$. Using the exact 2D ocp equation of state[3,4],

$$\beta p/n = 1 - (\gamma/4), \tag{9}$$

Equations (5) and (6) become:

$$\lim_{\underset{\sim}{k} \to 0} \alpha'(\underset{\sim}{k} 0) = \alpha_0'(\underset{\sim}{k} 0) \left[1 - (\gamma/4) \right]^{-1}, \tag{10}$$

$$\lim_{\underset{\sim}{k}_1 \to 0} \lim_{\underset{\sim}{k}_2 \to 0} \alpha''(\underset{\sim}{k}_1 0; \underset{\sim}{k}_2 0) = \alpha_0''(\underset{\sim}{k}_1 0; \underset{\sim}{k}_2 0) \left[1 - (\gamma/4) \right]^{-2}. \tag{11}$$

(Equations (10) and (11) suggest the structure

$$\lim_{\underset{\sim}{k}_1 \to 0} \cdots \lim_{\underset{\sim}{k}_n \to 0} \alpha(\underset{\sim}{k}_1 0; \cdots \underset{\sim}{k}_n 0) = \alpha_0(\underset{\sim}{k}_1 0; \cdots \underset{\sim}{k}_n 0)\left[1 - (\gamma/4)\right]^{-n} \tag{12}$$

for the $n^{\underline{th}}$ order polarizability function. However, Eq.(12) has yet to be rigorously established). From Eqs.(10), (11) and the Kramers-Kronig formulae

$$P\int_{-\infty}^{\infty} \frac{d\omega}{\omega} \alpha''(\underset{\sim}{k}\omega) = \pi\alpha'(\underset{\sim}{k}0), \tag{13}$$

$$P\int_{-\infty}^{\infty} \frac{d\omega_1}{\omega_1} \alpha'(\underset{\sim}{k}_1 \omega_1; \underset{\sim}{k}_2 0) = \pi\alpha''(\underset{\sim}{k}_1 0; \underset{\sim}{k}_2 0), \tag{14}$$

one obtains the desired sum rules for the internal polarizabilities,

$$\lim_{\underset{\sim}{k} \to 0} P\int_{-\infty}^{\infty} \frac{d\omega}{\omega} \alpha''(\underset{\sim}{k}\omega) = \pi\alpha_0'(\underset{\sim}{k}0)\left[1 - (\gamma/4)\right]^{-1}, \tag{15}$$

$$\lim_{\underset{\sim}{k}_1 \to 0} \lim_{\underset{\sim}{k}_2 \to 0} P\int_{-\infty}^{\infty} \frac{d\omega_1}{\omega_1} \alpha'(\underset{\sim}{k}_1 \omega_1; \underset{\sim}{k}_2 0) = \pi\alpha_0''(\underset{\sim}{k}_1 0; \underset{\sim}{k}_2 0)\left[1-(\gamma/4)\right]^{-2}. \tag{16}$$

The corresponding sum rules for the collisional corrections ($\alpha_{\text{coll}} = \alpha - \alpha_0$) must then be

$$\lim_{\underset{\sim}{k} \to 0} P\int_{-\infty}^{\infty} \frac{d\omega}{\omega} \alpha''_{coll}(\underset{\sim}{k}\omega) = \pi\alpha_0'(\underset{\sim}{k}0)(\gamma/4)\left[1 - (\gamma/4)\right]^{-1}, \tag{17}$$

$$\begin{aligned} &\lim_{\underset{\sim}{k}_1 \to 0} \lim_{\underset{\sim}{k}_2 \to 0} P\int_{-\infty}^{\infty} \frac{d\omega_1}{\omega_1} \alpha'_{coll}(\underset{\sim}{k}_1 \omega_1; \underset{\sim}{k}_2 0) = \\ &\quad = \pi\alpha_0''(\underset{\sim}{k}_1 0; \underset{\sim}{k}_2 0)(\gamma/4)\left[2-(\gamma/4)\right]\left[1-(\gamma/4)\right]^{-2}. \end{aligned} \tag{18}$$

For the linear external polarizability, one can readily derive from its Kramers-Kronig formula and Eq.(10) the long wavelength "perfect screening" sum rule[8],

$$\lim_{\underset{\sim}{k}\to 0} P\int_{-\infty}^{\infty}\frac{d\omega}{\omega}\hat{\alpha}''(\underset{\sim}{k}\omega) = \pi. \tag{19}$$

This sum rule is completely exhausted by $\hat{\alpha}_0''(\underset{\sim}{k}\omega)$ so that

$$\lim_{\underset{\sim}{k}\to 0} P\int_{-\infty}^{\infty}\frac{d\omega}{\omega}\hat{\alpha}''_{coll}(\underset{\sim}{k}\omega) = 0. \tag{20}$$

Similar long wavelength perfect screening sum rules for the external quadratic polarizability can be obtained only by taking frequency moments of the dynamical nonlinear fluctuation-dissipation theorem[9] relating the space-time Fourier transformed equilibrium ternary correlation of microscopic charge densities to the external quadratic polarizability. Sum rules for the 2D ocp, valid at all wavelengths, have been derived in this way; their structure is identical to those for the 3D ocp displayed as Eqs.(24) of Ref. 7. Applying Eq.(10) to these and going to the long wavelength limit, we obtain for the 2D ocp:

$$\lim_{\underset{\sim}{k}_1\to 0}\lim_{\underset{\sim}{k}_2\to 0}\int_{-\infty}^{\infty}d\omega_1\int_{-\infty}^{\infty}d\omega_2\Big[\hat{\alpha}''(\underset{\sim}{k}_1\omega_1;\underset{\sim}{k}_2\omega_2) - \frac{\omega_2}{\omega_1+\omega_2}\hat{\alpha}''(\underset{\sim}{k}_1\omega_1;-\underset{\sim}{k}_1-\underset{\sim}{k}_2,-\omega_1-\omega_2)$$
$$-\frac{\omega_1}{\omega_1+\omega_2}\hat{\alpha}''(-\underset{\sim}{k}_1-\underset{\sim}{k}_2,-\omega_1-\omega_2;\underset{\sim}{k}_2\omega_2)\Big] = -(\pi^2 q/m)(\underset{\sim}{k}_1\cdot\underset{\sim}{k}_2)|\underset{\sim}{k}_1+\underset{\sim}{k}_2|/(k_1k_2), \tag{21}$$

$$\lim_{\underset{\sim}{k}_1\to 0}\lim_{\underset{\sim}{k}_2\to 0}\int_{-\infty}^{\infty}d\omega_1\int_{-\infty}^{\infty}d\omega_2(\omega_1+\omega_2)^2\Big[\frac{\hat{\alpha}''(\underset{\sim}{k}_1\omega_1;\underset{\sim}{k}_2\omega_2)}{\omega_1\omega_2} - \tag{22}$$
$$\frac{\hat{\alpha}''(\underset{\sim}{k}_1\omega_1;-\underset{\sim}{k}_1-\underset{\sim}{k}_2,-\omega_1-\omega_2)}{\omega_1(\omega_1+\omega_2)} - \frac{\hat{\alpha}''(-\underset{\sim}{k}_1-\underset{\sim}{k}_2,-\omega_1-\omega_2;\underset{\sim}{k}_2\omega_2)}{\omega_2(\omega_1+\omega_2)}\Big] =$$
$$= -(\pi^2 q/m)(\underset{\sim}{k}_1+\underset{\sim}{k}_2)\cdot\big[(k_1\underset{\sim}{k}_2/k_2)+(k_2\underset{\sim}{k}_1/k_1)\big]/|\underset{\sim}{k}_1+\underset{\sim}{k}_2|.$$

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[5] Symbols are defined as follows: $\tilde{E}$ is the induced electric field response of the plasma particles to the weak external field $\hat{E}$; $E = \tilde{E} + \hat{E}$ is the total electric field. Superscripts refer to the order (of smallness) in powers of $\hat{E}$ (see, e.g., Eqs.(3), (4)). $\epsilon(\underset{\sim}{k}\omega) = 1 + \alpha(\underset{\sim}{k}\omega)$ is the linear dielectric function.

$$\not{\Sigma}_{\underset{\sim}{q}\mu} = (2\pi\Lambda)^{-1}\sum_{\underset{\sim}{q}}\int_{-\infty}^{\infty} d\mu$$, where Λ is the 2D volume.

[6] We note the following relations between the internal and external polarizabilities:

$$\hat{\alpha}(\underset{\sim}{k}\omega) = \frac{\alpha(\underset{\sim}{k}\omega)}{\epsilon(\underset{\sim}{k}\omega)} \quad , \quad \hat{\alpha}(\underset{\sim}{k}_1\omega_1;\underset{\sim}{k}_2\omega_2) = \frac{\alpha(\underset{\sim}{k}_1\omega_1;\underset{\sim}{k}_2\omega_2)}{\epsilon(\underset{\sim}{k}_1\omega_1)\epsilon(\underset{\sim}{k}_2\omega_2)\epsilon(\underset{\sim}{k}_1+\underset{\sim}{k}_2,\omega_1+\omega_2)}$$

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