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PROPAGATION OF SOLITARY WAVES IN A TWO ION SPECIES PLASMA WITH FINITE ION TEMPERATURE

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ABSTRACT - Using a Korteweg-de Vries equation, we study the propagation of ion acoustic solitary waves in a two ion species plasma with $T_i \neq 0$. As the dispersion relation presents two acoustic branches with quite different phase velocities, two types of soliton can propagate: the slow one which is subsonic with respect to one ion mode and supersonic to the other, has greater amplitude and smaller width than the fast one.

1. INTRODUCTION

For weakly non linear ion sound disturbances, dispersive effects prevents unlimited steepening due to non linearity: the resulting stationnary structure is a soliton. Theoritical models of solitary waves in a one component plasma have been discussed by several authors. WASHIMI and TANIUTI (1966) first showed that the propagation of weak solitary waves in a cold ion and hot isothermal plasma can be described by a Korteweg-de Vries (K.dV) equation. Considering a warm ion fluid, TAPPERT (1972) and TAGARE (1973) derived a modified K.dV equation whose stationnary solutions have smaller amplitude and width.

The presence of light ions strongly affects the soliton's amplitude as a too large potential can reflect the light impurities, even in the cold ion limit (WHITE et al., 1972). TRAN and HIRT (1974) used the fluid formalism to derive a KdV equation for a two ion species plasma with $T_i = 0$ and found the dependence of the soliton's amplitude on the light ion concentration. As the finite ion temperature yields two ion acoustic modes with quite different phase velocities, (FRIED et al., 1971), it can be expected that two types of solitons will emerge, which can be identified as fast one and slow one. Aside from their velocity, a striking difference is the increase in amplitude and decrease in width of the slow solitons as a consequence of the apparent reduction of electron temperature in the slow ion acoustic modes.

In section II the dispersion relation for the two acoustic modes will be derived using fluids equation. The section III will be devoted to the derivation of the K.dV equation using TANIUTI and WEI (1968) reductive perturbation method. Its stationnary solutions are the fast and slow solitons whose amplitude and width will be discussed in section IV.

II. THE LINEAR DISPERSION RELATION IN THE SMALL WAVE VECTOR LIMIT

We consider a two ion fluid with equal ion temperature T_i and a mass ratio $\mu = m_2/m_1 > 1$, neutralized by hot isothermal electrons. The basic equations are:

$$\frac{\partial n_1}{\partial t} + \frac{\partial n_1 u_1}{\partial x} = 0 \tag{1}$$

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} = \mu \left[E - \frac{\gamma \theta}{\alpha(\gamma-1)} n_1^{(\gamma-2)} \frac{\partial n_1}{\partial x} \right] \tag{2}$$

$$\frac{\partial n_2}{\partial r} + \frac{\partial n_2 u_2}{\partial x} = 0 \tag{3}$$

$$\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} = E - \frac{3\theta}{(1-\alpha)^2} n_2 \frac{\partial n_2}{\partial x} \tag{4}$$

$$\frac{\partial n_e}{\partial x} = - n_e E \tag{5}$$

$$\frac{\partial E}{\partial x} = \sum_{i=1}^{2} n_i - n_e \tag{6}$$

In these equation spatial distances are measured in Debye length unit $(\varepsilon_0 kT_e/n_{eo} e^2)^{\frac{1}{2}}$, velocity in a characteristic velocity unit $(kT_e/m_2)^{\frac{1}{2}}$;

times are normalized to $(n_{eo}e^2/\varepsilon_o m_2)^{-\frac{1}{2}}$, densities to unperturbed electron densities n_{eo} and electric potential to $k_B T_e/e$. The light ion concentration α is defined as n_{10}/n_{eo}.

For the heavy ions, an adiabatic equation of state was assumed, whereas the light ion one depends on the phase velocity of the wave. By eliminating n_2 and the electric field E, the linear part of (1) to (6) in the long wave length limit becomes:

$$\frac{\partial}{\partial t}\begin{bmatrix} n_1^{(1)} \\ u_1^{(1)} \\ n_e^{(1)} \\ u_2^{(1)} \end{bmatrix} + \begin{bmatrix} 0 & \alpha & 0 & 0 \\ \frac{\mu\gamma\theta}{\alpha} & 0 & \mu & 0 \\ 0 & \alpha & 0 & 1-\alpha \\ -\frac{3\theta}{1-\alpha} & 0 & 1+\frac{3\theta}{1-\alpha} & 0 \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} n_1^{(1)} \\ u_1^{(1)} \\ n_e^{(1)} \\ u_2^{(1)} \end{bmatrix} = 0$$

or:

$$\frac{\partial}{\partial t} U^{(1)} + A_o \frac{\partial}{\partial x} U^{(1)} = 0, \tag{7}$$

U being the column vector:

$$U = \begin{bmatrix} n_1 \\ u_1 \\ n_e \\ u_2 \end{bmatrix}$$

Looking for solution of the form $\tilde{u}^{(1)} \exp\{i(kx-\omega t)\}$, (7) becomes:

$$\left(A_0 - \frac{\omega}{k} I\right) \tilde{u}^{(1)} = 0 \tag{8}$$

The phase velocity ω/k is an eigenvalue λ of A_0 and $\tilde{U}^{(1)}$ the corresponding eigenvector. Two cases must be considered:

a) The phase velocity ω/k is much greater than the two ion thermal velocities $\sqrt{\theta}$ and $\sqrt{\theta\mu}$

Under these limits, the light ion can be considered as adiabatic with $\gamma = 3$. The eigenvalue λ A_0 which of satisfies the preceeding condition is given by:

$$\lambda = \frac{\omega}{k} = \left[1-\alpha+\alpha\mu + \frac{3\theta(1-\alpha+\alpha\mu^2)}{1-\alpha+\alpha\mu}\right]^{1/2} \tag{9}$$

b) The phase velocity ω/k is much greater than the heavy ion thermal velocity $\sqrt{\theta}$ but much smaller than the light ion one $\sqrt{\theta\mu}$

The light ion equation of state is therefore an isothermal one, which yields the following eigenvalue:

$$\lambda = \frac{\omega}{k} = \left[\frac{\mu\theta^2\left(1+\frac{\alpha}{\theta}\right)\left(1+2\alpha+3\theta\right)}{\mu\theta^2\left(1+\frac{\alpha}{\theta}\right)^2 + \alpha(1-\alpha)}\right]^{1/2} \tag{10}$$

A straight forward computation shows that in case b) the phase velocity λ is smaller than in case a): the isothermal light ion lowers the electron temperature (FRIED et al., 1971).

In the two cases the right and left eigenvectors can be expressed in term of λ :

$$\mathcal{R} = \begin{bmatrix} \dfrac{\mu\alpha}{\lambda^2-\mu\gamma\theta} \\ \dfrac{\lambda\mu}{\lambda^2-\mu\gamma\theta} \\ 1 \\ \dfrac{1}{\lambda}\left[1+\dfrac{3\theta}{1-\alpha}\left(1-\dfrac{\mu\alpha}{\lambda^2-\mu\gamma\theta}\right)\right] \end{bmatrix} \tag{11}$$

$$L = \left[\frac{\lambda^2-1+\alpha-3\theta}{\alpha\mu}-1,\ \frac{\lambda^2-1+\alpha-3\theta}{\lambda\mu},\ 1,\ \frac{1-\alpha}{\lambda}\right] \tag{12}$$

III. <u>DERIVATION OF THE KORTEWEG-de VRIES EQUATION</u>

Let us introduce the following stretched coordinates:

$$\xi = \varepsilon^{1/2}(x-\lambda t)$$

$$\eta = \varepsilon^{3/2}\ x$$

and develop all the quantities in power of the small parameter ε :

$$n_e = 1 + \varepsilon n_e^{(1)} + \varepsilon^2 n_e^{(2)} + \ldots\ldots.$$

$$n_1 = \alpha + \varepsilon n_1^{(1)} + \varepsilon^2 n_1^{(2)} + \ldots\ldots.$$

$$u_i = \varepsilon u_i^{(1)} + \varepsilon^2 u_i^{(2)} + \ldots\ldots \qquad i = 1.2$$

$$n_2 = 1 - \alpha + \varepsilon n_2^{(1)} + \varepsilon^2 n_2^{(2)} + \ldots\ldots$$

$$E = \varepsilon E^{(1)} + \varepsilon^2 E^{(2)}$$

$$U = U^o + \varepsilon U^{(1)} + \varepsilon^2 U^{(2)}$$

To the first order in ε the system (1) to (6) reduces to:

$$\begin{bmatrix} -\lambda & \alpha & 0 & 0 \\ \frac{\mu\gamma\theta}{\alpha} & -\lambda & \mu & 0 \\ 0 & \alpha & -\lambda & 1-\alpha \\ -\frac{3\theta}{1-\alpha} & 0 & 1+\frac{3\theta}{1-\alpha} & -\lambda \end{bmatrix} \frac{\partial}{\partial \xi} \begin{bmatrix} n_1^{(1)} \\ u_1^{(1)} \\ n_e^{(1)} \\ u_2^{(1)} \end{bmatrix} = 0$$

or

$$(A_o - \lambda I) \frac{\partial}{\partial \xi} U^{(1)} = 0$$

Under the condition

$$U^{(1)} = 0 \quad at \quad \xi = \pm\infty$$

one arrives at:

$$U^{(1)} = n_e^{(1)} R \tag{13}$$

To the second order in ε,(1) to (6) reduces to equation (14):

$$(A_o - \lambda I)\frac{\partial}{\partial\xi} U^{(2)} + A_o \frac{\partial}{\partial\eta} U^{(1)} + A_1' \frac{\partial}{\partial\xi} U^{(1)} + B_o \frac{\partial^3}{\partial\xi^3} U^{(1)} = 0 \tag{14}$$

where

$$A_1' = \begin{bmatrix} u_1^{(1)} & n_1^{(1)} & 0 & 0 \\ \text{Signum}\,(\gamma-2)\,\dfrac{\gamma\mu\theta}{\alpha^{(\gamma-1)}}\, n_1^{(1)} & u_1^{(1)} & -\mu n_e^{(1)} & 0 \\ u_1^{(1)} - u_2^{(1)} & n_1^{(1)} & u_2^{(1)} & n_e^{(1)} - n_1^{(1)} \\ \dfrac{-3\theta}{(1-\alpha)^2}\,(n_e^{(1)} - n_1^{(1)}) & 0 & -n_e^{(1)} + \dfrac{3\theta}{(1-\alpha)^2}\,(n_e^{(1)} - n_1^{(1)}) & u_2^{(1)} \end{bmatrix}$$

and

$$B_o = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & \dfrac{-3\theta}{1-\alpha} & 0 \end{bmatrix}$$

Using (13) to eliminate $n_1^{(1)}$, $u_1^{(1)}$ and $u_2^{(1)}$, A_1' can be written as

$$A_1' = n_e^{(1)} A_1$$

and (14) becomes

$$(A_o - \lambda I)\frac{\partial}{\partial \xi} U^{(2)} + A_o R \frac{\partial}{\partial \eta} n_e^{(1)} + A_1 R\, n_e^{(1)} \frac{\partial}{\partial \xi} n_e^{(1)} + B_o R \frac{\partial^3}{\partial \xi^3} n_e^{(1)} = 0 \quad (15)$$

As the determinant of ($A_o - \lambda I$) is null, a compatibility condition has to be found in order to solve for $\partial U^{(2)}/\partial \xi$ from (15). This condition is obtained by multiplying (15) by the left eigenvector L. The resulting equation is the K.dV equation:

$$\frac{\partial n_e^{(1)}}{\partial \eta} + P\, n_e^{(1)} \frac{\partial n_e^{(1)}}{\partial \xi} + \frac{1}{2} Q \frac{\partial^3 n_e^{(1)}}{\partial \xi^3} = 0 \quad (16)$$

where

$$P = \frac{L A_1 R}{L A_o R}$$

and

$$Q = \frac{2 L B_o R}{L A_o R}$$

IV. STATIONNARY SOLUTION OF THE K.dV EQUATION

In a frame moving at velocity $M = \lambda + \varepsilon a$, a stationnary solution of (16) is given by:

$$n_e^{(1)} = \frac{3a}{\lambda P} \operatorname{Sech}^2 \left[\left(\frac{a}{2\lambda Q} \right)^{1/2} (x - Mt) \right]$$

For $\lambda \gg \sqrt{\theta}, \sqrt{\theta\mu}$, the soliton's amplitude $3a/\lambda P$ and width $\sqrt{2\lambda Q/a}$ are shown in figures (1) and (2) for a mass ratio of 4 and 4 0 and various ion temperature*. As previously reported by TRAN and HIRT (1974) the presence of light ions greatly reduces the soliton's height. The finite ion effect is to decrease its height and width, as a consequence of an increase of the non linearity, or, alternatively, a decrease in dispersive effect (TAGARE, 1973).

An isothermal fluid of light ions gives striking by different results (Fig. 3 and 4). Under the velocity condition $\sqrt{\theta} \ll \lambda \ll \sqrt{\theta\mu}$, the amplitude of the soliton is increased and its width decreased roughly by an order of magnitude. Physically, the light ion fluid cools down the electrons: the electronic density being proportional to $\exp\left\{-e\,\Phi/kT_e\right\}$, the deviation from charge neutrality is thereby increased. On the other hand, as the phase velocity in this case $\left(\sqrt{\theta} \ll \lambda \ll \sqrt{\theta\mu}\right)$ is smaller than the fast acoustic mode one $\left(\lambda \gg \sqrt{\theta\mu}\right)$, the slow solitons are subsonic with respect to the fast ion mode.

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*In the paper of TRAN and HIRT (1974), the normalization of spatial distances, velocities and times was different. This explains the apparent contradiction between their results and the present one.

In conclusion the finite ion temperature yields two acoustic branches for the dispersion relation. The first one is a correction of the well known unique ion acoustic mode corresponding to $T_i = 0$ and the resulting soliton structure is only slightly modified. The second branch gives higher and sharper soliton, an effect resulting from an apparent decrease in electron temperature.

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FIGURE CAPTION

Figure 1 : Dependence of the soliton's amplitude versus light ion concentration for different mass ratio ($\mu = 4$, $\mu = 40$) and ion temperature ($T_i/T_e = 0$, 0.01, 0.05, 0.1, 0.25). An adiabatic equation of state is assumed for the two ion species with the arbitrary velocity condition $5\,\theta\mu < (\omega/k)^2$.

Figure 2 : Dependence of the soliton's width versus light ion concentration for different mass ratio and ion temperature. The two ion fluids follow an adiabatic equation of state.

Figure 3 : Dependence of the soliton's amplitude versus light ion concentration for a mass ratio 40 and different ion temperature T_i/T_e ($T_i/T_e =$ 0.01, 0.05, 0.1, 0.25). The light ion equation of state is isothermal with the arbitrary velocity condition $5\,\theta < (\omega/k)^2 < 5\,\theta\mu$.

Figure 4 : Dependence of the soliton's width versus light ion concentration for mass ratio 40 and different ion temperature. The light ion fluid is isothermal.

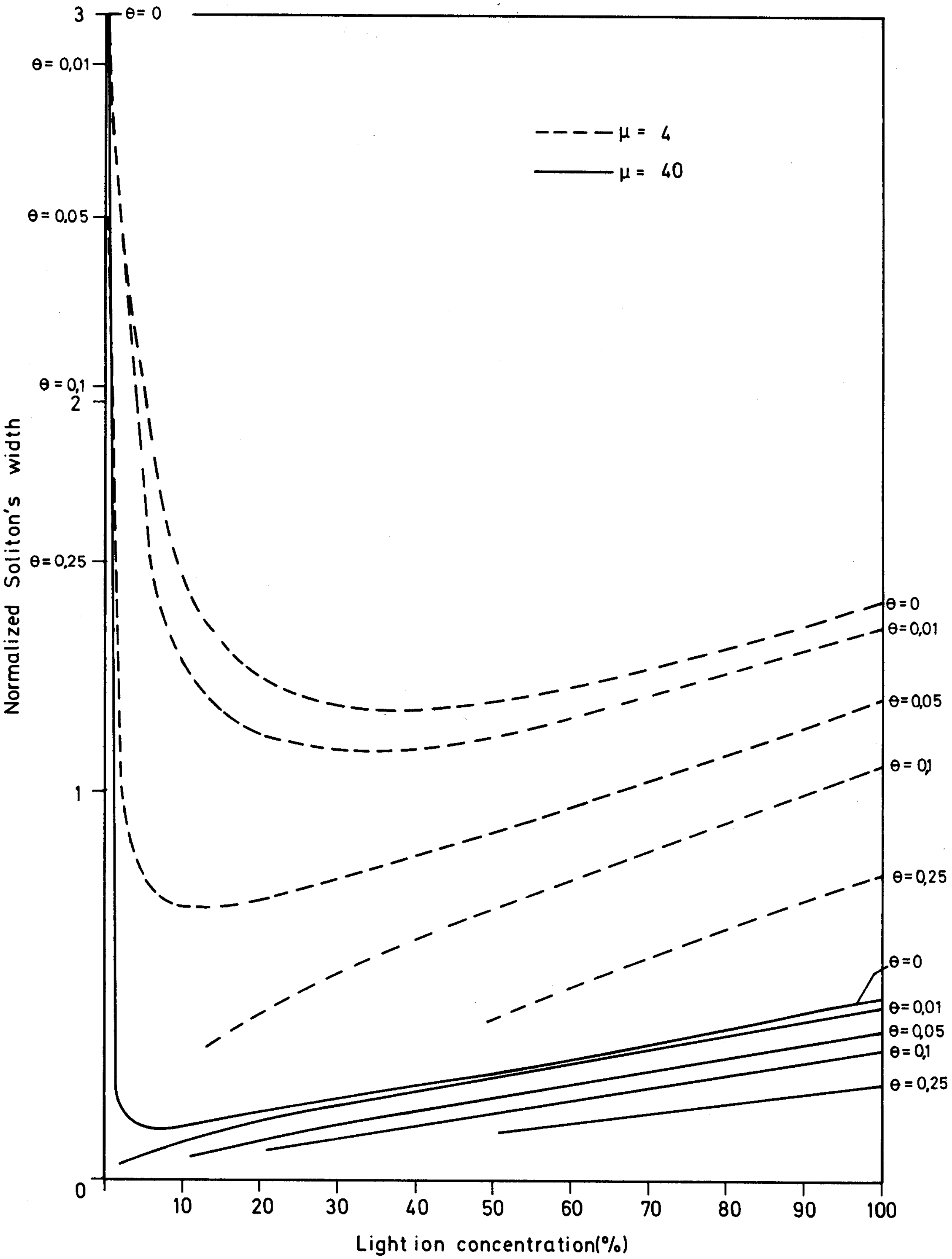

θ= 0
θ= 0,01
θ= 0,05
θ= 0,1
θ= 0,25
3
2
1
0
Normalized Soliton's width
μ = 4
μ = 40
θ=0
θ=0,01
θ=0,05
θ= 0,1
θ= 0,25
θ=0
θ=0,01
θ=0,05
θ=0,1
θ= 0,25
10
20
30
40
50
60
70
80
90
100
Light ion concentration(%)

Fig.1

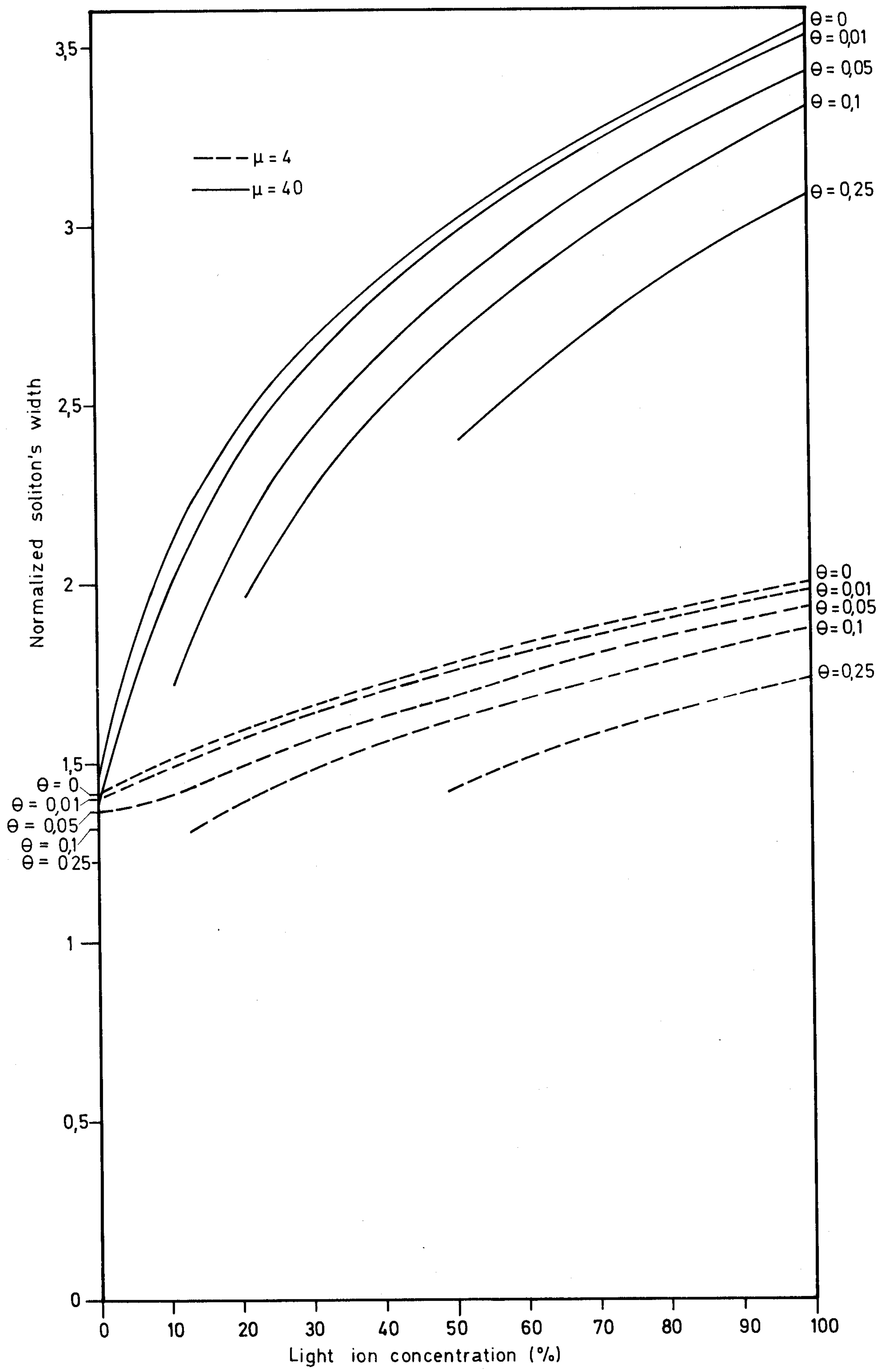
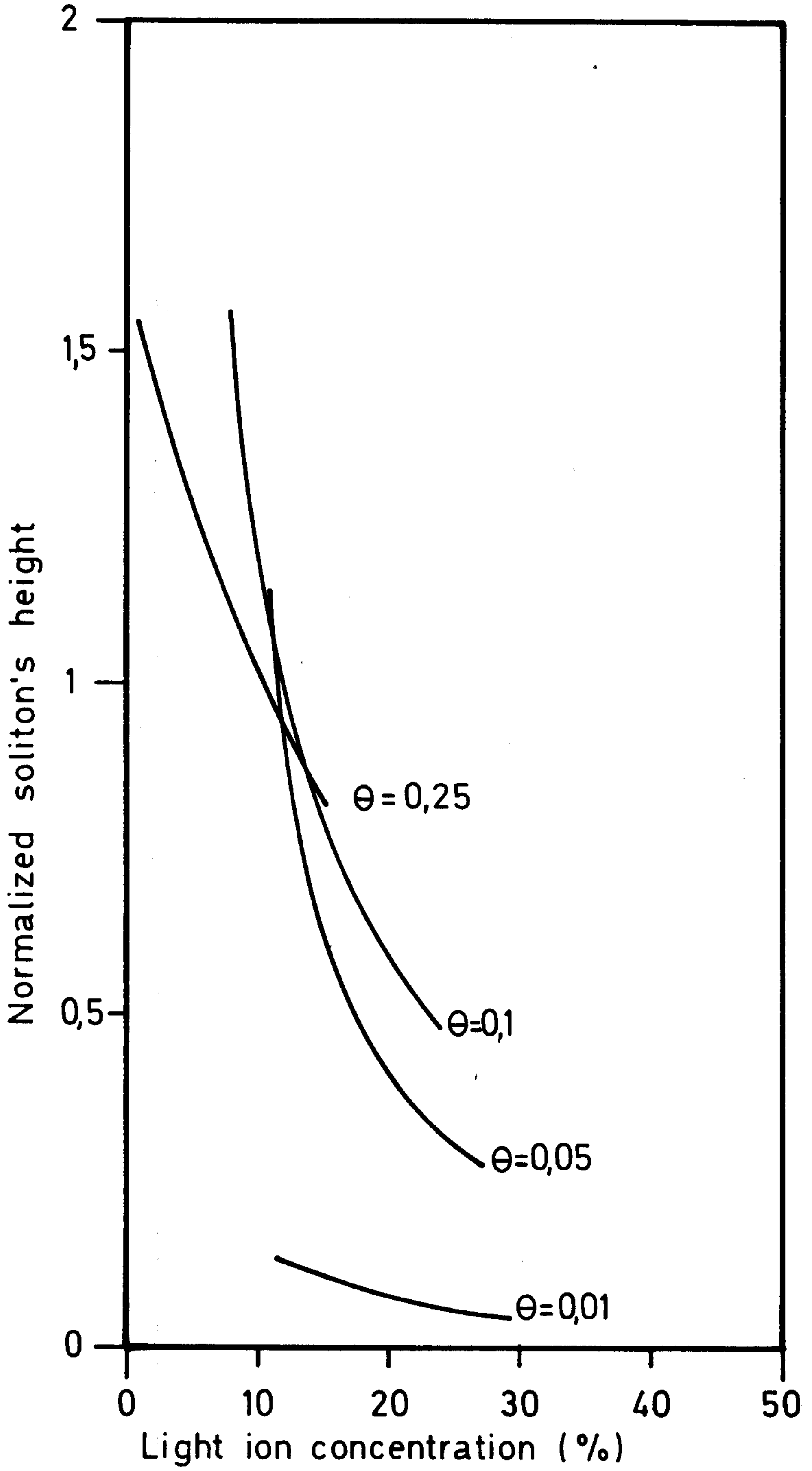

Normalized soliton's width
Light ion concentration (%)
μ = 4
μ = 40
θ = 0
θ = 0,01
θ = 0,05
θ = 0,1
θ = 0,25
0
0,5
1
1,5
2
2,5
3
3,5
0
10
20
30
40
50
60
70
80
90
100

Fig. 2



2
1,5
1
0,5
0
Normalized soliton's height
Θ = 0,25
Θ=0,1
Θ=0,05
Θ=0,01
0
10
20
30
40
50
Light ion concentration (%)

Fig. 3

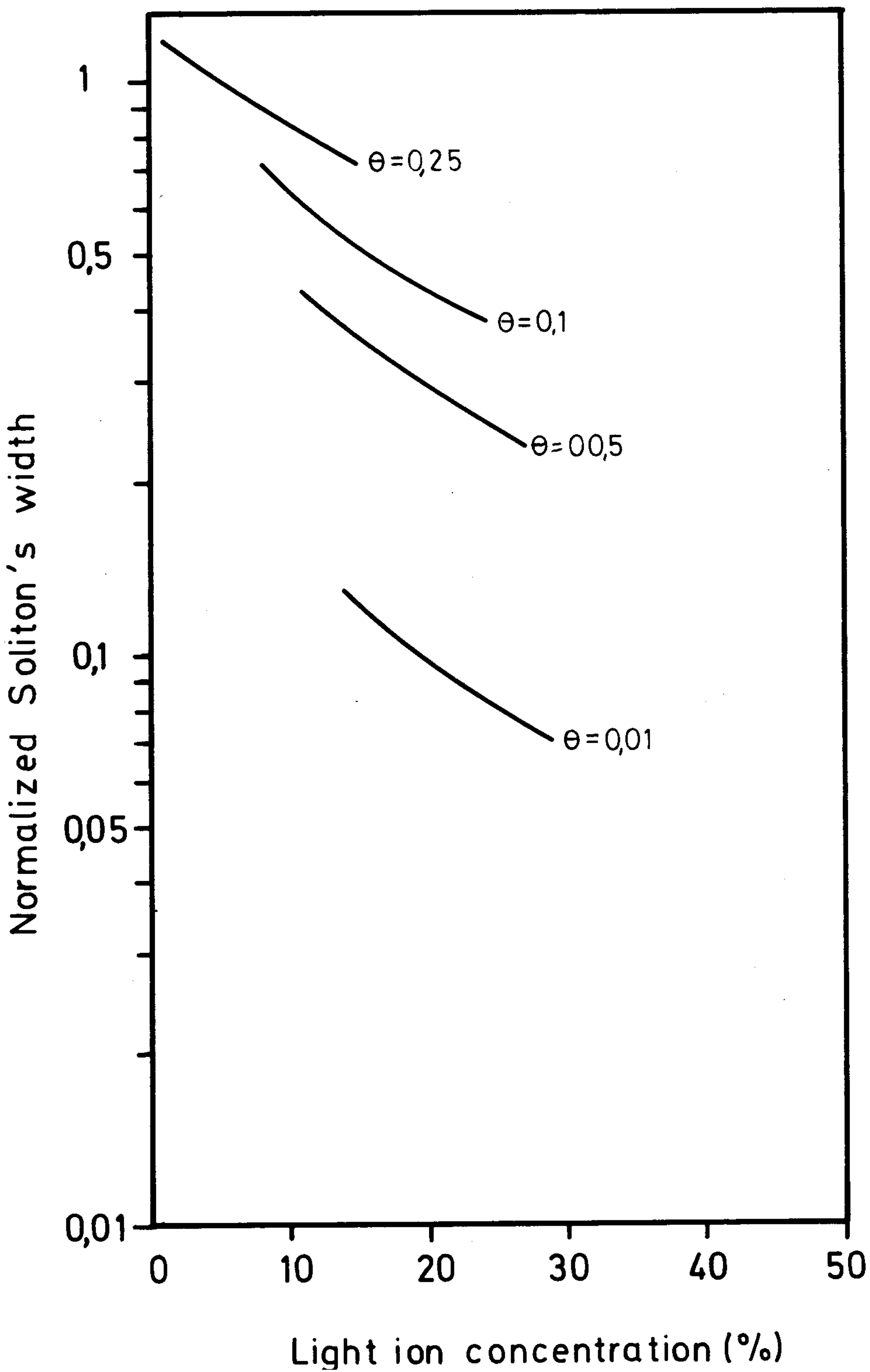

Normalized Soliton's width
1
0,5
0,1
0,05
0,01
θ=0,25
θ=0,1
θ=00,5
θ=0,01
0
10
20
30
40
50
Light ion concentration (%)

Fig. 4