July 1976 LRP 108/76

# PARAMETRIC EXCITATION OF "KINETIC" ALFVÈN WAVES BY WHISTLER WAVES

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## ABSTRACT

It is shown that a whistler wave can parametrically decay into an another whistler wave plus a kinetic Alfvèn wave. The parametric coupling occurs due to the electrostatic properties of the kinetic Alfvèn wave. Corresponding growth rate and threshold condition are obtained.

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## 1. INTRODUCTION

In recent years, parametric instabilities have received intensive studies due to their important roles in rf wave plasma heating schemes, laser plasma interactions and plasma turbulence theories. In this work, we present a new parametric process; in which a whistler wave decays into an another whistler wave and a kinetic Alfvèn wave (Hasegawa and Chen, 1975). Here, the parametric coupling occurs because, due to finite ion Larmor radius and electron inertia, the kinetic Alfvèn wave has an electrostatic component and, hence, associated density perturbations. We note that one possible application of this decay process is in the magnetosphere; where ULF (shear Alfvèn) oscillations of the earth's magnetic field lines can be excited by either artificially triggered or naturally existing whistler waves (e.g. VLF emissions).

In Section 2, we present the theoretical model and the basic set of equations describing parametric couplings among the waves. The dispersion relation is then derived in Section 3 and analyzed for the resonant decay instability. Section 4 contains the final conclusions.

## 2. THEORY OF PARAMETRIC COUPLINGS

The plasma is assumed to be infinite, spatially uniform with a small but finite $\beta$ value; $1 \gg \beta \gg m_e/m_i$. Here, $\beta = 8\pi n_0(T_e + T_i)/B^2$. $n_0$ is the plasma density, $T_e$ and $T_i$ are, respectively, electron and ion temperatures, and $\underset{\sim}{B} = B \underset{\sim}{e}_z$ is the static confining magnetic field. The pump field, $\underset{\sim}{E}_0(\underset{\sim}{x},t)$, is itself a self-consistent whistler wave; i.e.,

$$\underset{\sim}{E}_0(\underset{\sim}{x},t) = \underset{\sim}{E}_0 \exp[i(\underset{\sim}{k}_0 \cdot \underset{\sim}{x} - \omega_0 t)] + \text{c.c.} \quad (1)$$

Here, $\underset{\sim}{k}_0 = k_{0x} \underset{\sim}{e}_x + k_{0z} \underset{\sim}{e}_z$, $\omega_0 = c^2 k_0 k_{0z} \Omega_e/\omega_{pe}^2$, $\omega_{pi}^2 \ll \omega_0^2 \ll \Omega_e^2, \omega_{pe}^2$ and $|k_z/k|_0^2 \gg |\omega_0/\Omega_e|^2$. The notations here are standard. Subscript 0 denotes quantities associated with the pump. Note also that $(E_y/E_x)_0(\pm\omega_0) = \pm i |k_z/k|_0 \equiv \pm i\beta_0$, and $|E_z/E_x|_0 = |c^2 k_z k_x/\omega_{pe}^2|_0 \ll 1$.

To consider the parametric couplings, we assume $|\underset{\sim}{E}_0|$ to be sufficiently weak so that only interactions up to $O(|\underset{\sim}{E}_0|^2)$ need to be kept. That is, we only consider interactions among the pump wave $(\pm\omega_0, \pm\underset{\sim}{k}_0)$, the Stoke $(\omega - \omega_0, \underset{\sim}{k} - \underset{\sim}{k}_0) \equiv (\omega_-, \underset{\sim}{k}_-)$, the Anti-Stoke $(\omega + \omega_0, \underset{\sim}{k} + \underset{\sim}{k}_0) \equiv (\omega_+, \underset{\sim}{k}_+)$ and the low-frequency kinetic Alfvèn wave $(\omega, \underset{\sim}{k})$. Here, $|\omega| \ll |\omega_0|$ and the sidebands, $(\omega_\pm, \underset{\sim}{k}_\pm)$, are whistler waves.

The wave equations of the two whistler sidebands $(\omega_\pm, \underset{\sim}{k}_\pm)$ in the coordinates $(x_\pm, y_\pm, z)$ defined by $(\underset{\sim}{e}_x)_\pm = (\underset{\sim}{k}_\perp/k_\perp)_\pm$ and $(\underset{\sim}{e}_y)_\pm = \underset{\sim}{e}_z \times (\underset{\sim}{e}_x)_\pm$, can be written as

$$\begin{bmatrix} -n_z^2 & K_{xy} & n_x n_z \\ -K_{xy} & -n^2 & 0 \\ n_z n_x & 0 & -(n_x^2 + K_{zz}) \end{bmatrix}_{\pm} \cdot \underset{\sim}{E}_{\pm} = \mp \frac{i4\pi}{\omega_0} \underset{\sim}{j}_{\pm}^{(2)} \quad . \tag{2}$$

Here, $\underset{\sim}{n} = c\underset{\sim}{k}/\omega$, $K_{xy} = -i\omega_{pe}^2/\Omega_e\omega$ , $K_{zz} = \omega_{pe}^2/\omega^2$ and $\underset{\sim}{j}_{\pm}^{(2)}$ is the nonlinear contribution to the current of the sideband. The dominant contribution comes from coupling between the electron $\underset{\sim}{E} \times \underset{\sim}{B}$ drift induced by the pump and the low-frequency density perturbation $n_A$ of the $(\omega, \underset{\sim}{k})$ mode; i.e.,

$$\underset{\sim}{j}_{\pm}^{(2)} = -n_A \frac{ec}{B} (\pm i\beta_0 \underset{\sim}{e}_x - \underset{\sim}{e}_y) E_{0x}(\pm\omega_0) \quad . \tag{3}$$

Substituting Eq.(3) into Eq.(2), we obtain

$$(DE_x)_{\pm} = \mp \frac{i4\pi}{\omega_0} \frac{n_A ec}{B} \alpha_{\pm} E_{0x}(\pm\omega_0), \tag{4}$$

where

$$D_{\pm} = \left(\frac{\omega_{pe}^2}{\Omega_e c k_{\pm}}\right)^2 \left[ 1 - \left(\frac{c k_z \Omega_e n}{\omega_{pe}^2}\right)_{\pm}^2 \right], \tag{5}$$

$$\alpha_\pm = \mp i(\beta_0+\beta_\pm)\cos\theta_\pm + (1+\beta_0\beta_\pm)\sin\theta_\pm \quad , \tag{6}$$

with $\beta_\pm = \omega_{pe}^2\omega_0/c^2\Omega_e k_\pm^2$ and $\cos\theta_\pm = \underset{\sim}{e}_x \cdot (\underset{\sim}{e}_x)_\pm$ .

For the low-frequency kinetic Alfvèn wave, we follow the procedures of Hasegawa and Chen (1975) and employ the two self-consistent field quantities $E_z = -\partial\Psi/\partial z$ and $\underset{\sim}{E}_\perp = -\underset{\sim}{\nabla}_\perp\phi$ ($\phi \neq \Psi$) to decouple from the compressional Alfvèn wave; i.e., $b_z = 0$ (Kadomtsev, 1965). As to the low-frequency response to the high-frequency whistler waves, we use the concept of ponderomotive potential (Drake et al., 1974). The dominant nonlinear coupling comes from the parallel magnetic ponderomotive force $(-e/c)(\underset{\sim}{v}_\perp \times \underset{\sim}{b}_\perp)_z$ produced by the pump and sidebands, which acts on the electrons along $\underset{\sim}{B}$. Let the corresponding ponderomotive potential by $\Psi_p$; i.e.,

$$-\frac{\partial\Psi_p}{\partial z} = \frac{1}{c}\left(\underset{\sim}{v}_{0\perp}\times\underset{\sim}{b}_{\perp-} + \underset{\sim}{v}_{\perp-}\times\underset{\sim}{b}_{0\perp} + \underset{\sim}{v}_{0\perp}^*\times\underset{\sim}{b}_{\perp+} + \underset{\sim}{v}_{\perp+}\times\underset{\sim}{b}_{0\perp}^*\right). \tag{7}$$

Here, $\underset{\sim}{v}_\perp$ is due to electron $\underset{\sim}{E} \times \underset{\sim}{B}$ drift and $\underset{\sim}{b}_\perp$ is related to $\underset{\sim}{E}_\perp$ as

$$(\underset{\sim}{b}_\perp)_{0,+,-} \simeq c\,(k_z\underset{\sim}{e}_z \times \underset{\sim}{E}_\perp/\omega)_{0,+,-} \quad . \tag{8}$$

Equation (7) then reduces to

$$\psi_p = \frac{ic}{B\omega_0}\,( E_{0x} E_{x-} \alpha_-^{*} - E_{0x}^{*} E_{x+} \alpha_+^{*} ) . \tag{9}$$

We now use $\psi_p$ along with the self-consistent potentials $\phi$ and $\psi$ in the dynamics of the $(\omega,\ \underset{\sim}{k})$ mode. The quasi-neutrally condition and Ampere's law in the parallel to $\underset{\sim}{B}$ direction yield, respectively,

$$\lambda_{De}^{-2}(\psi+\psi_p) + \lambda_{Di}^{-2}(1 - I_0 e^{-\lambda_i})\phi = 0 , \tag{10}$$

and

$$\psi - \phi = \left(\frac{\omega^2}{k_z^2 V_A^2 \lambda_s}\right)(\psi + \psi_p) . \tag{11}$$

Here, the argument of $I_0$ is $\lambda_i = k_\perp^2 \rho_i^2$ and $\lambda_s = \lambda_i\, T_e/T_i$. Combining Eqs.(10) and (11), we have

$$\epsilon_A \phi = \psi_p , \tag{12}$$

where

$$\epsilon_A = \frac{\omega^2}{k_z^2 V_A^2} - \lambda_i \left[ \frac{T_e}{T_i} + (1 - I_0 e^{-\lambda_i})^{-1} \right] . \tag{13}$$

The density perturbation $n_A$ is then related to $\psi_p$ as

$$4\pi e\, \epsilon_A n_A = -(\lambda_s/\lambda_{De}^2)\psi_p . \tag{14}$$

Eq.(14) shows that the coupling coefficient is proportional to $\lambda_s = k_\perp^2 \rho_i^2 T_e/T_i$. Hence, the coupling occurs due to the electrostatic properties associated with the finite ion Larmor radius and electron inertia.

## 3. THE DISPERSION RELATION

Equations (4) and (14) along with Eq.(9) are the parametrically coupled equations; from which we derive the following dispersion relation

$$\epsilon_A = \lambda_s \left(\frac{\omega_{pi}}{\omega_0}\right)^2 \left|\frac{c E_{0x}}{B C_s}\right|^2 \left( \frac{|\alpha_-|^2}{D_-} + \frac{|\alpha_+|^2}{D_+} \right) . \tag{15}$$

We now examine the above dispersion relation for the resonant decay instability. For that purpose, we ignore the upper sideband, $(\omega_+, \underset{\sim}{k}_+)$, as being off-resonant and treat both the lower sideband, $(\omega_-, \underset{\sim}{k}_-)$, and the low-frequency $(\omega, \underset{\sim}{k})$ wave as resonant normal modes. That is, we let $\omega = \omega_A + i\gamma$ and $\omega_- = \omega_A - \omega_0 + i\gamma = -\omega_w + i\gamma$, where

$$\begin{aligned} \omega_A &= k_z V_A \lambda_i^{1/2} [T_e/T_i + (1 - I_0 e^{-\lambda_i})^{-1}]^{1/2} \\ &\simeq k_z V_A [1 + \lambda_i (T_e/T_i + 3/4)]^{1/2}, \text{ for } \lambda_i < 1 \end{aligned} \tag{16}$$

and

$$\omega_w = c^2 |k_z k|_- \Omega_e / \omega_{pe}^2 . \tag{17}$$

The dispersion relation, Eq.(15), then reduces to

$$(\gamma+\Gamma_A)(\gamma+\Gamma_w)\left(\frac{\partial\epsilon_A}{\partial\omega_A}\right)\left(\frac{\partial D_-}{\partial\omega_w}\right)=\lambda_s\left(\frac{\omega_{pi}}{\omega_0}\right)^2\left|\frac{cE_{0x}}{BC_s}\right|^2|\alpha_-|^2 , \tag{18}$$

where

$$\frac{\partial\epsilon_A}{\partial\omega_A}=\frac{2\lambda_i}{\omega_A}\left[T_e/T_i+(1-I_0e^{-\lambda_i})^{-1}\right] \simeq \frac{2}{\omega_A}\ , \ \text{for } \lambda_i,\lambda_s<1 \tag{19}$$

and

$$\frac{\partial D_-}{\partial\omega_w}=\frac{2}{\omega_w}\left(\frac{\omega_{pe}^2}{\Omega_e c k_-}\right)^2 . \tag{20}$$

$\Gamma_w$ and $\Gamma_A$ are linear damping rates of the whistler wave and the kinetic Alfvèn wave, respectively. $\Gamma_w$ is mainly due to electron collisional damping. $\Gamma_A$ consists of electron collisional and collisionless dampings as well as ion viscous damping (Hasegawa and Chen, 1975). We then obtain the following threshold condition for $\lambda_i, \lambda_s < 1$

$$\left|\frac{cE_{0x}}{BC_s}\right|^2_{th}=\left(\frac{4}{\lambda_s}\right)(n_z)^2_w\left(\frac{\omega_0}{\omega_{pi}}\right)^2|\alpha_-|^2\left(\frac{\Gamma_A}{\omega_A}\right)\left(\frac{\Gamma_w}{\omega_w}\right) . \tag{21}$$

Since $|\alpha_-|\simeq 1$ and $|\Gamma_A/\omega_A|, |\Gamma_w/\omega_w| \ll 1$, Eq.(21) indicates that for moderate values of $\lambda_s$ the threshold electron $\tilde{E}\times\tilde{B}$ drift speed is generally much less than the ion-sound speed and, hence, this decay instability should be readily realized in space and laboratory plasmas. Far above the threshold $|\gamma| \gg |\Gamma_A|, |\Gamma_w|$, the growth rate is given gy

$$\gamma = \frac{1}{2} \lambda_s^{1/2} (n_\gamma)_w^{-1} \left(\frac{\omega_{pi}}{\omega_0}\right) |\alpha_-| \left|\frac{c E_{0x}}{B C_s}\right| (\omega_A \omega_w)^{1/2} . \qquad (22)$$

## 4. CONCLUSION

We have presented in this paper theoretical calculations, which indicate that a whistler pump wave can parametrically decay into an another whistler wave and a kinetic Alfvèn wave (shear Alfvèn wave with perpendical wavelength of the order of ion Larmor radius). The decay occurs because, due to the effects of finite ion Larmor radius and electron inertia, the modified shear Alfvèn wave contains an electrostatic component and, hence, is accompanied by density perturbations. A general dispersion relation including both upper and lower sidebands is derived. Expressions for the threshold condition and growth rate are then obtained for the resonant decay instability. The results indicate that this decay process should be readily excited in space and laboratory plasmas.

As we remark in the beginning of this paper, this decay instability may be of interest to the magnetosphere, where ULF oscillations of the earth's magnetic field lines can be excited by either artifically triggered or naturally existing whistler turbulence (such as VLF emissions) through the process described here. Detailed comparisons with available experimental observations are, however, needed in the future to verify the theoretical predictions.

The author acknowledges useful discussions with Dr. L.C. Bernard.

This work was supported by the Swiss National Science Foundation.

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