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EMISSION OF LONGITUDINAL WAVES FROM THE INTERACTION OF TWO BEAMS OF TRANSVERSE RADIATION

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ABSTRACT

When two beams of electromagnetic radiation having finite diameter intersect within a plasma they generate in the interaction volume a second order charge density which is proportional to the amplitudes of both beams. Provided that a resonance condition is satisfied this charge density acts as directional antenna emitting longitudinal waves. In this process power is transferred from the beam of higher frequency to the beam of lower frequency with the excess going into the longitudinal waves. The power emitted is $\mathcal{P} = \mathcal{P}_1 \mathcal{P}_2 / \mathcal{P}^*$ where $\mathcal{P}_i$ are the powers of the electromagnectic beams. $\mathcal{P}^*$ is computed for both Langmuir and sound waves. In the first case this process can probably be used as a means to measure the plasma density locally.

I. INTRODUCTION AND RESULTS

We consider two beams of electromagnetic radiation which intersect within a plasma. The beams have finite diameters so that the interaction volume is of finite extent. They propagate at right angles, their electric vector are parallel, and their frequencies are above the plasma frequency to ensure transmission. Two fluid equations are used to describe the plasma which is uniform, field free and hot. From the electric field of the two beams the second order charge density in space-time is calculated using the second order susceptibility. We consider that Fourier component of the charge which oscillates at the difference frequency, $\omega_0 = \omega_2 - \omega_1$. The spatial dependence of this density is characterized by the difference of the propagation vectors of the beams $\underline{k}_0 = \underline{k}_2 - \underline{k}_1$, but its amplitude is substantially different from zero only in a finite volume. At resonance, that is if ω_0 and $\underline{k}_0$ satisfy the dispersion equation of a longitudinal wave this charge density acts as a highly directive antenna for such a wave. In this paper we consider Langmuir and sound waves. The emitted of longitudinal field is computed by means of the Green's function which gives the electric potential of the wave due to a point source. We assume that the energy radiated out of the interaction region is far greater than the energy absorbed within this region by collisions or Landau damping. This is usually the case. The total power radiated can be computed by integrating the energy flow over all angles in the far field approximation. We find that the emitted power is proportional to both incident powers

$$\mathcal{P} = \mathcal{P}_1 \mathcal{P}_2 / \mathcal{P}^* \tag{1}$$

and does not involve the fields of the beams themselves. $\mathcal{P}^*$ depends on whether we consider Langmuir or sound wave emission. At resonance

$$\mathcal{P}^*_L = \frac{24\, m_e T_e}{e^2}\, \eta(\nu) \tag{2}$$

$$\mathcal{P}_S^* = \frac{4 m_e m_i^{1/2} (T_e + 3T_i)^{1/2}}{e^2} \quad \nu^2 \tag{3}$$

where

$$\nu = \omega_1 / \omega_{pe} > 1$$

and

$$\eta(\nu) = \nu(\nu+1)(\nu^2+2\nu)^{1/2} (\nu^2-1)^{1/2} (2\nu^2+2\nu-1)^{-1/2}$$

Written in the international system of units these formulae have the form

$$\mathcal{P}_L^* = 24\, c^3 \varepsilon_0 K_B \frac{m_e}{e^2} T_e \; \eta(\nu) = 2.78\; T_e\; \eta(\nu) \tag{2'}$$

$$\mathcal{P}_S^* = 4\, c^4\; \varepsilon_0\; K_B^{1/2}\; \frac{m_e\, m_e^{1/2}}{e^2}\; (T_e + 3T_i)^{1/2}\; \nu^2 \tag{3'}$$

$$= 1.58\; 10^6\; (T_e+3T_i)^{1/2}\; \nu^2$$

where $C = 3 \cdot 10^8$ m/sec, $\mathcal{E}_0 = 8.86 \cdot 10^{-12}$ As/Vm, $K_B = 1.38 \cdot 10^{-23}$ Joule/degK.

The result suggests that different beam profiles would lead to the same formulae for P except for a numerical factor.

The calculation is not self consistent, since the field strengths of the incident beams are held constant. However, as long as $\mathcal{P}_i \ll \mathcal{P}^*$ one can obtain the changes of the power in each beam from the Manley-Rowe[1)] relations:

$$-\frac{\Delta \mathcal{P}_2}{\omega_2} = \frac{\Delta \mathcal{P}_1}{\omega_1} = \frac{\mathcal{P}}{\omega_0} \;. \tag{4}$$

The quantity $\mathcal{P}^*$ gives a measure of the power required in the incident beams to obtain a measurable out-put. If we choose $\mathcal{P}_1 = \mathcal{P}_2$ and require an out-put $\mathcal{P}$ which is a fraction $\varkappa$ of the power of one incident beam then we must have $\mathcal{P}_1 = \varkappa \mathcal{P}^*$.

In principle this process can be used as the basis of a local plasma diagnostic method. To implement the method one of the two beams should be frequency modulated over a range that includes the perfect match. The power of the second beam and of the generated longitudinal waves will then be amplitude modulated and can therefore be synchroneously detected. Peak power in the emitted wave occurs for the perfect match, $\varepsilon(\underline{k}_2-\underline{k}_1,\omega_2-\omega_1)=0$. From this equation one can then determine $3\,T_i + T_e$ in the case of sound emission or ω_{pe} in the case of Langmuir wave emission. Due to the large value of $\mathcal{P}^*_s$ however the former measurement seems difficult. Here one should exploit the large ratio ω_2/ω_0 and attempt to measure the modulation of one of the injected beams $\Delta\mathcal{P}_2 = (\omega_2/\omega_0)\mathcal{P}$ rather than the acoustic wave of power P. The measurement of plasma density, however, is probably feasible.

II. PREVIOUS WORK

The interaction of crossed beams has been considered previously by a number of authors. One of the first appears to be H. Dreicer[2] who worked out the Compton scattering of a beam of photons, enhanced due to the presence of a second beam. He neglects collective effects since he uses the Thomson cross-section. Without giving results he also mentions the possibility of enhanced scattering off phonons and plasmons. He describes these processes by a transition probability σ :

$$dn_0/dt = \sigma\left[n_2(n_1+1)(n_0+1) - (n_2+1)n_1 n_0\right]$$

where $n_j(\underline{k},r)$ is the density in phase space of the boson of type j. If $n_1 \gg 1, n_2 \gg 1$ and $n_0 \ll n_1, n_2$, then $dn_0/dt = \sigma n_1 n_2$. This leads directly to a relation of the form (1). To find the coefficient $\mathcal{P}^*$, however, it would be necessary to integrate the Boltzman equation for the plasmons, interpreting d/dt as $\partial/\partial t + v_g \partial/\partial r$. We have not followed this proceedure which seems equivalent to ours. In this formulation the Manley-Rowe relations (4) are obvious, since the destruction of one photon in the first beam produces one photon in the second beam and one longitudinal boson.

N.M. Kroll, A. Ron and N. Rostoker[3)] compute the second order charge density produced by the interaction of two plane transverse wave fields whose frequencies lie far above the plasma frequency. The interaction produces a second order charge density $\rho^{(2)}$ of constant amplitude throughout space and time, driving plasma oscillations according to the equation

$$k^2 \phi_0 - \rho_0 = \rho^{(2)}$$

where ϕ_0 and ρ_0 are related by the linear susceptibility χ:

$$\rho_0 = - k^2 \chi \, \phi_0$$

Hence at resonance the total charge density

$$\rho_{tot} = \rho_0 + \rho^{(2)} = \rho^{(2)}/(1+\chi)$$

depends critically on the damping and diverges when there is none. The power delivered to the plasma oscillations is absorbed uniformly throughout space since the energy flow, which is everywhere constant cannot carry away any energy. This calculation therefore applies to the situation in which the absorption length is much smaller that the diameter of the interaction region. If we were to consider a finite cubic interaction volume of size L^3 we would obtain for the power of the beams $P_i \sim L^2 E_i^2$ while the power transferred to and dissipated in the excited waves would become $P \sim L^3 E_0^2 \sim L^3 E_1^2 E_2^2$. This then leads to a relation of the form $P \sim P_1 P_2 / L$ involving the dimension of the interaction volume and differing from (1).

The same remarks apply to the work of G. Weyl[4)] who applied a similar approach to the case of a magnetized plasma.

Both papers 4), 5) discuss the possibility of using the beam interaction as a diagnostic tool. They propose to measure the charge density ρ_{tot} by means of scattering of a third beam. Therefore they give as end result the total induced charge density (4) but not the power transferred to the oscillation.

L. Kuhn, R.F. Leheny, R.C. Marshall[5] examine theoretically and experimentally the mixing of two waves within a bounded magnetized plasma. In this situation, the interaction volume is quite a bit smaller than the cube of the wave lengths so that wave vector matching is not necessary. The resonance therefore is broad and the efficiency of power transfer low. They find theoretically and verify experimentally a relation of the type (1) for the incident and emitted power.

In the analysis we use a natural system of units in which all coefficients in Maxwell's and Newton's equations are equal to unity. The final result is also presented in MKS units. The necessary transformation has been reported previously[6].

III. ANALYSIS

1. The Charge Density due to the Interaction

The two beams are propagating along the x- and y-axis respectively, each beam having its electric vector parallel to the z-axis. For later convenience we assume the following form for the two beams:

$$E_{1z} = E_1 \cos(k_1 x - \omega_1 t) \; \psi_1(x,y,z)$$

$$E_{2z} = E_2 \cos(k_2 y - \omega_2 t) \; \psi_2(y,x,z)$$

where k_i, ω_i satisfy the dispersion relation $\omega^2 = \omega_p^2 + k^2$ and

$$\psi_i(x,y,z) = \exp\left[-\frac{y^2}{2(\alpha + ix/k_i)} - \frac{z^2}{2(2\alpha + ix/k_i)}\right] .$$

The quantities k_1, k_2, ω_1, ω_2 and $\omega_2 - \omega_1$ are all positive. These fields are valid solutions of the linear wave equation provided that the width of the beams is much larger than their wave lengths $\alpha k_i^2 \gg 1$. In the region of interaction the dependence of ψ_i on the first variable -which describes the spreading of the beam- can be neglected, so that

$$\psi_i(x,y,z) = \exp\left[-\frac{x^2}{2\alpha} - \frac{z^2}{4\alpha}\right]$$

Although each beam is a superposition of plane waves of different $\underline{k}$ vectors the amplitude distribution

$$\sqrt{8}\,\pi\,\alpha\, E_1 \exp\left(-\frac{1}{2}\alpha k_y^2 - \alpha k_z^2\right)$$

is so narrow that it will suffice to consider the single wave vectors $\underline{k}_1 = (k_1, 0, 0)$ and $\underline{k}_2 = (0, k_2, 0)$ for each beam respectively. The total power transmitted by each beam is

$$P_i = 2^{-1/2}\,\pi\,\alpha\,\frac{k_i}{\omega_i}\,E_i^2 \tag{5}$$

The charge density produced by the interaction of the two beams can be obtained from the second order susceptibility, given in the Appendix (A 13).

$$\rho^{(2)}(r,t) = \rho^{(2)}(r)\exp(-i\omega_0 t) + cc$$

with

$$\rho^{(2)}(r) = E_1 E_2 \sum_{\alpha = e,i} \frac{q_\alpha^3\, n_0}{4\, m_\alpha^2}\,\frac{k_0^2}{\omega_1\omega_2(\omega_0^2 - c_\alpha^2 k_0^2)} \exp\left(-\frac{r^2}{2\alpha} + i\underline{k}_0\cdot\underline{r}\right) \tag{6}$$

In this expression

$$\omega_0 = \omega_2 - \omega_1 \tag{7}$$

$$\underline{k}_0 = \underline{k}_2 - \underline{k}_1 \tag{8}$$

and

$$c_\alpha^2 = \gamma_\alpha T_\alpha / m_\alpha \tag{9}$$

The adiabatic exponent γ_α is 1 or 3 depending on whether $c_\alpha k_0/\omega_0$ is much greater or smaller than unity for the mode considered. The spherically symmetric amplitude distribution is of course a result of the particular choice of the beam profiles. The ion contribution in (6) can always be neglected, even for an ion acoustic wave with $\omega_0^2/k_0^2 = (T_e + 3T_i)/m_i$.

2. The Green's Function for Longitudinal Wave Emission

We describe longitudinal waves by their electrostatic potential ϕ which in Fourier space satisfies the equation

$$k^2 \varepsilon \phi = \rho_{source} \tag{10}$$

In this equation ε is the linear dielectric function

$$\varepsilon = 1 + \frac{\omega_{pe}^2}{c_e^2 k^2 - \omega^2} + \frac{\omega_{pi}^2}{c_i^2 k^2 - \omega^2} \quad . \tag{11}$$

The source emitting the longitudinal waves is $\rho^{(2)}$ given in the previous section. To calculate ϕ it is advantageous to make use of the Green's function which obeys (10) with a point source oscillating at the fixed frequency ω_0.

$$\varrho_{source} = 2\pi\,\delta\,(\omega-\omega_0)$$

The expression for the Green's function

$$\phi_{Green}(\underline{r},t) = \int \frac{d\omega\, d^3k}{(2\pi)^4} \quad \frac{2\pi\,\delta(\omega-\omega_0)}{k^2\varepsilon(\omega,k)} \quad \exp(i\,\underline{k}\cdot\underline{r} - i\omega t)$$

can be integrated explicitely

$$\phi_{Green}(r,t) = \frac{\exp(-i\omega_0 t)}{4\pi r\,\varepsilon(\omega_0, k=0)} + \sum_{j=1,2} \left[\frac{\exp(-i\omega_0 t + ikr)}{4\pi r\; k^2(\partial\varepsilon/\partial k^2)}\right]_{k_j}$$

where k_j are the two roots of $\varepsilon(\omega_0, k_j) = 0$. The first term is a capacitive near field term giving an electric field proportional to r^{-2} and no energy radiation. We note in passing that for $\omega_0 = 0$, $\gamma_\alpha = 1$ the capacitive term disappears, while the other two terms combine to give the shielded Coulomb-Debye field. The other two terms represent the emission of Langmuir and sound waves. Of course, in reality the two waves will never be excited simultaneously even for $\omega_0 > \omega_{pe}$, since Landau damping, not included in this model, prevents the excitation of the sound waves at such high frequencies.

3. The Potential of the Emitted Wave

The potential of the emitted longitudinal wave field is

$$\phi(\underline{r},t) = \int \varrho^{(2)}(\underline{r}') \quad \phi_{Green}(\underline{r}-\underline{r}', t)\, d^3r$$

$$= -\frac{E_1E_2}{16\pi}\;\frac{e}{m_e}\;\frac{\omega_{pe}^2\, k_0^2\, \exp(-i\omega_0 t)}{\omega_1\omega_2(\omega_0^2 - c_e^2 k^2)\,k^2\,(\partial\varepsilon/\partial k^2)} \;\cdot$$

$$\cdot \int \frac{d^3r'}{|r-r'|}\; \exp\left[-\frac{r'^2}{2\alpha} + i\underline{k}_0\cdot\underline{r}' + ik\,|\underline{r}-\underline{r}'|\right] + cc \qquad (12)$$

The wave number k is one or the other positive solution of $\varepsilon(\omega_0, k) = 0$. To compute the emitted power we only need the far field approximation for ϕ, valid for $r/r' \gg 1$ and $kr \gg 1$. To obtain it, we define first the vector

$$\underline{k}(r) = k\,\underline{r}/r$$

The expression $k\,|\underline{r} - \underline{r}'|$ appearing as argument of the exponential in (12) can now be approximated as $\underline{k}\cdot(\underline{r} - \underline{r}')$, while in the denominator it suffices to replace $|\underline{r} - \underline{r}'|$ by r. With these simplifications the integration can be carried out explicitely and yields:

$$\int \exp\left[-(r')^2/2\alpha + i\underline{k}_0\cdot\underline{r}' + i\underline{k}(\underline{r} - r')\right] d^3r'$$
$$= (2\pi\alpha)^{3/2} \exp\left[i\underline{k}\cdot\underline{r} - \frac{1}{2}\alpha(\underline{k} - \underline{k}_0)^2\right]$$

Hence

$$\phi(\underline{r},t) = 2^{-5/2}\, \pi^{1/2}\, \alpha^{3/2}\, E_1 E_2 (e/m_e) \cdot$$
$$\cdot \frac{\omega_{pe}^2 k_0^2 \exp\left[-i\omega_0 t + ikr - \frac{1}{2}\alpha(\underline{k} - \underline{k}_0)^2\right]}{\omega_1\omega_2(\omega_0^2 - c_e^2 k_0^2)\, k^2\, (\partial\varepsilon/\partial k^2)\, r} + CC \qquad (13)$$

Remembering that $\underline{k}$ is parallel to $\underline{r}$ and that $\alpha k^2 \gg 1$ we see that the emission is peaked in the direction of $\underline{k}_0$. The emission is strongest for $k = k_0$, that is if the frequency ω_0 and the wave number k_0 satisfy the dispersion relation $\varepsilon(\omega_0, \underline{k}_0) = 0$.

4. The Energy Flow

The energy density carried in a monochromatic longitudinal wave is $W = \omega(\partial\varepsilon/\partial\omega)\langle\frac{1}{2}E^2\rangle$ where the brackets $\langle\ \rangle$ indicate time averaging. The energy flow $\underline{S}$ is obtained by multiplying W by the group velocity

$$\partial\omega/\partial\underline{k} = -(\partial\varepsilon/\partial\underline{k})/(\partial\varepsilon/\partial\omega)$$

Hence

$$\underline{S} = -\omega\underline{k}\,(\partial\varepsilon/\partial k^2)\,\langle E^2\rangle \tag{14}$$

When calculating the total power passing through a sphere of radius r

$$P = \int |S|\, r^2 \sin\vartheta\, d\vartheta\, d\varphi \tag{15}$$

the far field approximation only is necessary since the terms decreasing faster than r^{-1} give a vanishing contribution as r goes to infinity. Therefore when computing $E = -\nabla\phi$ it suffices to differentiate the factors $\exp(ikr)$ and $\exp(-ikr)$. Thus one finds

$$\langle E^2\rangle = (\pi\alpha^3/16)\, E_1^2\, E_2^2\, (e/m_e)^2 \cdot$$
$$\cdot \frac{\omega_{pe}^4\, k_o^4 \exp[-\alpha(\underline{k}-\underline{k}_o)^2]}{\omega_1^2\,\omega_2^2\,(\omega_o^2 - c_e^2 k^2)^2 k^2 (\partial\varepsilon/\partial k^2)^2 r^2}$$

and

$$\underline{S} = -(\pi\alpha^3/16)\, E_1^2\, E_2^2\, (e/m_e)^2 \cdot$$
$$\cdot \frac{\omega_{pe}^2\,\omega_o\, k_o^4\,\underline{k} \exp[-\alpha(\underline{k}-\underline{k}_o)^2]}{\omega_1^2\,\omega_2^2\,(\omega_o^2 - c_e^2 k_o^2)^2 k^2 (\partial\varepsilon/\partial k^2)\, r^2}$$

Carrying out the integration indicated in (15) we obtain the total power radiated

$$\mathcal{P} = -(\pi^2 \alpha^2/16)\, E_1^2\, E_2^2\, M\,(e/m_e)^2 \,\cdot$$

$$\cdot\ \frac{\omega_{pe}^4\, \omega_0\, k_0^3}{\omega_1^2\, \omega_2^2 \left(\omega_0^2 - c_e^2 k_0^2\right)^2 k^2 (\partial \varepsilon/\partial k^2)} \qquad (16)$$

where

$$M = \exp\left[-\alpha(k_0 - k)^2\right] - \exp\left[-\alpha(k_0 + k)^2\right]$$

The second term of the last expression is negligible since $\alpha k_0^2 \gg 1$. At resonance, $\varepsilon(\omega_0, k_0) = 0$, that is for $k = k_0$ we have $M = 1$. Expressing the field amplitudes of the beam in terms of the beam powers (Eq. 5), we obtain finally

$$\mathcal{P} = -\frac{1}{8} M \mathcal{P}_1 \mathcal{P}_2 \left(\frac{e}{m_e}\right)^2 \frac{\omega_{pe}^4\, \omega_0\, k_0^3}{\omega_1\, \omega_2\, k_1 k_2 \left(\omega_0^2 - c_e^2 k^2\right)^2 k^2 (\partial \varepsilon/\partial k^2)} \qquad (17)$$

5. The Resonance Condition

Using the dispersion relations for Langmuir and sound waves

$$\omega^2 = \omega_{pe}^2 + c_e^2\, k^2$$

and

$$\omega^2 = c_s^2\, k^2$$

with

$$c_s^2 = (3T_i + T_e)/m_i$$

we find from (11)

$$-k^2 \frac{\partial \varepsilon}{\partial k^2} = \begin{cases} \dfrac{c_e^2 k^2}{\omega_{pe}^2} & \text{Langmuir} \\[2ex] \left(1 + 3\dfrac{T_i}{T_e}\right)^2 \dfrac{\omega_{pi}}{\omega_0^2} & \text{sound} \end{cases}$$

The emitted power is maximized by setting $k = k_0$ in equation (17). Substituting the above expression into (17) one obtains in the two cases

$$\mathcal{P}_L = \frac{e^2}{24 m_e T_e} \frac{\omega_{pe}^6 k_0}{\omega_1 \omega_2 k_1 k_2 \omega_0^3} \mathcal{P}_1 \mathcal{P}_2 \tag{18}$$

$$\mathcal{P}_s = \frac{e^2 m_i^{1/2}}{8 m_e (T_e + 3T_i)^{3/2}} \frac{\omega_{pe}^2 \omega_0^2}{\omega_1 \omega_2 k_1 k_2} \mathcal{P}_1 \mathcal{P}_2 \tag{19}$$

The condition $k_0 = k$ means that the dispersion equation must be satisfied for $\omega_0 = \omega_2 - \omega_1$ and $\underline{k}_0 = \underline{k}_2 - \underline{k}_1$. This implies a relation between the two frequencies of the incident beams. We consider ω_1 as given and calculate ω_2. We find in the case of Langmuir waves

$$\omega_2 = \omega_p + \omega_1 + c_e^2 \left(\frac{\omega_1^2}{\omega_{pe}} + \omega_1 - \frac{\omega_{pe}}{2}\right) \tag{20}$$

and in the case of sound waves

$$\omega_2 = \omega_1 + \sqrt{2}\ c_s \left(\omega_1^2 - \omega_{pe}^2\right)^{1/2} \tag{21}$$

Using the dispersion relation for the incident beams and the results (20) and (21) we write the radiated power in the final form

$$\mathcal{P} = \mathcal{P}_1 \mathcal{P}_2 / \mathcal{P}^*$$

$$\mathcal{P}_L^* = \frac{24\, m_e T_e}{e^2}\, \eta(\nu)$$

$$\mathcal{P}_s^* = \frac{4 m_e\, m_i^{1/2} \left(T_e + 3T_i\right)^{1/2}}{e^2}\, \nu^2$$

In these formulae

$$\nu = \omega_1 / \omega_{pe}$$

and

$$\eta(\nu) = \nu(\nu+1)\left(\nu^2 + 2\nu\right)^{1/2}\left(\nu^2 - 1\right)^{1/2}\left(2\nu^2 + 2\nu - 1\right)^{-1/2}$$

For practical purposes it is convenient to write these equations in the international system of units, using a transformation given previously[6]:

$$\mathcal{P}_L^* = 24\ c^3\ \varepsilon_0\ K_B\ \frac{m_e}{e^2}\ T_e\ \eta_L(\nu)$$

$$\mathcal{P}_s^* = 4\ c^4\ \varepsilon_0\ K_B^{1/2}\ \frac{m_e m_i^{1/2}}{e^2}\left(T_e + 3T_i\right)^{1/2}\ \nu^2$$

These are the formulae given in the introduction.

APPENDIX: The Second Order Susceptibility in the Two Fluid Model

The second order susceptibility is well known in the cold plasma approximation. M.V. Goldman[7] has given the susceptibility for a hot unmagnetized plasma in the special case in which one may assume an isothermal equation of state.

We take the opportunity to present here the general form of the second order susceptibility in the two fluid model of a magnetized plasma having finite temperatures.

We start from the equations of motion and continuity for one species

$$\frac{\partial n}{\partial t} + \underline{\nabla}(n\underline{v}) = 0 \tag{A1}$$

$$\frac{\partial \underline{v}}{\partial t} + (\underline{v}\underline{\nabla})\underline{v} + \frac{\underline{\nabla}p}{mn} = \frac{q}{m}\left[\underline{E} + \underline{v}\times\underline{B}\right] \tag{A2}$$

Pressure and density shall be related by the adiabatic equation of state

$$p = A\, n^{\gamma}$$

so that

$$\frac{\underline{\nabla}p}{mn} = \frac{\gamma A}{(\gamma-1)m}\,\underline{\nabla}n^{\gamma-1}$$

We now replace n by $n_0 + n$, B by $B_0 + B$, where n_0 and B_0 are the equilibrium values and n and B are the perturbations.

We expand $(n_0+n)^{\gamma-1}$ in powers of n. This leads to

$$\frac{\underline{\nabla}p}{mn} = \frac{c_0^2}{n_0}\,\underline{\nabla}n + \frac{\gamma-2}{2}\,\frac{c_0^2}{n_0^2}\,\underline{\nabla}n^2 + \frac{(\gamma-2)(\gamma-3)}{6}\,\frac{c_0^2}{n_0^3}\,\underline{\nabla}n^3 + \ldots$$

where

$$c_0^2 = \gamma T_0 / m$$

We now insert the above expression for $\nabla p / mn$ into (A1) and Fourier transform both (A1) and (A2). The non-linear terms give rise to convolution integrals. Putting $\varkappa = \{\underline{k}, \omega\}$ and

$$d\varkappa_{12} = (2\pi)^{-4} \, d\varkappa_1 \, d\varkappa_2 \, \delta(\varkappa - \varkappa_1 - \varkappa_2)$$

we write the transform of AB as

$$\int A(\varkappa_1) \, A(\varkappa_2) \, d\varkappa_{12}$$

or simply

$$\int A_1 B_2 \, d\varkappa_{12}$$

With this notation (A1,A2) become

$$n - n_0 (\underline{k} \cdot \underline{v}) / \omega = N \tag{A3}$$

$$\underline{v} + i \underline{\Omega} \times \underline{v} / \omega - (c_0^2 / n_0) \underline{k} n / \omega = \underline{V} \tag{A4}$$

where

$$\underline{\Omega} = q \underline{B}_0 / m \tag{A5}$$

$$N = (\underline{k} / \omega) \cdot \int d\varkappa_{12} \, n_1 \underline{v}_2 \tag{A6}$$

$$\begin{aligned} \underline{V} &= i (q / m\omega) \underline{E} + (1/\omega) \int d\varkappa_{12} (\underline{v}_1 \cdot \underline{k}_2) \underline{v}_2 \\ &\quad + i (q / m\omega) \int d\varkappa_{12} \left[\underline{k}_2 (\underline{v}_1 \cdot \underline{E}_2) - \underline{E}_2 (\underline{v}_1 \cdot \underline{k}_2) \right] / \omega_2 \\ &\quad + \tfrac{1}{2} (\gamma - 2) (c_0^2 / n_0^2) (\underline{k} / \omega) \int d\varkappa_{12} \, n_1 n_2 + \ldots\ldots \end{aligned} \tag{A7}$$

Using (A3) we eliminate n from (A4):

$$\underline{v} + i\,\frac{\dot{\underline{\Omega}}}{\omega} \times \underline{v} - c_0^2\,\frac{\underline{k}(\underline{k}\cdot\underline{v})}{\omega^2} = \underline{V} + \frac{c_0^2}{n_0}\,\frac{\underline{k}}{\omega}\,N \tag{A8}$$

We introduce the operator $\mathcal{U}$by the definitaion

$$\mathcal{U}^{-1}\underline{v} = \underline{v} + i\,\frac{\dot{\underline{\Omega}}}{\omega} \times \underline{v} - c_0^2\,\frac{\underline{k}(\underline{k}\cdot\underline{v})}{\omega^2} \tag{A9}$$

so that we can write (A3) and (A8) in the form

$$\underline{v} = \mathcal{U}\left(\underline{V} + \frac{c_0^2}{n_0}\,\frac{\underline{k}}{\omega}\,N\right) \tag{A10}$$

$$n = N + \frac{n_0}{\omega}\,\underline{k}\,\mathcal{U}\left(\underline{V} + \frac{c_0^2}{n_0}\,\frac{\underline{k}}{\omega}\,N\right) \tag{A11}$$

The right hand sides of these two equations (A10, 11) do not contain any linear term except the driving term $\underline{V}^{(1)} = iq\underline{E}/m\omega$. These two equations can there fore be solved for $\underline{v}$ and n by iteration in ascending orders of $\underline{E}$.

Omitting the rather tedious algebra we give the polarisation to second order

$$\underline{P}^{(2)} = \frac{i}{\omega}\,\dot{\underline{j}}^{(2)} = \frac{i}{\omega}\,q\left[n_0\,\underline{v}^{(2)} + n^{(1)}\,\underline{v}^{(1)}\right]$$

in the form:

$$\begin{aligned}\underline{P}^{(2)} = -\frac{iq^3 n_0}{m^2\omega}\int d\varkappa_{12}\Big\{&(\omega_1^2\,\omega_2)^{-1}\,(\underline{k}_1\mathcal{U}_1\underline{E}_1)(\mathcal{U}_2\underline{E}_2)\\ &+(\omega\,\omega_1\omega_2)^{-1}\,(\underline{k}_2\,\mathcal{U}_1\underline{E}_1)\,\mathcal{U}\,\mathcal{U}_2\,\underline{E}_2\\ &+(\omega\,\omega_1\omega_2)^{-1}\,(\underline{E}_2\,\mathcal{U}_1\,\underline{E}_1)\,\mathcal{U}\,\underline{k}_2\end{aligned}$$

$$
\begin{aligned}
&-(\omega\,\omega_1\,\omega_2)^{-1}\,(\underline{k}_2\,U_1\,\underline{E}_1)\;U\,\underline{E}_2 \\
&+\tfrac{1}{2}(\gamma-2)\,c_0^2\,(\omega\,\omega_1^2\,\omega_2^2)^{-1}\,(\underline{k}_1\,U_1\,\underline{E}_1)(\underline{k}_2\,U_2\,\underline{E}_2)\,U\underline{k} \\
&+c_0^2\,(\omega^2\,\omega_1^2\,\omega_2)(\underline{k}_1\,U_1\,\underline{E}_1)\,(\underline{k}\,U_2\,\underline{E}_2)\,U\,\underline{k}\;\Big\}
\end{aligned}
\tag{A12}
$$

From this expression one can easily extract -after symmetrizing it- the second order susceptibility.

In the special case of a plasma without a magnetic field, $\underline{B}_0 = 0$, that is $\underline{\Omega} = 0$ one has

$$
U^{-1} = 1 - \frac{c_0^2}{\omega^2}\,\underline{k}\circ\underline{k} \qquad\qquad U = 1 + \frac{c_0^2}{\omega^2 - c_0^2 k^2}\,\underline{k}\circ\underline{k}
$$

Where $\underline{a}\circ\underline{b}$ is our notation for a dyadic. This form of U allows the contraction of the second, third and fourth term of (A12) into one. Thus we find

$$
\begin{aligned}
\underline{P}^{(2)} = -\frac{i q^3 n_0}{2m^2}\int \frac{d\varkappa_{12}}{\omega\,\omega_1\,\omega_2}\Big\{ &\frac{1}{\omega}(\underline{E}_1\,U_1\,U_2\,\underline{E}_2)\,U\underline{k} \\
&+\frac{1}{\omega_1}(\underline{k}_1\,U_1\,\underline{E}_1)\,U_2\,\underline{E}_2 + \frac{1}{\omega_2}(\underline{k}_2\,U_2\,\underline{E}_2)\,U_1\,\underline{E}_1 \\
&+\frac{(\gamma-1)c_0^2}{\omega\,\omega_1\,\omega_2}\,(\underline{k}_1\,U_1\,\underline{E}_1)\,(\underline{k}_2\,U_2\,\underline{E}_2)\,U\underline{k} \\
&+\frac{c_0^2}{\omega^2}\Big[\frac{1}{\omega_1}(\underline{k}_1U_1\underline{E}_1)(\underline{k}_1U_2\underline{E}_2)+\frac{1}{\omega_2^2}(\underline{k}_2U_2\underline{E}_2)(\underline{k}_2U_1\underline{E}_1)\Big]U\underline{k}\Big\}
\end{aligned}
\tag{A13}
$$

This expression which is already symmetrized agrees with the result obtained by M.V. Goldman for the special case of $\gamma=1$. Some algebra, however, is required to prove it.

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