

November 1975

LRP 98/75

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FOR WAVE EMISSION IN PLASMAS

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ABSTRACT

Green's function for the emission of radiation from an oscillating point source within a plasma is examined in the limit of large distances from the source. The plasma is characterized by its susceptibility tensor $\underline{\chi}$ leading to the dispersion relation $D(\omega, \underline{k}) = 0$. Asymptotic formulae are derived which state that only those points in \underline{k} -space contribute to the far field which satisfy the dispersion relation and for which the group velocity points towards the observer. The intensity of the emission is inversely proportional to the Gaussian curvature of the surface $D(\omega, \underline{k}) = 0$ at that point. The result is generalized for the case in which this curvature vanishes.

INTRODUCTION

The plasma as a carrier of waves shall be described by its linear susceptibility tensor $\underline{\underline{\chi}}$. The electric field shall be designated by \underline{E} the polarization by \underline{P} . The Fourier transforms in space and time of A is written as \hat{A} . In frequency and wave number space Maxwell's equations take the form

$$\underline{\underline{H}} \hat{\underline{E}} = -\omega^2 \hat{\underline{P}}_{\text{source}} \quad (1)$$

where $\hat{\underline{P}}_{\text{source}}$ designates any externally imposed polarization, while the operator $\underline{\underline{H}}$ stands for

$$\underline{\underline{H}} = (1 + \underline{\underline{\chi}}) \omega^2 - \underline{1} k^2 + k_0 k$$

We designate the determinant of $\underline{\underline{H}}$ by D

$$D(\omega, \underline{k}) = \text{Det}(\underline{\underline{H}})$$

Thus the inverse of the operator $\underline{\underline{H}}$ can be written as

$$\underline{\underline{H}}^{-1} = \underline{\underline{h}} / D$$

where $\underline{\underline{h}}$ stands for the transposed matrix of the minors of $\underline{\underline{H}}$. Thus the solution of (1) can be given formally as

$$\hat{\underline{E}} = -\omega^2 \underline{\underline{h}} \hat{\underline{P}} / D$$

If the source is oscillating in time at a fixed real frequency

$$\hat{\underline{P}}_{\text{source}}(r, t) = \underline{P}(r) \exp(-i\omega t)$$

then the electric field in space time becomes

$$\underline{E}(r, t) = \exp(-i\omega t) \int \underline{\underline{G}}(\omega, r-r') \underline{P}(r') d^3r$$

where the matrix

$$\underline{G}(\omega, \underline{r}) = -\frac{\omega^2}{(2\pi)^3} \int \frac{\underline{h}}{D} \exp(i\underline{k} \cdot \underline{r}) d^3k \quad (2)$$

is Green's function. The integrations extend over the entire space of real \underline{k} vectors. They are difficult or impossible to do analytically in most cases. But, as we shall see, they can be carried out in the far field limit by means of the methods of the stationary phase and of the residues.

We shall assume that the plasma is only slightly dissipative so that for real arguments the imaginary part of D is small and negligible except near $D = 0$.

$$D = \Delta + i\Gamma$$

The small imaginary part, Γ , is essential since it defines the singular integral (2) such that causality is respected.

Asymptotic expressions for Green's function have been obtained fifteen years ago by M.J. Lighthill¹⁾. The only excuse for this report -prepared in ignorance of Lighthill's work- is that the present version is some what more general and more concise, specially in the treatment of the radiation (or causality) condition.

More recently these results have been applied to the emission of waves from antennas in magnetized plasmas²⁻⁵⁾ with particular attention to the emission in the directions of resonance (resonance cones).

ANALYSIS

The integrations implied in (2) shall be carried out in two steps of which the first is an integration over surfaces of constant Δ and the second an integration over Δ . It is necessary, therefore, to introduce new variables in \underline{k} space one of which must be the determinant Δ ; the other two may be chosen at will but may be thought of as specifying the direction of \underline{k}

$$\underline{r} = \underline{r}(s_1, s_2, \Delta) \quad (3)$$

This transformation, which may be multivalued, defines a family of surfaces Σ_Δ on which Δ is constant. In these coordinates the integral (2) assumes the form

$$\underline{G} = -\frac{\omega^2}{(2\pi)^3} \int \frac{d\Delta}{\Delta + i\Gamma} \int \underline{J} \underline{r} \exp(i\underline{r} \cdot \underline{k}) ds_1 ds_2 \quad (4)$$

where

$$\underline{J} = \left[\frac{d\underline{r}}{ds_1} \times \frac{\partial \underline{r}}{\partial s_2} \right] \cdot \frac{\partial \underline{r}}{\partial \Delta} > 0 \quad (5)$$

is the Jacobian of the transformation.

For what follows it is necessary to introduce the metric tensor on the surfaces Σ_Δ

$$g_{\alpha\beta} = \frac{\partial \underline{r}}{\partial s_\alpha} \cdot \frac{\partial \underline{r}}{\partial s_\beta} \quad \alpha, \beta = 1, 2 \quad (6)$$

and its determinant

$$g = \text{Det}(g_{\alpha\beta}) \quad (7)$$

We shall also need the vector normal to the surface Σ_Δ

$$\underline{n} = g^{-1/2} \frac{\partial \underline{r}}{\partial s_1} \times \frac{\partial \underline{r}}{\partial s_2} \quad (8)$$

and the curvature pseudo tensor

$$\chi_{\alpha\beta} = -\underline{n} \cdot \frac{\partial^2 \underline{r}}{\partial s_\alpha \partial s_\beta} \quad (9)$$

Expressing the increment of Δ as

$$d\Delta = \frac{\partial \Delta}{\partial \underline{k}} \cdot \left(\frac{\partial \underline{k}}{\partial \Delta} d\Delta + \frac{\partial \underline{k}}{\partial s_1} ds_1 + \frac{\partial \underline{k}}{\partial s_2} ds_2 \right) \quad (10)$$

we see that

$$\frac{\partial \Delta}{\partial \underline{k}} \cdot \frac{\partial \underline{k}}{\partial \Delta} = 1 \quad (11)$$

and

$$\frac{\partial \Delta}{\partial \underline{k}} \cdot \frac{\partial \underline{k}}{\partial s_a} = 0 \quad a=1,2 \quad (12)$$

Therefore the vector normal \underline{n} can also be expressed as

$$\underline{n} = \frac{\partial \Delta}{\partial \underline{k}} \left| \frac{\partial \Delta}{\partial \underline{k}} \right|^{-1} \quad (13)$$

Since r is large by hypothesis the integral over the variables s_1 and s_2 is determined by the contributions from those points on which lie in the immediate vicinity of the points on which the phase of the exponential is stationary

$$\frac{\partial}{\partial s_a} (\underline{r} \cdot \underline{k}) = 0 \quad (14)$$

The solutions s_{ip} of these equations define the points of stationary phase

$$\underline{k}_p = \underline{k}(s_{ip}, s_{ip}, \Delta) \quad (15)$$

Obviously \underline{r} and \underline{n} are parallel so that we can write

$$\underline{r} = \lambda \frac{\partial \Delta}{\partial \underline{k}} = \text{sign}(\lambda) \underline{n} r \quad (16)$$

Following the method of the stationary phase we expand $\underline{k}(s_1, s_1, \Delta)$ about the stationary point in powers of $s_a - s_{ip}$ and write the argument of the exponential in (4) as

$$i \underline{k} \cdot \underline{r} = i \left[\underline{k}_p \cdot \underline{r} + \frac{1}{2} \underline{r} \cdot \frac{\partial^2 \underline{k}}{\partial s_a \partial s_p} (s_a - s_{ip})(s_p - s_{ip}) \right] \quad (17)$$

Remembering (9) and (16) we may write (17) as

$$i\mathbf{k} \cdot \mathbf{r} = i\mathbf{k}_p \cdot \mathbf{r} + i \operatorname{sign}(\lambda) \frac{1}{2} \alpha_{\alpha\beta} (s - s_{\alpha p})(s - s_{\beta p}) r \quad (18)$$

so that the inner integral of (4) can be approximated as

$$\underline{Y} = \exp(i\mathbf{k}_p \cdot \mathbf{r}) \int \underline{L} \exp[-i \operatorname{sign}(\lambda) \frac{1}{2} \alpha_{\alpha\beta} (s_1 - s_{\alpha p})(s_1 - s_{\beta p}) r] ds_1 ds_2 \quad (19)$$

Without restricting the generality of the method we now assume that the coordinates s_1 and s_2 are locally cartesian at the stationary point and that the axes are aligned with the principal directions. In this coordinate system the curvature tensor is diagonal and its eigenvalues are the principal curvatures, \bar{K}_α :

$$g_{\alpha\beta} = \delta_{\alpha\beta} \quad \alpha_{\alpha\beta} = \bar{K}_\alpha \delta_{\alpha\beta} \quad (\text{no sum}) \quad (20)$$

It will be convenient later to define a new set of curvatures K_α which differ from the old ones only by the sign of λ :

$$K_\alpha = \operatorname{sign}(\lambda) \bar{K}_\alpha \quad (21)$$

This convention uniquely defines the curvatures at the stationary point.

K_α is positif if it is convex towards the point r .

The asymptotic value of the integral (19) is according to the method of the stationary phase

$$\underline{Y} = 2\pi \left\{ \bar{\alpha} \int \underline{L} |K_1 K_2|^{-\frac{1}{2}} \exp(i\mathbf{k}_p \cdot \mathbf{r}) / r \right\}_p \quad (22)$$

where

$$\bar{\alpha} = \begin{cases} -i & \text{if } K_1 > 0, K_2 > 0 \\ i & \text{if } K_1 K_2 < 0 \\ i & \text{if } K_1 < 0, K_2 < 0 \end{cases} \quad (23)$$

The expression (22) is to be evaluated at the stationary point.

Fortunately it is not necessary to actually carry out the transformation onto principal axes. In any coordinate system the principal curvatures are the solutions of

$$\text{Det}(\alpha_{ap} - \bar{K} g_{ap}) = 0 \quad (24)$$

The product of the principle curvatures -which is the Gaussian curvature squared- is simply

$$\bar{K}_1 \bar{K}_2 = \frac{\text{Det}(\alpha_{ap})}{\text{Det}(g_{ap})} \quad (25)$$

In Jacobian J also can be brought into a form which is independent of the choice of the coordinates Δ :

$$J = \underline{n} \cdot \frac{\partial \underline{k}}{\partial \Delta} = \left| \frac{\partial \Delta}{\partial \underline{k}} \right|^{-1} \quad (26)$$

Therefore

$$\underline{G}(\omega, \underline{r}) = - \frac{\omega^2}{(2\pi)^2} \sum_p \left\{ 2 \underline{k} \frac{\exp(i \underline{k}_p \cdot \underline{r})}{|\partial \Delta / \partial \underline{k}| \cdot |K_1 K_2|^{1/2} r} \right\} \frac{d\Delta}{\Delta + i\Gamma} \quad (27)$$

The integral over Δ can be carried out by again taking advantage of the large value of r . The main contribution clearly comes from a small interval around $\Delta = 0$. We therefore expand the argument of the exponential

$$\underline{k}_p \cdot \underline{r} = \underline{k}_j \cdot \underline{r} + \left(\frac{\partial \underline{k}}{\partial \Delta} \right)_j \cdot \underline{r} \Delta \quad (28)$$

where \underline{k}_j is a stationary point for the phase $\underline{k} \cdot \underline{r}$ on the surface Σ_0 . Such a point shall be called a conjugate point with respect to \underline{r} . Thus

$$\underline{G}(\omega, \underline{r}) = - \frac{\omega^2}{(2\pi)^2} \sum_j \left\{ 2 \underline{k} \frac{\exp(i \underline{k}_j \cdot \underline{r})}{|K_1 K_2|^{1/2} |\partial \Delta / \partial \underline{k}| r} \right\} S_j \quad (29)$$

where

$$S_j = \int \frac{\exp(i\lambda_j \Delta) d\Delta}{\Delta + i\Gamma_j} = \begin{cases} i2\pi & \text{if } \lambda_j > 0, \Gamma_j < 0 \\ 0 & \text{if } \lambda_j \Gamma_j > 0 \\ -i2\pi & \text{if } \lambda_j < 0, \Gamma_j > 0 \end{cases} \quad (30)$$

Since S_j depends on the signs of λ_j and Γ_j we must determine the conditions under which the various combinations of signs may occur.

Let ω_0 , \underline{k}_0 be a real solution of the approximate dispersion relation $\Delta = 0$. Taking dissipation into account and keeping \underline{k}_0 real and fixed we find

$$\omega = \omega_0 - i\Gamma / (\partial\Delta / \partial\omega) \quad (31)$$

Since the plasma is slightly dissipative, $\text{Im}\omega < 0$, we must necessarily have

$$\Gamma \frac{\partial\Delta}{\partial\omega} > 0 \quad (32)$$

On the other hand according to (16)

$$\frac{\partial\Delta}{\partial\underline{k}} \cdot \underline{r} = \left| \frac{\partial\Delta}{\partial\underline{k}} \right|^2 \lambda \quad (33)$$

This relation together with the equation for the group velocity

$$\underline{v}_g = -(\partial\Delta / \partial\underline{k}) / (\partial\Delta / \partial\omega) \quad (34)$$

provides a link of the sign of $\underline{v}_g \cdot \underline{r}$ with those of λ and Γ . Indeed

$$\underline{v}_g \cdot \underline{r} = -\left(\frac{\partial\Delta}{\partial\underline{k}} \cdot \underline{r} \right) / \frac{\partial\Delta}{\partial\omega} = -\left| \frac{\partial\Delta}{\partial\underline{k}} \right|^2 \lambda \frac{\partial\Delta}{\partial\omega} \quad (35)$$

so that

$$\text{sign}(\underline{v}_j \cdot \underline{r}) = \text{sign}\left(\lambda \frac{\partial \Delta}{\partial \omega}\right) = -\text{sign}(\lambda \Gamma) \quad (36)$$

Therefore

$$S_j = \begin{cases} i \text{sign}(\lambda) 2\pi & \text{if } \underline{v}_j \cdot \underline{r} > 0 \\ 0 & \text{if } \underline{v}_j \cdot \underline{r} < 0 \end{cases} \quad (37)$$

When this result is substituted into (29) the following groups of factors appear

$$\frac{\text{sign}(\lambda)}{|\partial \Delta / \partial \underline{k}| r} = \frac{1}{(\partial \Delta / \partial \underline{k}) \cdot \underline{r}} \quad (38)$$

and

$$-i\bar{\alpha} = \alpha = \begin{cases} -1 & k_1 > 0, k_2 > 0 \\ -i & k_1 k_2 < 0 \\ 1 & k_1 < 0, k_2 < 0 \end{cases} \quad (39)$$

Thus Green's function can be written in the form

$$\underline{\underline{G}}(\omega, \underline{r}) = \frac{\omega^2}{2\pi} \sum_j' \left\{ \alpha \underline{k} \frac{\exp(i \underline{k} \cdot \underline{r})}{|k_1 k_2|^{1/2} (\partial \Delta / \partial \underline{k}) \cdot \underline{r}} \right\}_j \quad (40)$$

where the summation extends over the points \underline{k}_j for which $\underline{v}_j \cdot \underline{r} > 0$. In other words \underline{k}_j must satisfy the conditions

$$\Delta(\omega, \underline{k}_j) = 0 \quad (41)$$

and

$$\frac{\partial \Delta}{\partial \underline{R}} = - \eta \frac{\partial \Delta}{\partial \omega} \underline{r} \quad (42)$$

with

$$\eta > 0 \quad (43)$$

The asymptotic formulae (40) fails when at the conjugate point \underline{k}_j the curvature $K_1 K_2$ vanishes or even when $K_1 K_2 r^2$ is small.

Let \underline{k}_0 be such a point. We assume $K_2 \neq 0$ and we develop \underline{k} in powers of s_1 and s_2 about \underline{k}_0 and for $\Delta=0$. This development has the form

$$\underline{k} = \underline{k}_0 + \{ s_1, s_2, -\frac{1}{3} L s_1^3 + \frac{1}{2} \bar{K}_2 s_2^2 \} \quad (44)$$

Since at \underline{k}_0 , $K_1 = 0$ we must retain the cubic term in s_1 . The normal vector is

$$\underline{n} = \{ L s_1^2, \bar{K}_2 s_2, 1 \} \quad (45)$$

The vector \underline{r} shall be nearly parallel to the normal \underline{n}_0

$$\underline{r} = \text{sign}(\lambda) \{ \varphi, \psi, 1 \} r \quad (46)$$

where φ and ψ are small angles. The conjugate point is given by

$$s_{1j} = \pm \sqrt{\frac{\varphi}{L}} \quad s_{2j} = \frac{\psi}{\bar{K}_2} \quad (47)$$

Thus there are either two conjugate points or none depending on the sign of φ (Fig. 1). Assuming $L\varphi > 0$ the curvatures at the conjugate points

$$\bar{K}_1 = \pm 2(L\varphi)^{1/2} \quad (48)$$

while the value of $\frac{k_j \cdot r}{j}$ becomes

$$\begin{aligned} \frac{k_j \cdot r}{j} &= \underline{k}_0 \cdot r + \text{sign}(\lambda) \left(\pm \frac{2}{3} \frac{\varphi^{3/2}}{L^{1/2}} + \frac{1}{2} \frac{g^2}{\bar{K}_2} \right) r \\ &= \underline{k}_0 \cdot r + \left(\pm \frac{2}{3} \frac{\varphi^{3/2}}{L^{1/2}} + \frac{1}{2} \frac{g^2}{K_2} \right) r \end{aligned} \quad (49)$$

where the upper sign applies to the point in which $K_1 = \text{sign}(\lambda) K_1$ is positive.

Instead of (19) the integral \underline{Y} has now the form

$$\underline{Y} = \exp(i \underline{k}_0 \cdot r) \left(\int \underline{h} \right)_0 \int \exp \left[i \text{sign}(\lambda) \left(\varphi s_1 - \frac{1}{3} L s_1^3 \right) r \right] ds_1 \quad (50)$$

$$\times \int \exp \left[i \text{sign}(\lambda) \left(\varphi s_2 - \frac{1}{2} \bar{K}_2 s_2^2 \right) r \right] ds_2 \quad (51)$$

The integral over s_1 can be expressed in terms of an Airy function $A(x)$ defined by

$$A(x) = \int \exp \left(tx - \frac{t^3}{3} \right) dt \quad (52)$$

with the integration going from $i^{-1/3} \infty$ to $i^{1/3} \infty$. It can also be defined by the equation $A'' - xA = 0$ with the initial conditions

$$A(0) = i 3^{5/6} \left(\frac{1}{3} \right)! \quad A'(0) = -i 3^{7/6} \left(\frac{2}{3} \right)! \quad (53)$$

For large arguments the following asymptotic expressions are valid

$$A(-x) = \sqrt{\pi} x^{-1/4} \exp\left[i\left(\frac{\pi}{4} + \frac{2}{3}x^{3/2}\right)\right] + cc, \quad x \gg 1 \quad (54)$$

$$A(x) = i\sqrt{\pi} x^{-1/4} \exp\left[-\frac{2}{3}x^{3/2}\right], \quad x \gg 1 \quad (55)$$

The integration over s_2 as well as all further steps of the analysis can be carried out exactly as in the case $K_1 \neq 0$ and lead to the result

$$\underline{g}_0 = \frac{\omega^2}{(2\pi)^{3/2}} \left\{ \beta \underline{h} \frac{r^{1/6} A(-L^{1/3} r^{2/3} \varphi) \exp\left[i\frac{k_0}{2}r + i\frac{\varphi^2}{2K_2}\right]}{|L|^{1/3} |K_2|^{1/2} (\partial\Delta/\partial\underline{k}) \cdot \underline{r}} \right\}_0 \quad (56)$$

where

$$\beta = \begin{cases} i^{3/2} & \text{if } K_2 > 0 \\ i^{-3/2} & \text{if } K_2 < 0 \end{cases} \quad (57)$$

The cube root in the argument of the Airy function must be interpreted as

$$L^{1/3} = \text{sign}(L) |L|^{1/3} \quad (58)$$

The subscript 0 indicates that the expression must be evaluated at the point in which $K_2 = 0$ and not at the conjugate points.

The emission of waves now depends very strongly on the angle φ . If $L\varphi > 0$ is positive, the field oscillates as a function of φ . This is due to the interference of the emission from the two conjugate points lying in the

vicinity of \underline{k}_0 . If $L\psi$ is negative there are no conjugate points and the field decays exponentially as a function of ψ .

For large values of r with $L\psi$ fixed and positive the asymptotic expression (54) for the Airy function can be used. The two terms of the asymptotic expansion represent the contribution from the two stationary points. The exponential factors coming from the Airy function

$$\exp\left(\pm i \frac{2}{3} L^{-1/2} \psi^{3/2} r\right) \quad (59)$$

combine with the exponential of (56) to give

$$\exp\left(i \underline{k}_0 \cdot \underline{r} + i \frac{\psi^2}{2K_2}\right) \exp\left(\pm i \frac{2}{3} L^{-1/2} \psi^{3/2} r\right) = \exp(i \underline{k}_j \cdot \underline{r}) \quad (60)$$

according to equation (49). Making use of (48) asymptotic form (40) can be recovered.

RESULTS

Green's function is an integral over all of \underline{k} -space. At very large distances from the source, however, only a few isolated points and their immediate vicinity contribute to the emission. These are those points \underline{k}_j which satisfy the dispersion relation $\Delta(\omega, \underline{k}_j) = 0$ and for which the group velocity points in the same direction as the vector \underline{r} and which we call conjugate points with respect to \underline{r} .

The intensity of the emitted radiation is inversely proportional to the total curvature of the surface $\Delta = 0$ at the conjugate point.

If a conjugate point lies near a point of zero curvature a more accurate asymptotic form must be used involving an Airy function. In this case the emission varies rapidly with angle showing a behaviour which is similar

to the diffraction of light on an edge: there exists a plane which separates two domains in one of which the intensity oscillates as a function of angle while in the other it decays exponentially. The field strengths decrease as $r^{-5/6}$ with distance from the source.

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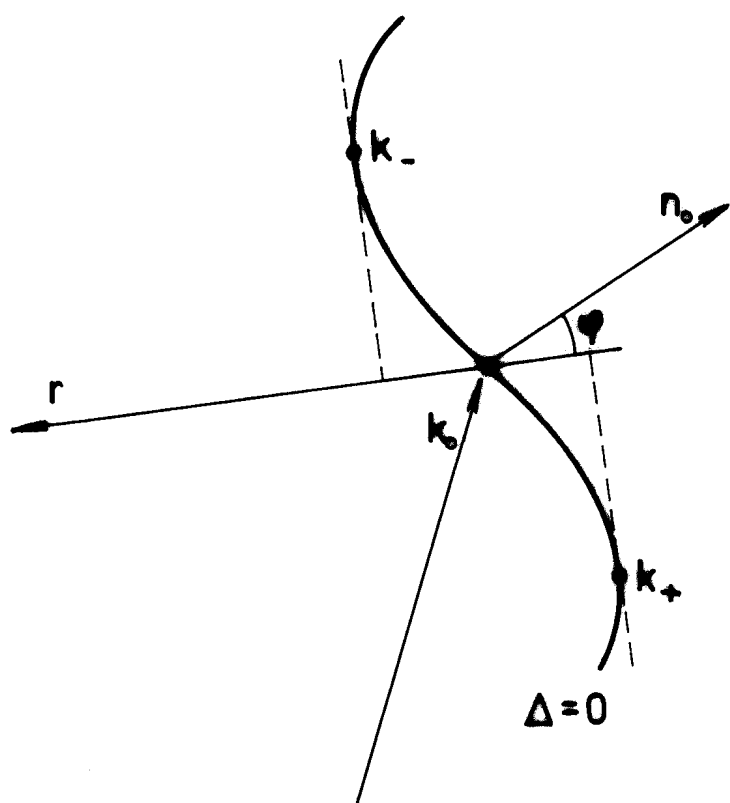


Fig. 1: Stationary points \underline{k}_+ , \underline{k}_- in the vicinity of a point \underline{k} of zero curvature $K_1 = 0$.