ANOMALOUS SPATIAL DEPLETION OF LOWER-HYBRID
CONE THROUGH PARAMETRIC DECAY

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ABSTRACT

Analytic solutions for the envelope structures of two nonlinearly coupled lower-hybrid waves propagating along their respective cone trajectories are obtained. The coupling occurs through induced scatterings by particles. The results indicate anomalous spatial pump depletion. Implications to plasma heating are also discussed.

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1. INTRODUCTION

Recently, there has been active theoretical as well as experimental efforts in studying the r.f. plasma heating scheme using waves in the lower-hybrid frequency range. It is now widely believed that nonlinear effects, such as pump self-induced filamentation and parametric decays, play important roles in this heating scheme [1,2]. One of the major concerns, which motivates the present work, is then how these nonlinear processes affect the transport of r.f. energy into the interior of a plasma. The nonlinear process considered here is the decay instability due to induced scatterings by particles; i.e., nonlinear electron and/or ion Landau dampings [3].

In Section 2, we first describe the theoretical model and approach used in this work. We then derive the set of nonlinearly coupled equations in terms of the action variables. The solutions to these equations are presented in Section 3. Section 4 contains a summary of the theoretical results as well as a discussion on the implications of these results to the plasma heating process.

2. THE NONLINEARLY COUPLED EQUATIONS

The plasma is assumed to be spatially homogeneous and uniformly magnetized with $B = B_0 z$. In the steady state, we assume that the electrostatic potential consists of two parts, $\phi_0(x, t)$ and $\phi_1(x, t)$, oscillating at,
respectively, frequencies $\omega_0$ and $\omega_1$; i.e.,

$$\phi_j(x,t) = \tilde{\phi}_j(x) \exp(-i\omega_j t) + \text{c.c.}; \ j = 0, 1. \quad (1)$$

Here, $\omega_0$ and $\omega_1$ are of the order of lower-hybrid frequency, $\omega_{lh} = \omega_{pi}(1 + \omega_{pe}^2/\omega_{ce}^2)^{-\frac{1}{2}}$ and $\omega_{pe}^2 \sim \omega_{ce}^2$. The notations used here have their standard meanings. $\phi_0$ and $\phi_1$ are coupled to each other through a low-frequency mode at frequency $\omega_s = \omega_0 - \omega_1 > 0$. Thus, $\phi_0$ corresponds to the pump wave excited by an external structure and $\phi_1$ is the daughter wave excited by the pump wave $\phi_0$ through the decay process. The dominant coupling between $\phi_0$ and $\phi_1$ comes from the electron density perturbation of the $\omega_s$ mode. Let

$$\hat{n}_{es}(x,t) = \tilde{\hat{n}}_{es}(x) \exp(-i\omega_s t) + \text{c.c.} \quad (2)$$

It is then straightforward to derive the following set of coupled equations for $\tilde{\phi}_0$ and $\tilde{\phi}_1$:

$$\left( \nabla_1 \cdot \epsilon_{10} \nabla_1 - \nabla_3 \epsilon_{30} \nabla_3 \right) \tilde{\phi}_0 = -i\frac{4\pi e}{\omega_0} \nabla \cdot (\tilde{\hat{n}}_{es} \Upsilon_1), \quad (3)$$

$$\left( \nabla_1 \cdot \epsilon_{11} \nabla_1 - \nabla_3 \epsilon_{31} \nabla_3 \right) \tilde{\phi}_1 = -i\frac{4\pi e}{\omega_1} \nabla \cdot (\tilde{\hat{n}}_{es}^* \Upsilon_0); \quad (4)$$

where with $j = 0, 1$

$$\epsilon_{1j} = 1 + \frac{\omega_{pe}^2}{\omega_{ce}^2} - \frac{\omega_{pi}^2}{\omega_j^2}, \quad (5)$$
\[
\xi_{3j} = \frac{\omega_{pe}^2}{\omega_j^2} = 1 \approx \frac{\omega_{pe}^2}{\omega_j^2},
\]

(6)

and \( u_j \) is mainly due to the electron \( \nabla \times B \) drift,

\[
u_j \sim - \frac{c}{B} (\nabla_\perp \vec{\phi}_j \times \vec{e}_3).
\]

(7)

Also, we note that \( \nabla_\perp = \frac{2}{\partial x} \xi_x + \frac{2}{\partial y} \xi_y \) and \( \nabla_\parallel = \frac{2}{\partial y} \).

For toroidal devices, \( x, y \) and \( z \) correspond to the (minor) radial, poloidal and toroidal directions, respectively. Furthermore, in deriving Eqs. (3) and (4) we have ignored the thermal effects; which is justified so long as

\[|v_{i_e}^2|, |v_{e}^2| \ll |\omega_j^2|; |v_{e}^2| \ll |\omega_{ce}^2| \text{ and } \omega_j \text{ is not close to } \omega_{1h} .
\]

For regions near the plasma surface, these conditions are easily satisfied in typical experiments.

As to the low-frequency \( \omega_s \) mode, the dominant coupling is due to the parallel ponderomotive force, \(-m_e (u \cdot \vec{v}) u_z\), produced by \( \Phi_0 \) and \( \Phi_1 \); which acts on the electrons along \( \vec{B} \). Denoting the corresponding potential be \( \vec{\Psi}_p \); i.e.,

\[\nabla_\parallel \vec{\Psi}_p = -\left(\frac{m_e}{e}\right) \left[ (\xi_0 \cdot \vec{v}) u_j^* + (\xi_1 \cdot \vec{v}) u_{j0} \right],
\]

(8)

we obtain, using Eq. (7) and noting \( u_{zj} = (ie/m_e \omega_j) \nabla_\parallel \vec{\Phi}_j \),

\[\vec{\Psi}_p (x) = -\frac{i c}{B \omega_0} \left[ (\nabla_\perp \vec{\Phi}_1^* \times \nabla_\perp \vec{\Phi}_0) \cdot \vec{e}_3 \right].
\]

(9)
Because $\omega_s \ll \omega_{ce}$, electron dynamics is mainly along $\tilde{z}$. We then can use $\tilde{\psi}_p$ along with the self-consistent potential $\tilde{\Phi}_s$ in the electron one-dimensional (in $\tilde{z}$ direction) Vlasov equation to obtain the perturbed electron distribution function $\tilde{f}_{es}$;

$$(-i\omega_s + \nabla_{\tilde{z}} \cdot \nabla_{\tilde{z}}) \tilde{f}_{es}(\tilde{x}, \nabla_{\tilde{z}}) = -\frac{e}{m_e} \frac{df_{oe}}{d\nabla_{\tilde{z}}} \nabla_{\tilde{z}} (\tilde{\Phi}_s + \tilde{\psi}_p).$$ (10)

As to the effectively unmagnetized ions ($\omega_i^2 \gg \omega_{ci}^2$), they only respond linearly to $\tilde{\Phi}_s$. Hence, we have

$$(-i\omega_s + \nabla \cdot \nabla) \tilde{f}_{is}(\tilde{x}, \nabla) = \frac{e}{m_i} \frac{df_{io}}{d\nabla} \cdot \nabla \tilde{\Phi}_s. \quad (11)$$

Poisson's equation

$$-\nabla^2 \tilde{\Phi}_s = 4\pi e (\tilde{n}_{is} - \tilde{n}_{es}) \quad (12)$$

then relates $\tilde{\Phi}_s$ and, hence, $\tilde{n}_{es} = n_0 \int dv \tilde{f}_{es}$ to $\psi_p$. Equations (3), (4), (9), (10), (11) and (12), thus, provide a complete description of the nonlinearly coupled system.

To proceed further analytically, we make the following WKB approximation

$$\tilde{\Phi}_j(\tilde{x}) = \tilde{\Phi}_j(\epsilon x, \epsilon \tilde{z}) \exp \left( i \frac{\epsilon}{h} \tilde{z} \cdot \tilde{x} \right); \quad j = 0, 1 \quad (13)$$

and

$$[\tilde{n}_{es}, \tilde{\psi}_p, \tilde{\Phi}_s](\tilde{x}) = [N_{es}, \psi_p, \tilde{\Phi}_s](\epsilon x, \epsilon \tilde{z}) \exp(\epsilon \tilde{z} \cdot \tilde{x}). \quad (14)$$
Here, \( \varepsilon \ll 1 \), and 

\[
N = N_0 \varepsilon_x + N_z \varepsilon_z, \quad k = k_{x1} \varepsilon_x + k_{y1} \varepsilon_y + k_{z1} \varepsilon_z, \quad N_s = N_0 - k_1
\]

and \((\omega_j, k_j)\) satisfies the lower-hybrid dispersion relation

\[
\omega_j = \omega_{lh} \left( 1 + \frac{k_{ji}^2}{k_{j1}^2} \frac{m_i}{m_e} \right)^{1/2} ; \ j = 0, 1
\]

(15)

Note, in Eqs. (13) and (14), because we are mainly interested in the spatial structures in the x-z plane, we have assumed the envelopes to be independent of y. Equations (3) and (4) then become

\[
\left[ k_x \varepsilon_x \frac{\partial}{\partial x} - k_y \varepsilon_y \frac{\partial}{\partial y} \right] \Phi_0 = \lambda N_e \varepsilon_1 \Phi_0
\]

(16)

\[
\left[ k_x \varepsilon_x \frac{\partial}{\partial x} - k_y \varepsilon_y \frac{\partial}{\partial y} \right] \Phi_1 = -\lambda N_e^* \Phi_0
\]

(17)

where

\[
\lambda = \frac{2 \pi e c}{B \omega_0} \left[ (k_1 \times k_0) \cdot \varepsilon_x \right].
\]

(18)

For the induced scattering processes considered here, \( \omega_s \sim k_v v_i \) or \( k z v_e \) and the low-frequency mode is non-resonant. Thus, we can replace \( \nabla \) by \( ik_s \) in Eqs. (10), (11) and (12). We then obtain

\[
4 \pi e N_e = \beta \Psi_p
\]

(19)

where

\[
\beta(\omega_s, k_s) = \frac{k_s^2 \lambda e (1 + \chi_i)}{\varepsilon_e}
\]

(20)
\[ \varepsilon_5 = 1 + \chi_e + \chi_i , \]
\[ \chi_e = - \frac{\omega^2_{pe}}{k_s^2} \int d\nu \frac{d\nu}{\nu - \omega_s / k_s} \frac{df_{pe} / d\nu}{\nu} , \]
\[ \chi_i = - \frac{\omega^2_{pi}}{k_s^2} \int d\nu \frac{d\nu}{\nu - \omega_s / k_s} \frac{df_{pi} / d\nu}{\nu} , \]

and, from Eq.(9),
\[ \psi_p = - \frac{i c}{k_s} \left[ \left( \xi_1 \times \xi_0 \right) \cdot \varepsilon_3 J \right] \overline{\Phi_0} \overline{\Phi_1}^* . \]

Substituting Eq.(19) into Eqs.(16) and (17) and expressing in terms of the action variables \[ I_j = k_j^2 \left| \Phi_j \right|^2 / \omega_j \] with \( j = 0,1 \), we obtain
\[ \nabla \cdot \left( \nabla g_0 I_0 \right) = - \alpha I_0 I_1 , \]
\[ \nabla \cdot \left( \nabla g_1 I_1 \right) = \alpha I_0 I_1 , \]

where
\[ \alpha = \frac{1}{2} \left( \frac{k_{ys}}{k_{1B}} \right)^2 (1 + \frac{\omega^2_{pe}}{\omega^2_{ce}})^{-1} \text{Im} \beta > 0 , \]

and
\[ \nabla g_j = \left( \frac{\partial \omega}{\partial k_x} \xi_x + \frac{\partial \omega}{\partial k_y} \xi_y \right) , \quad j = 0,1 \]

is the group velocity. Note Eqs.(25) and (26) exhibit the conservation of action flux; i.e., \[ \sum_{j=0,1} \nabla \cdot \left( \nabla g_j I_j \right) = 0 . \] Solutions to this set of non-linearly coupled equations are discussed in the next section.
3. SOlutions of the Coupled Equations

Equations (25) and (26) can be written as

\[ \frac{\partial G_0}{\partial x} + C_0 \frac{\partial G_0}{\partial \delta} = -G_0 G_1, \]  \hspace{1cm} (25)'

\[ \frac{\partial G_1}{\partial x} + C_1 \frac{\partial G_1}{\partial \delta} = G_0 G_1. \]  \hspace{1cm} (26)'

Here, \( C_0 = \alpha I_0 / V \) \( g x_1 \), \( C_1 = \alpha I_1 / V \) \( g x_0 \) and \( C_j = (V - V_j) / V \) \( g x_j \) with \( j = 0, 1 \). Equations (25)' and (26)' indicate that if nonlinear coupling is absent, \( G_0(I_0) \) and \( G_1(I_1) \) just propagate along their respective cone trajectories; i.e., \( G_0 \) and \( G_1 \) are only functions of, respectively, \( z - C_0x \) and \( z - C_1x \). Nonlinear couplings, thus, make \( G_0 \) and \( G_1 \) functions of both \( z - C_0x \) and \( z - C_1x \). There exists a class of exact solutions to Eqs.(25)' and (26)'

and is given by \([4]\)

\[ G_0(\tau, \xi) = \frac{T(\tau)}{Z(\xi) - T(\tau)}, \]  \hspace{1cm} (29)

\[ G_1(\tau, \xi) = \frac{Z(\xi)}{Z(\xi) - T(\tau)}, \]  \hspace{1cm} (30)

where

\[ \tau = -(\xi - C_0x) / V, \]  \hspace{1cm} (31)

\[ \xi = (\xi - C_1x) / V. \]  \hspace{1cm} (32)
and

\[ V = C_0 - C_1 \]  \hspace{1cm} (33)

Subscripts \( T \) and \( Z \) denote derivatives. Formally, \( T \) and \( Z \) are two arbitrary, differentiable functions; which, however, are uniquely defined by the boundary conditions at \( x = 0 \). Denoting \( G_0(x = 0, z) = \overline{G}_0(z) \) and \( G_1(x = 0, z) = \overline{G}_1(z) \), we then have

\[ T(t) = -\frac{1}{2} \frac{1}{V} \int_0^t ds \overline{G}_0(s) \exp \left[ \int_0^s dr G_t(r)/V \right] , \]  \hspace{1cm} (34)

\[ Z(s) = \frac{1}{2} + \frac{1}{V} \int_0^s ds \overline{G}_1(s) \exp \left[ \int_0^s dr G_t(r)/V \right] , \]  \hspace{1cm} (35)

with \( G_t(z) = \overline{G}_0(z) + \overline{G}_1(z) \).

To illuminate the physical meanings of the solutions, let us take the following simple boundary conditions

\[ \overline{G}_0(z) = \begin{cases} A_0, & |z| \leq \alpha \\ 0, & |z| > \alpha \end{cases} \]  \hspace{1cm} (36)

and, similarly,

\[ \overline{G}_1(z) = \begin{cases} A_1, & |z| \leq \alpha \\ 0, & |z| > \alpha \end{cases} \]  \hspace{1cm} (37)
Let $A_t = A_0 + A_1$, the solution then is;

for $|z - C_0 x|, |z - C_1 x| \leq a$,

$$G_0 (\tau, \xi) = \frac{A_t A_0}{A_0 + A_1 \exp(A_t x)} ,$$  \hspace{1cm} (38)

$$G_1 (\tau, \xi) = \frac{A_t A_1 \exp(A_t x)}{A_0 + A_1 \exp(A_t x)} ;$$ \hspace{1cm} (39)

for $|z - C_0 x| \leq a, |z - C_1 x| > a$,

$$G_0 (\tau, \xi) = \frac{A_t A_0}{A_0 + A_1 \exp[A_t (\tau + a/v)]} ,$$  \hspace{1cm} (40)

$$G_1 (\tau, \xi) = 0 ;$$ \hspace{1cm} (41)

and for $|z - C_0 x| > a, |z - C_1 x| \leq a$,

$$G_0 (\tau, \xi) = 0 ,$$ \hspace{1cm} (42)

$$G_1 (\tau, \xi) = \frac{A_t A_1 \exp[A_t (\xi + a/v)]}{A_0 + A_1 \exp[A_t (\xi + a/v)]} .$$ \hspace{1cm} (43)

The above solution, thus, suggests a typical scale length of pump depletion (or growth of the decay wave) in $x$ given by

$$\chi_d = 1/A_t = [\alpha(I_{00}/v_{g1} + I_{10}/v_{g0})]^{-1} .$$ \hspace{1cm} (44)
Here, $I_{j0}$ is the value of $I_j$ at $x = 0$ for $j = 0, 1$. Note, from the definition of $\alpha$ in Eq. (27), $x_d$ has a simple physical interpretation; i.e., $x_d = \frac{\text{effective group velocity in } x}{\text{effective parametric growth rate}}$. To estimate $x_d$, we assume $v_{gx0} \sim v_{gx1}$ and $I_{00} \geq I_{10}$, then $x_d$ becomes approximately, using Eq. (27),

$$x_d \approx \frac{\omega_0 v_{gx0}}{\alpha |E_0|^{2/3}}$$

$$\approx \left| \frac{cE_0}{B_c \alpha} \right|^{2/3} \left[ \frac{2}{n_{g0}} \left( \frac{\omega_0^2}{\omega_{1h}^2} - 1 \right)^{3/2} \frac{c/\omega_{1h}}{\omega_0/\omega_{1h}} \left( \frac{m_e}{m_i} \right)^{1/2} \right]. \quad (45)$$

Here, $|E_0| = |k_0 E_0|$, $n_{g0} = C k_{z0}/\omega_0$, $C_s$ is the ion-acoustic speed and we have assumed $\omega_s \approx k_{z_0} v_e$ or $k_{z_1} v_i$ for maximum coupling. For typical tokamak experiments, $n_{z0} \approx 2-3$, $\omega_0/\omega_{1h} \sim 3$ and $\omega_{1h} \sim 10^{10}$, we have

$$x_d \approx \left| \frac{cE_0}{B_c \alpha} \right|^{2/3} \left( cm \right). \quad (46)$$

Currently, lower-hybrid heating experiments are running with $\left| \frac{cE_0}{B_c \alpha} \right|^{2/3} \lesssim 0(1)$, thus, the depletion length is in the centimeter range. The anomalous pump depletion, however, does not occur until the pump wave penetrates a typical distance $x_p$ into the plasma; which is given by

$$x_p = x_d \ln \left( \frac{I_{00} v_{gx0}}{I_{10} v_{gx1}} \right). \quad (47)$$

Thus, $x_p$ depends on the wave intensity of the decay wave, $I_{10}$. To estimate $(I_{10} / I_{00})$, we note that the results of both theoretical [5] as well as numerical [6] investigations on the nonlinear saturation of
the decay instability has shown that, in general, $I_{10} \sim I_{00}$ at saturation such that the parametric growth rate is balanced by the linear and nonlinear damping rates. $x_p$ is then estimated to be

$$x_p \sim O(x_d).$$

(48)

The above estimates show that for typical tokamak parameters and lower-hybrid wave heating experiments with $| CE_0/BC_s|^2 \lesssim 0(1)$, the pump penetration and depletion lengths in $x$(the minor radius) are in the centimeter range. As pointed out to us by F.W. Perkins, the most pessimistic observation is that the scaling law (which, from Eq.(45), depends mainly on density and electron temperature) does not improve with the size of the device. It, therefore, indicates that for present (e.g. ATC and PLT) and future generations of tokamaks, nonlinear decay processes will anomalously deplete the pump wave energy near the surface of the plasma.

4. SUMMARY AND DISCUSSION

In this paper, we consider the effects of parametric decays on the transport of lower-hybrid pump wave energy into the interior of a plasma. Using the WKB approximation, a set of equations describing two nonlinearly coupled lower-hybrid waves propagating along their respective cone trajectories are derived in terms of the action variables. The two waves correspond to, respectively, the pump and the decay waves. The nonlinear
couplings considered here are due to nonlinear electron and/or ion Landau
dampings. Exact solutions are then constructed from the boundary condi-
tions. Using simple but not totally unrealistic boundary conditions,
scale lengths for pump depletion and penetration are obtained; which
for typical tokamak parameters and pump power \( \left| \frac{C_E}{B_C} \right|^2 \lesssim 0(1) \), are
of the order of centimeters. The results, thus, indicate that for present
and future large-scale tokamaks, the lower-hybrid pump wave tend to
anomalously depletes its energy close to the plasma surface.

We now briefly discuss the implications of the above results to the
plasma heating process. We note that the depleted pump wave energy goes
into the decay wave, which is parametrically amplified. However, due to
the fact that the decay wave has a lower frequency and generally a finite
\( k_y \) component, its cone trajectory in the x-z plane is closer to the
plasma surface than that of the pump wave; i.e., \( C_1 > C_0 \) using the
notation of Eqs.(25)' and (26)'. This observation suggests that the decay
wave will have further problems in penetrating into the plasma and will
deposit most of its energy near the surface; which implies, therefore,
anomalous heating at the plasma surface.

Finally, we remark that the analysis presented here can be extended to
include the effects due to plasma inhomogeneities. However, since the
scale lengths due to nonlinear effects are shown to be of the order of
centimeters and are generally comparable or shorter than those associ-
ted with inhomogeneities, the essential features of the results obtained
here, therefore, are expected to remain valid.
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REFERENCES


