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ABSTRACT

Using the Kubo sum rule theorem, external conductivity frequency moment sum rules are derived for the relativistic two-component plasma in a constant external magnetic field.

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Using the Kubo sum rule theorem [1], conductivity sum rules are derived for the relativistic tcp (two component plasma) in a constant external magnetic field. This is a generalization of the relativistic ocp rules derived in 1969 by Golden and Kalman [2].

Consider a plasma of $N_e = N$ electrons (each having mass m_e and charge $e_e = -e$) and $N_i = N/Z$ ions (each having mass m_i and charge $e_i = Ze$) confined in the large but bounded volume V at the temperature $\beta^{-1} = \kappa T$. Let

$$\Omega = \exp(-\beta H)/\left[\int d\Gamma \exp(-\beta H)\right] \tag{1}$$

be the macrocanonical distribution of the equilibrium system; $d\Gamma$ is an element of hypervolume in the Γ phase space spanned by the coordinates and momenta of the plasma particles and transverse radiation field oscillators. The Hamiltonian H for this system is

$$H = \sum_{A=e,ion} \sum_{i=1}^{N_A} \gamma_i^A m_A c^2 + (1/2) \sum_{A,B=e,ion} \sum_{i\neq j} \frac{e_A e_B}{|\underset{\sim}{x}_i^A - \underset{\sim}{x}_j^B|} \tag{2}$$
$$+ (1/4\pi V) \sum_{\underset{\sim}{q}} \left[(1/2) \underset{\sim}{E}_{\underset{\sim}{q}} \cdot \underset{\sim}{E}_{\underset{\sim}{q}}^* + (1/2) q^2 c^2 \underset{\sim}{A}_{\underset{\sim}{q}} \cdot \underset{\sim}{A}_{\underset{\sim}{q}}^*\right],$$

$$\gamma_i^A = \left[1 - (v_i^A)^2/c^2\right]^{-1/2}, \tag{3}$$

$$\underset{\sim}{v}_i^A = \frac{\left[\underset{\sim}{p}_i^A - (e_A/V)\sum_{\underset{\sim}{q}} \underset{\sim}{A}_{\underset{\sim}{q}} \exp(i\underset{\sim}{q}\cdot\underset{\sim}{x}_i^A) - e_A \underset{\sim}{A}_0(\underset{\sim}{x}_i^A)\right]/m_A}{\left\{1 + (m_A c)^{-2}\left[\underset{\sim}{p}_i^A - (e_A/V)\sum_{\underset{\sim}{q}} \underset{\sim}{A}_{\underset{\sim}{q}} \exp(i\underset{\sim}{q}\cdot\underset{\sim}{x}_i^A) - e_A \underset{\sim}{A}_0(\underset{\sim}{x}_i^A)\right]^2\right\}^{1/2}}, \tag{4}$$

where $\underset{\sim}{x}_i^A$, $\underset{\sim}{p}_i^A$ are the canonical coordinate and momentum of the ith particle member of type A plasma particles, $\underset{\sim}{v}_i^A$ is its velocity, and $\underset{\sim}{A}_{\underset{\sim}{q}}$, $\underset{\sim}{E}_{\underset{\sim}{q}}$ are the canonical coordinate and momentum of the $\underset{\sim}{q}$th mode radiation field oscillator; $\underset{\sim}{A}_o(\underset{\sim}{x}_i^A)$ is the vector potential (acting at the coordinate point $\underset{\sim}{x}_i^A$) due to the constant external magnetic field $\underset{\sim}{B}_o$ $(=\underset{\sim}{\nabla}_i^A \times \underset{\sim}{A}_o)$ pervading the plasma.

We turn now to the derivation of the frequency moment sum rules for the dissipative part of the external conductivity tensor $\underset{\approx}{\hat{\sigma}}^A$, the proportionality coefficient in the Ohms law relating the average induced current density response of type A plasma particles to a weak external electric field excitation. We note that for the symmetric elements of $\underset{\approx}{\hat{\sigma}}^A$ (denoted by $\hat{\sigma}^A_{(\mu\nu)}$), dissipation is reflected by the real part, whereas for the antisymmetric elements (denoted by $\hat{\sigma}^A_{[\mu\nu]}$), it is reflected by the imaginary part. The aim of this paper can now be precisely stated: we wish to determine the constants $C_{(\mu\nu)}$, $D_{[\mu\nu]}$ for the lowest order frequency moment rules*

$$\int_{-\infty}^{\infty} d\omega \, Re \, \hat{\sigma}^A_{(\mu\nu)}(\underset{\sim}{k}\omega) = C_{(\mu\nu)} \, , \int_{-\infty}^{\infty} d\omega \, \omega \, Im \, \hat{\sigma}^A_{[\mu\nu]}(\underset{\sim}{k}\omega) = D_{[\mu\nu]} .$$

The basic equation for the derivations is the fluctuation-dissipation theorem (valid for the relativistic tcp in an external magnetic field)

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*Without loss of generality, our final expressions will be valid in either the $\underset{\sim}{B}_o$- system $[\underset{\sim}{k} = (k_x,0,k_z), \underset{\sim}{B}_o = (0,0,B_o)]$ or the $\underset{\sim}{k}$-system $[\underset{\sim}{k} = (0,0,k), \underset{\sim}{B}_o = (B_{ox},0,B_{oz})]$. In either system, the 11,22,33,13,31 elements are the symmetric ones and the 12,21,23,32 elements are the antisymmetric ones.

[2],

$$\hat{\sigma}^{A}_{\nu\mu}(\underset{\sim}{k}\omega) = \beta\int_0^\infty dt\, e^{i\omega t}\left[Q^{AA}_{\mu\nu}(\underset{\sim}{k},t) + Q^{AB}_{\mu\nu}(\underset{\sim}{k},t)\right], \begin{pmatrix} A,B = e, ion; \\ B \neq A \end{pmatrix} \tag{5}$$

connecting $\underset{\approx}{\hat{\sigma}}^{A}$ to equilibrium correlations*

$$Q^{AB}_{\mu\nu}(\underset{\sim}{k},t) = (1/V) < j^{A}_{\underset{\sim}{k}\nu}(t)\, j^{B}_{-\underset{\sim}{k}\mu}(0) > \tag{6}$$

of microscopic current densities, e.g.,

$$j^{B}_{-\underset{\sim}{k}\mu}(0) = e_B \sum_{j=1}^{N_B} V^{B}_{j\mu}(t=0)\exp\left[i\underset{\sim}{k}\cdot\underset{\sim}{x}^{B}_{j}(t=0)\right]. \tag{7}$$

For the evaluation of $C_{(\mu\nu)}$ and $D_{[\mu\nu]}$, one begins by taking ω^{0} and ω^{1} frequency moments of Eq.(5). This gives

$$\int_{-\infty}^{\infty} d\omega\, \mathrm{Re}\, \hat{\sigma}^{A}_{(\nu\mu)}(\underset{\sim}{k}\omega) = \pi\beta\, Q^{AA}_{(\mu\nu)}(\underset{\sim}{k}, t=0), \tag{8}$$

$$\int_{-\infty}^{\infty} d\omega\, \omega\, \mathrm{Im}\, \hat{\sigma}^{A}_{[\nu\mu]}(\underset{\sim}{k}\omega) = \pi\beta\, \dot{Q}^{AA}_{[\mu\nu]}(\underset{\sim}{k}, t=0). \tag{9}$$

The r.h.s. of Eqs.(8) and (9) were obtained by use of the parity rules [2]

$$Q^{AA}_{(\mu\nu)}(\underset{\sim}{k},t) = Q^{AA}_{(\mu\nu)}(\underset{\sim}{k},-t), \quad Q^{AA}_{[\mu\nu]}(\underset{\sim}{k},t) = -Q^{AA}_{[\mu\nu]}(\underset{\sim}{k},-t), \tag{10a,b}$$

and by observing that $Q^{AB}_{(\mu\nu)}(\underset{\sim}{k},t=0) = 0$, $\dot{Q}^{AB}_{[\mu\nu]}(\underset{\sim}{k},t=0) = 0$, since for $B \neq A$ these amount to odd velocity averages which must vanish in view of the isotropy of the velocity distributions.

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*The correlation braces $<\ldots>$ denote averaging over the equilibrium ensemble, viz: $<\ldots> = \int d\Gamma\, \Omega\, (\cdots)$.

From Eqs.(6), (7), and (1),

$$Q^{AA}_{(\mu\nu)}(\underset{\sim}{k}, t=0) = (e_A^2/V) \sum_{i,j} \langle v_{i\mu}^A v_{j\nu}^A \exp[i\underset{\sim}{k}\cdot(\underset{\sim}{x}_i^A - \underset{\sim}{x}_j^A)] \rangle =$$

$$= (e_A^2/V\beta) \sum_{i,j} \int d\Gamma\, \Omega\, (\partial v_{i\mu}^A/\partial p_{j\nu}^A) \exp[i\underset{\sim}{k}\cdot(\underset{\sim}{x}_i^A - \underset{\sim}{x}_j^A)], \qquad (11)$$

and from Eq.(4),

$$\partial v_{i\mu}^A/\partial p_{j\nu}^A = (1/\gamma_i^A m_A)\, \delta_{ij} [\delta_{\mu\nu} - (v_{i\mu}^A v_{i\nu}^A/c^2)], \qquad (12)$$

so that

$$Q^{AA}_{(\mu\nu)}(\underset{\sim}{k}, t=0) = [e_A^2/(\beta m_A V)] \sum_i \int d\Gamma\, \Omega\, [\delta_{(\mu\nu)} - (v_{i\mu}^A v_{i\nu}^A/c^2)] (\gamma_i^A)^{-1} =$$

$$= [\omega_{pA}^2/(4\pi\beta)] \langle \gamma_A^{-1}(1 - v_A^2/3c^2) \rangle \delta_{(\mu\nu)}, \qquad (13)$$

where $\omega_{pA}^2 = 4\pi n_A e_A^2/m_A$ is the plasma frequency of type A particles. It is convenient to express the relativistic correction in terms of so-called G functions defined in Ref. [2] as

$$G_n(\beta mc^2) = \frac{(\beta mc^2) \int_1^\infty d\gamma\, \gamma^n (\gamma^2 - 1)^{1/2} \exp(-\beta mc^2 \gamma)}{K_2(\beta mc^2)}, \qquad (14)$$

$K_2(\beta mc^2)$ being the modified Bessel function of order two. One obtains

$$\langle \gamma_A^{-1}(1 - v_A^2/3c^2) \rangle = (2/3)\, G_0(\beta m_A c^2) + (1/3)\, G_{-2}(\beta m_A c^2) \qquad (15)$$

for the relativistic correction.

The calculation of $\dot{Q}^{AA}_{[\mu\nu]}(\underset{\sim}{k}, t = 0)$ is somewhat more involved:

$$\dot{Q}^{AA}_{[\mu\nu]}(\underset{\sim}{k}, t=0) = (1/V) < (d/dt) j^A_{\underset{\sim}{k}\nu}(t) j^A_{-\underset{\sim}{k}\mu}(0) >_{t=0} =$$

$$= (e_A^2/V) \sum_{i,j} \Big\{ < v^A_{i\mu}(0)\, \dot{v}^A_{j\nu}(0) \exp[i \underset{\sim}{k}\cdot(\underset{\sim}{x}^A_i(0) - \underset{\sim}{x}^A_j(0))] > \qquad (16)$$

$$- i k_\lambda < v^A_{i\mu}(0)\, v^A_{j\nu}(0)\, v^A_{j\lambda}(0) \exp[i \underset{\sim}{k}\cdot(\underset{\sim}{x}^A_i(0) - \underset{\sim}{x}^A_j(0))] > \Big\}.$$

The second r.h.s. correlation of microscopic velocities vanishes because of its odd velocity parity. The first r.h.s. correlation can be calculated by use of the microscopic equation of motion (in the Coulomb gauge)

$$\dot{\underset{\sim}{v}}^A_j = [e_A/(\gamma^A_j m_A)][\underset{\approx}{1} - (\underset{\sim}{v}^A_j \underset{\sim}{v}^A_j/c^2)]\cdot \Big\{ \underset{\sim}{E}^{\ell}(|\underset{\sim}{x}^A_1 - \underset{\sim}{x}^A_j|, \ldots, |\underset{\sim}{x}^B_{N_B} - \underset{\sim}{x}^A_j|)$$

$$- (1/V) \Sigma_{\underset{\sim}{q}} \dot{\underset{\sim}{A}}_{\underset{\sim}{q}} \exp(i \underset{\sim}{q}\cdot \underset{\sim}{x}^A_j) + \qquad (17)$$

$$+ (1/V)(\underset{\sim}{v}^A_j/c) \times \Sigma_{\underset{\sim}{q}}\, i\underset{\sim}{q} \times \underset{\sim}{A}_{\underset{\sim}{q}} \exp(i \underset{\sim}{q}\cdot\underset{\sim}{x}^A_j) + (\underset{\sim}{v}^A_j/c) \times \underset{\sim}{B}_0 \Big\},$$

where $\underset{\sim}{E}^{\ell}$ is the electrostatic field component. Adopting for the moment the notion of kinetic $(x, v, A, \dot{A})$ rather than canonical (x, p, A, E) coordinates, we see by inspection of Eqs.(16) and (17) that only the last r.h.s. term in (17) contributes to $\dot{Q}^{AA}_{[\mu\nu]}(\underset{\sim}{k}, t = 0)$, i.e.,

$$\dot{Q}^{AA}_{[\mu\nu]}(\underset{\sim}{k}, t=0) = (e_A^2/V)\, \omega_{cA}\, \epsilon_{\alpha\beta\gamma}\, \hat{b}_{o\gamma} \sum_{i,j} < (\gamma^A_j)^{-1} [\delta_{\nu\alpha} - (v^A_{j\nu} v^A_{j\alpha}/c^2)]\cdot$$

$$\cdot\, v^A_{j\beta}\, v^A_{i\mu} \exp[i\underset{\sim}{k}\cdot(\underset{\sim}{x}^A_i - \underset{\sim}{x}^A_j)] > =$$

$$= (e_A^2/V)\, \omega_{cA}\, \epsilon_{\nu\beta\gamma}\, \hat{b}_{o\gamma} \sum_i < (\gamma^A_i)^{-1} v^A_{i\beta}\, v^A_{i\mu} >, \qquad (18)$$

where $\omega_{cA} = e_A B_o/m_A c$ is the cyclotron frequency, $\epsilon_{\nu\beta\gamma}$ is the unit permutation pseudo tensor component, and $\hat{b}_o$ is the unit vector in the direction of $\underset{\sim}{B}_o$. Returning to canonical formalism and taking account of

Eq.(1), Eq.(18) becomes

$$\dot{Q}^{AA}_{[\mu\nu]}(\underset{\sim}{k}, t=0) = (e_A^2/\beta V)\,\omega_{cA}\,\epsilon_{\nu\beta\gamma}\hat{b}_{o\gamma}\Sigma_i \int d\Gamma\,\Omega(\partial/\partial p^A_{i\mu})[(\gamma_i^A)^{-1}V^A_{i\beta}], \qquad (19)$$

which, in virtue of Eq.(4), is

$$= (1/4\pi\beta)\,\omega_{pA}^2\,\omega_{cA}\,\epsilon_{[\nu\mu]\alpha}\hat{b}_{o\alpha} < \gamma_A^{-2}(1-2V_A^2/3c^2) > .$$

Here we note that the relativistic correction, expressed in terms of the G functions, is

$$< \gamma_A^{-2}(1-2V_A^2/3c^2) > = (1/3)G_{-1}(\beta m_A c^2) + + (2/3)\,G_{-3}(\beta m_A c^2). \qquad (20)$$

From Eqs.(8),(9),(13),(15),(19), and (20), the desired conductivity sum rules can now be stated. They are:

$$C_{(\mu\nu)} = (1/4)\,\omega_{pA}^2\,[(2/3)G_0(\beta m_A c^2) + (1/3)G_{-2}(\beta m_A c^2)]\,\delta_{(\mu\nu)}, \qquad (21)$$

$$D_{[\mu\nu]} = (1/4)\,\omega_{pA}^2\,\omega_{cA}[(1/3)G_{-1}(\beta m_A c^2) + (2/3)G_{-3}(\beta m_A c^2)]\,\epsilon_{[\mu\nu]\alpha}\hat{b}_{o\alpha}. \qquad (22)$$

When $\underset{\sim}{B}_o = 0$, the conductivity tensor is diagonal; its longitudinal projection is given by

$$(k_\mu k_\nu/k^2)\,\mathrm{Re}\,\hat{\sigma}^{AA}_{(\mu\nu)}(\underset{\sim}{k}\omega) = (\omega/4\pi)\,\mathrm{Im}\,\frac{\alpha_A(\underset{\sim}{k}\omega)}{\epsilon(\underset{\sim}{k}\omega)}. \qquad (23)$$

It is at once apparent that the so-called f-sum rules [2] for the relativistic tcp must be

$$\int_{-\infty}^{\infty} d\omega\,\omega\,\mathrm{Im}\,\frac{\alpha_A(\underset{\sim}{k}\omega)}{\epsilon(\underset{\sim}{k}\omega)} = \pi\omega_{pA}^2[(2/3)G_0(\beta m_A c^2) + + (1/3)G_{-2}(\beta m_A c^2)]. \qquad (24)$$

Finally, it is interesting to note that, starting from the correlation functions for the non-relativistic tcp in an external magnetic field,

$$Q^{AA}_{(\mu\nu)}(\underset{\sim}{k}, t=o) = (1/4\pi\beta)\,\omega^2_{pA}\,\delta_{(\mu\nu)}\,, \tag{25}$$

$$\dot{Q}^{AA}_{[\mu\nu]}(\underset{\sim}{k}, t=o) = (1/4\pi\beta)\,\omega^2_{pA}\,\omega_{cA}\,\epsilon_{[\nu\mu]\alpha}\,\hat{b}_{o\alpha}\,, \tag{26}$$

one could partially reconstruct their relativistic counterparts simply by replacing ω^2_{pA} and ω_{cA} in (25) and (26) by $\tilde{\omega}^2_{pA} = 4\pi n_A e^2_A/\tilde{m}_A$, $\tilde{\omega}_{cA} = e_A B_o/c\tilde{m}_A$, where $\tilde{m}_A = \gamma_A m_A$. What would be, of course, lacking in this naive approach are the $\left[1 - (v^2_A/3c^2)\right]$, $\left[1 - (2\,v^2_A/3c^2)\right]$ factors appearing in the correct expressions Eqs.(13) and (19).

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