August 1976

LRP 116/76

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# CONDUCTIVITY SUM RULES IN TWO COMPONENT PLASMAS

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ABSTRACT

Conductivity sum rules are established for the linear and quadratic polarizabilities in the two component non-relativistic plasma.

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The conductivity and long wavelength compressibility sum rules of plasma physics are model-independent and are therefore useful for checking the accuracy of response functions calculated in various approximations from kinetic equations. By now key sum rules have been formulated for a host of linear [1] and nonlinear [2] response functions for the one component plasma (ocp). Sum rules for tcp (electron-ion two component) systems are, however, still lacking. In this paper we shall partially fill in this gap by reporting new tcp conductivity rules for the linear and quadratic polarizabilities, ${}_1\alpha_A$ and ${}_2\alpha_A$ (A = electrons, ions) defined by the relations,

$$\tilde{E}_A^{(1)}(\underset{\sim}{k}\,\omega) = -{}_1\alpha_A(\underset{\sim}{k}\,\omega)\,E^{(1)}(\underset{\sim}{k}\,\omega), \tag{1}$$

$$\tilde{E}_A^{(2)}(\underset{\sim}{k}\,\omega) = -(1/V)\sum_{\underset{\sim}{q}}\int_{-\infty}^{\infty}(d\mu/2\pi)\,\frac{{}_2\alpha_A(\underset{\sim}{q}\mu;\underset{\sim}{k}-\underset{\sim}{q},\omega-\mu)}{1+{}_1\alpha_A(\underset{\sim}{k}\,\omega)}\cdot E^{(1)}(\underset{\sim}{q}\mu)\,E^{(1)}(\underset{\sim}{k}-\underset{\sim}{q},\omega-\mu), \tag{2}$$

connecting the $n^{\underline{th}}$ order induced electric field $\tilde{E}_A^{(n)}$ and first order (of smallness in the excitation field $\hat{E}$) total electric fields $E^{(1)}=\hat{E}+\sum_A \tilde{E}_A^{(1)}$; note that $E^{(n)}=\sum_A \tilde{E}_A^{(n)}$. A derivation of the tcp compressibility rules is deferred to a later paper [3].

Consider a plasma free of externally applied magnetic fields; longitudinal and transverse* elements of the polarizability matrix are therefore uncoupled. The field quantities in Eqs.(1) and (2) are taken to be along $\underset{\sim}{k}$ so that the polarizabilities there are purely longitudinal. We begin with the derivation of the sum rules for ${}_1\alpha_A$. Upon expanding the r.h.s. denominator of the Kramers-Kronig formula,

$$\mathrm{Re}\,{}_1\alpha_A(\underset{\sim}{k}\,\omega) = -(1/\pi)\,P\int_{-\infty}^{\infty}\frac{d\omega'}{\omega-\omega'}\,\mathrm{Im}\,{}_1\alpha_A(\underset{\sim}{k}\,\omega'), \tag{3}$$

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*With respect to the wave vector $\underset{\sim}{k}$.

in the $\omega \to \infty$ limit, one obtains

$$Re\,{}_{I}\alpha_A(\underset{\sim}{k}\omega \to \infty) = -(1/\pi\omega^2)\int_{-\infty}^{\infty} d\omega'\omega'\, Im\,{}_{I}\alpha_A(\underset{\sim}{k}\omega') - \ldots \quad (4)$$

The l.h.s. can be determined from the macroscopic equation of motion,

$$(\partial/\partial t)\underset{\sim}{V}_A + \underset{\sim}{V}_A\cdot(\partial/\partial\underset{\sim}{X}_A)\underset{\sim}{V}_A = (e_A/m_A)\underset{\sim}{E}, \quad (A = \text{electrons, ions}) \quad (5)$$

valid at high frequencies where the electrons and ions behave like collections of non-interacting particles*. From (5), one obtains

$${}_{I}\alpha_A(\omega \to \infty) \approx -\omega_{pA}^2/\omega^2, \quad (6)$$

where $\omega_{pA} = (4\pi n_A e_A^2/m_A)^{\frac{1}{2}}$. Eqs. (4) and (6) then combine to give the desired conductivity sum rule,

$$\int_{-\infty}^{\infty} d\omega\, \omega\, Im\,{}_{I}\alpha_A(\underset{\sim}{k}\omega) = \pi\omega_{pA}^2 . \qquad (A = \text{electron, ion}) \quad (7)$$

Equation (7) shows that the sum rule always involves the dissipative part of ${}_{I}\alpha_A$. We note that the tcp result [7], while necessary to report, is, nevertheless, a trivial generalization of the well-known ocp conductivity rule [1]. This will be the case for all the conductivity rules reported in this paper and is not surprising in view of the fact that at high frequencies, the non-local collision terms are negligible in the electron and ion macroscopic momentum equations leaving them uncoupled and therefore identical to the ocp $\omega \to \infty$ momentum equation.

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*For the derivation of the compressibility sum rules, it is the static ($\omega = 0$) value of the polarizability which is relevant. In this case, Eq.(5) must be modified by including (i) the pressure gradient term on its l.h.s. to take account of isothermal processes and (ii) non-local collision terms on its r.h.s. [3].

If the tcp is in a constant external magnetic field $\underset{\sim}{B}_0$, then for the symmetric elements of the linear polarizability tensor ${}_1\underset{\approx}{\alpha}^A$ (denoted by ${}_1\alpha^A_{(\mu\nu)}$)* dissipation is reflected by the imaginary part, whereas for the antisymmetric elements (denoted by ${}_1\alpha^A_{[\mu\nu]}$)*, it is reflected by the real part. Thus from $\omega \to \infty$ denominator expansions of the Kramers-Kronig formulae |2|,

$$\mathrm{Re}\,{}_1\alpha^A_{(\mu\nu)}(\underset{\sim}{k}\omega) = -(1/\pi)\,P\int_{-\infty}^{\infty}\frac{d\omega'}{\omega-\omega'}\,\mathrm{Im}\,{}_1\alpha^A_{(\mu\nu)}(\underset{\sim}{k}\omega'), \tag{8}$$

$$\begin{aligned}\mathrm{Im}\,{}_1\alpha^A_{[\mu\nu]}(\underset{\sim}{k}\omega) = {} & (1/\pi)\,P\int_{-\infty}^{\infty}\frac{d\omega'}{\omega-\omega'}\,\mathrm{Re}\,{}_1\alpha^A_{[\mu\nu]}(\underset{\sim}{k}\omega') \\ & -(1/\pi\omega)\int_{-\infty}^{\infty}d\omega'\,\mathrm{Re}\,{}_1\alpha^A_{[\mu\nu]}(\underset{\sim}{k}\omega'),\end{aligned} \tag{9}$$

and the high frequency equations of motion,

$$(\partial/\partial t)\underset{\sim}{v}_A + \underset{\sim}{v}_A\cdot(\partial/\partial\underset{\sim}{x}_A)\underset{\sim}{v}_A \approx (e_A/m_A)\left[\underset{\sim}{E} + (\underset{\sim}{v}_A/c)\times(\underset{\sim}{B}+\underset{\sim}{B}_0)\right], \tag{10}$$

one obtains the tcp linear sum rules:

$$\int_{-\infty}^{\infty}d\omega\,\omega\,\mathrm{Im}\,{}_1\alpha^A_{(\mu\nu)}(\underset{\sim}{k}\omega) = \pi\omega_{pA}^2\,\delta_{(\mu\nu)}, \tag{11}$$

$$\int_{-\infty}^{\infty}d\omega\,\omega^2\,\mathrm{Re}\,{}_1\alpha^A_{[\mu\nu]}(\underset{\sim}{k}\omega) = -\pi\omega_{pA}^2\,\omega_{cA}\,\epsilon_{[\mu\nu]\alpha}\,\hat{b}_{0\alpha}, \tag{12}$$

where $\omega_{cA} = e_A B_0/m_A c$, $\epsilon_{[\mu\nu]\alpha}$ is the unit permutation pseudotensor and $\hat{b}_0$ is the unit vector along $\underset{\sim}{B}_0$.

We next derive sum rules for the quadratic polarizability in the magnetic field-free tcp. Here the relevant Kramers-Kronig formula *is*

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* In the $B_0$-system $[\underset{\sim}{k} = (k_x,0,k_z), \underset{\sim}{B}_0 = (0,0,B_0)]$ or in the $\underset{\sim}{k}$-system $[\underset{\sim}{k} = (0,0,k), \underset{\sim}{B}_0 = (B_{0x},0,B_{0z})]$, the 11,22,33,13,31 elements are the symmetric ones and the 12,21,23,32 elements are the antisymmetric ones.

$$\mathrm{Im}\ {}_2\alpha_A(\underset{\sim}{k}_1\omega_1;\underset{\sim}{k}_2\omega_2) = (1/\pi)\,P\int_{-\infty}^{\infty}\frac{d\omega_1'}{\omega_1-\omega_1'}\,\mathrm{Re}\,{}_2\alpha_A(\underset{\sim}{k}_1\omega_1';\underset{\sim}{k}_2\omega_2). \tag{13}$$

As was pointed out in Ref. [2], useful conductivity rules can be derived only for the limiting case $\omega_2\to\infty$ (or $\omega_1\to\infty$ ). In this limit, $\mathrm{Re}\ {}_2\alpha_A\ (\underset{\sim}{k}_1\omega_1;\ \underset{\sim}{k}_2\omega_2\to\infty) = -\ \mathrm{Re}\ {}_2\alpha_A\ (\underset{\sim}{k}_1 -\omega_1;\ \underset{\sim}{k}_2,\omega_2\to\infty)$ so that even frequency moments are eliminated from further consideration. Then from the asymptotic expression,

$$(\underset{\sim}{k} = \underset{\sim}{k}_1 + \underset{\sim}{k}_2\,,\ \omega = \omega_1 + \omega_2)$$

$$\mathrm{Im}\ {}_2\alpha_A(\underset{\sim}{k}_1\omega_1\to\infty;\ \underset{\sim}{k}_2\omega_2\to\infty) \approx \tag{14}$$

$$\approx \frac{e_A\,\omega_{pA}^2}{2m_A\,\omega\omega_1\omega_2}\left[\frac{k(\underset{\sim}{k}_1\cdot\underset{\sim}{k}_2)}{\omega\,k_1k_2} + \frac{k_1(\underset{\sim}{k}\cdot\underset{\sim}{k}_2)}{\omega_1\,k\,k_2} + \frac{k_2(\underset{\sim}{k}\cdot\underset{\sim}{k}_1)}{\omega_2\,k\,k_1}\right],$$

derived from Eq.(5), and the high frequency denominator expansion of (13),

$$\lim_{\omega_1\to\infty}(1/\pi)\,P\int_{-\infty}^{\infty}\frac{d\omega_1'}{\omega_1-\omega_1'}\,\mathrm{Re}\,{}_2\alpha_A\ (\underset{\sim}{k}_1\omega_1';\ \underset{\sim}{k}_2\ \omega_2\to\infty) \approx \tag{15}$$

$$\approx \lim_{\omega_1\to\infty}\left(\frac{1}{\pi\omega_1^2}\int_{-\infty}^{\infty}d\omega_1'\,\omega_1'\,\mathrm{Re}\,{}_2\alpha_A(\underset{\sim}{k}_1\omega_1';\underset{\sim}{k}_2\ \omega_2\to\infty) + \cdots\right),$$

one obtains the desired conductivity sum rule,

$$\int_{-\infty}^{\infty}d\omega_1\,\omega_1\ \mathrm{Re}\ {}_2\alpha_A(\underset{\sim}{k}_1\omega_1;\underset{\sim}{k}_2\ \omega_2\to\infty) = \frac{\pi e_A}{2m_A}\left(\frac{\omega_{pA}}{\omega_2}\right)^2\frac{k_1(\underset{\sim}{k}\cdot\underset{\sim}{k}_2)}{k\,k_2}. \tag{16}$$

Similarly, one obtains

$$\int_{-\infty}^{\infty}d\omega_2\,\omega_2\ \mathrm{Re}\ {}_2\alpha_A(\underset{\sim}{k}_1\omega_1\to\infty;\underset{\sim}{k}_2\omega_2) = \frac{\pi e_A}{2m_A}\left(\frac{\omega_{pA}^2}{\omega_1}\right)^2\frac{k_2(\underset{\sim}{k}\cdot\underset{\sim}{k}_1)}{k\,k_1}. \tag{17}$$

Unlike the compressibility sum rules, all the conductivity rules -both linear and quadratic- are completely exhausted by the Vlasov polarizabilities. Finally, we expect that the sum rules derived here will be equally valid for quantum mechanical plasmas.

This research was supported by the Swiss National Science Foundation.

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