NON-LINEAR WAVE PROPAGATION FOR A SLIGHTLY

INHOMOGENEOUS PLASMA

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A medium which may be slightly inhomogeneous in space and time is charac-
terized by its current response. We derive a system of equations describing
the linear and non-linear coupling of WKB modes.

Introduction

DuBois has derived previously a system of equations for the non-linear wave propagation in an inhomogeneous plasmas[1]. In this paper we treat the same problem in a somewhat more general and concise form, allowing the plasma to vary slowly in both space and time. We find that the analysis is simplified if we split the linear Maxwell operator into its hermitian and anti-hermitian parts and defining the zero order WKB modes by the hermitian part alone. The ikonal thus becomes real. The amplitude variations are then not only due to geometric optic effects, but are also due to the anti-hermitian part of the polarization operator and to the non-linear coupling. These amplitude variations are determined in the first WKB order. The effects of damping do not appear in the form of a complex propagation vector, and the use of bi-orthogonal sets of polarization vectors is avoided.

Analysis

Let us assume that the current density j produced by the plasma in response to the applied electromagnetic field is known. The propagation of waves is then governed by Maxwell's equations

$$\nabla \times (\nabla \times E) - \ddot{E} = \partial j / \partial t \qquad (1)$$

$$\dot{B} = -\nabla \times E \qquad (2)$$

and the initial conditions $\nabla B = \nabla E - S = 0$ which must be satisfied at some time. The E field determines the B field up to a constant field $B_0(r)$. Therefore the current response j of the plasma must be expressible in terms of $E(r,t)$ and $B_0(r)$. The most general response can be written in the form

$$\frac{\partial j_\alpha(R)}{\partial t} = \int s_{\alpha\beta}(R, R-R_1) E_\beta(R_1)\, dR_1$$

$$+ \int Q_{\alpha\beta c}(R, R-R_1, R-R_2) E_\beta(R_1) E_c(R_2)\, dR_1\, dR_2 \qquad (3)$$

$$+ \int T_{\alpha\beta cd}(R, R-R_1, R-R_2, R-R_3) E_\beta(R_1) E_c(R_2) E_d(R_3)\, dR_1\, dR_2\, dR_3$$

$$+ \ \text{------}$$

where

$$R \equiv \{t, \underline{r}\} \equiv \{R_\alpha\} \qquad , \quad \alpha = 0, 1, 2, 3 \qquad (4)$$

The dependence of s, Q, T on $B_0(r)$ is not explicitely indicated. These res-
ponse functions depend on R through the dependence of $B_0(r)$ and of the other
plasma parameters which determine its equilibrium. The plasma need not be
in an exact equilibrium state, it suffices that this state varies slowly in
time. Thus we assume that the functions s, Q, T depend only weakly on the
first variable R. This dependence disappears altogether in a uniform plasma.

It will be convenient to introduce the following transforms:

$$\hat{s}_{\alpha\beta}(R, \varkappa) = \int s_{\alpha\beta}(R, X) e^{-i\varkappa X}\, dX \qquad (5)$$

$$\hat{Q}_{\alpha\beta c}(R, \varkappa_1, \varkappa_2) = \int Q_{\alpha\beta c}(R, X_1, X_2) e^{-i(\varkappa_1 X_1 + \varkappa_2 X_2)}\, dX_1\, dX_2 \qquad (6)$$

where

$$\varkappa = \{\omega, \underline{k}\} = \{\varkappa_\alpha\} \qquad , \quad \alpha = 0, 1, 2, 3 \qquad (7)$$

and[*]

$$\varkappa R = -\omega t + \underline{\kappa} \underline{r} \tag{8}$$

The linear part of $\partial j / \partial t$ can now be written as

$$\frac{\partial j_a^{(lin)}(R)}{\partial t} = \int \delta_{a\beta}(R, \varkappa) E_\beta(\varkappa) e^{i\varkappa R} d\varkappa$$

$$= \delta_{a\beta}\left(R, -i\frac{\partial}{\partial R}\right) E_\beta(R) \tag{9}$$

Maxwell's equations (1) take the form

$$M_{a\beta}\left(R, -i\frac{\partial}{\partial R}\right) E(R) = \frac{\partial}{\partial t} j_a^{(n\ell)}(R) \tag{10}$$

where

$$M_{a\beta}\left(R, -i\frac{\partial}{\partial R}\right) = \left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r_c \partial r_c}\right) \delta_{a\beta} - \frac{\partial^2}{\partial r_a \partial r_\beta} - \delta_{a\beta}\left(R, -i\frac{\partial}{\partial R}\right) \tag{11}$$

is the linear Maxwell operator including the linear conductivity and

$$\frac{\partial j_q^{(n\ell)}(R)}{\partial t} = \int Q_{qrs}(R, R-R_1, R-R_2) E_r(R_1) E_s(R_2) dR_1 dR_2 + \ldots \tag{12}$$

represents the non-linear part of the current response.

We make now the additional assumption: decomposing M into its hermitian and its anti-hermitian part

$$M = H + ia \tag{13}$$

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[*]Note that: $\partial/\partial R = \{-\partial/\partial t, \partial/\partial \underline{r}\}$ and $\partial/\partial \varkappa = \{-\frac{\partial}{\partial \omega}, \frac{\partial}{\partial \underline{\kappa}}\}$

we suppose that the hermitian operator a is much smaller than the hermitian operator H (that is: $(E^*, aE)/(E^*, HE) \ll 1$ for all E).

We now assume that equation (10) possesses approximate solutions which can be expressed as a sum of WKB modes:

$$E_\Delta(R) = \sum_N A^N_\Delta(R)\, exp\left[i\, S^N(R)\right] \qquad (14)$$

In this expression the amplitude $A^N_S(R)$ is assumed to vary very slowly, and the eikonal $S^N(R)$ is real. Since the electric field must be real we require $S^{-N} = -S^N$, $A^{-N} = A^{N\,*}$, $A^0 = 0$. Considering that

$$-i\,\frac{\partial}{\partial R}\left\{A_s(R)\,exp\left[i\,S(R)\right]\right\} = exp\left[i\,S(R)\right]\left\{\left[\frac{\partial S}{\partial R} - i\,\frac{\partial}{\partial R}\right]A_\Delta(R)\right\} \qquad (15)$$

equation (10) takes the form

$$\sum_N \left\{M_{r\Delta}(R, \frac{\partial S^N}{\partial R} - i\,\frac{\partial}{\partial R})A^N_\Delta(R)\right\}e^{iS^N(R)} = \partial j_r^{(n\ell)}(R)/\partial t \qquad (16)$$

Sofar no approximation has been made. We now examine equation (16) in two successive approximations. In the zeroth order we neglect: the non-linear term, the anti-hermitian part of M and we neglect the derivatives $\partial A/\partial R$ compared to $A\,\partial S/\partial R$. Furthermore, we require that each term of the series (14) satisfy separately the linearized equations:

$$H_{a\ell}(R, \frac{\partial S^N}{\partial R})\,A^N_\ell(R) = 0 \qquad (17)$$

The solution of (17) requires that a condition

$$D(R, \frac{\partial S^N}{\partial R}) = 0 \qquad (18)$$

be satisfied, implying that the determinant of H vanish (D need not itself be the determinant). Solutions $S^N(R)$ of (18) can be constructed according to the classic theory of Hamilton and Jacobi by solving the canonical equations

$$\frac{dR}{d\tau} = \frac{\partial D}{\partial \varkappa} \qquad , \qquad \frac{d\varkappa}{d\tau} = - \frac{\partial D}{\partial R}$$

for the trajectories $R(\tau), \varkappa(\tau)$. τ is a parameter which depends on the choice of D: Had we solved $D(R,\varkappa)$ for $\omega = \Omega(R,\underline{k})$ and used $\tilde{D} = \Omega(R,\underline{k}) - \omega$ then $\tilde{\tau} = t$. The ray velocity

$$V = \frac{dr}{dt} = \frac{dr/d\tau}{dt/d\tau} = - \frac{\partial D/\partial \underline{k}}{\partial D/\partial \omega} = \frac{\partial \Omega}{\partial R}.$$

is the group velocity of the waves. The eikonal can, in principle be found from the action integral

$$S(R) = \int^{R} \varkappa \, \frac{dR}{d\tau} \, d\tau$$

taken along the rays[2]. Each solution reprensents a wave train. To each solution belongs a real wave vector

$$\varkappa^{N}(R) = \frac{\partial S^{N}}{\partial R} \tag{19}$$

To each solution S^{N} also belongs an eigen-vector of H, which we assume normalized:

$$H_{\alpha\beta}\left(R, \frac{\partial S^{N}}{\partial R}\right) e_{\beta}^{N}(R) = 0 \qquad , \qquad e_{c}^{N*} e_{c}^{N} = 1 \tag{20}$$

Thus the electric field has the form

$$E_{\alpha}(R) = \sum_{N} e_{\alpha}^{N}(R) \, E^{N}(R) \, \exp\left[i \, S^{N}(R)\right] \tag{21}$$

In this (lowest) order the scalar amplitudes $E^{N}(R)$ are yet undetermined.

The variation of these amplitudes can be determined in the next higher approximation: We return to equation (16) and include the non-linear current, the anti-hermitian part of M and the derivatives $\partial/\partial R$ occuring in H to first order. Thus:

$$i \sum_M e^{iS^M(R)} \left[a_{pq}(R,\varkappa) + \frac{1}{2} \frac{\partial^2 H_{pq}(R,\varkappa)}{\partial \varkappa_\alpha \partial \varkappa_\beta} \frac{\partial \varkappa_\alpha}{\partial R_\beta} + \frac{\partial H_{pq}(R,\varkappa)}{\partial \varkappa_\alpha} \frac{\partial}{\partial R_\alpha} \right] \cdot$$
$$\cdot e_q^M(R) E^M(R) = \partial j_p^{(nt)}(R)/\partial t \qquad \varkappa = \frac{\partial S^M}{\partial R} \qquad (22)$$

The right hand side of equation (22) must now be expressed in terms of the WKB modes. We limit our considerations to the second order term of the non-linear current. Introducing (14) into the second term on the right of (3) we obtain:

$$\frac{\partial}{\partial t} j_p^{(2)}(R) = \sum_{MK} \int Q_{pqr}(R, X_1, X_2) e_q^M(R-X_1) e_r^K(R-X_2) \cdot$$
$$\cdot E^M(R-X_1) E^K(R-X_2) \exp i \left[S^M(R-X_1) + S^K(R-X_2) \right] dX_1 dX_2 \qquad (23)$$

We develop the argument of the exponential function to bring out explicitely the rapid variation of the phase

$$S^M(R-X_1) + S^K(R-X_2) = S^M(R) + S^K(R) - \varkappa^M(R) X_1 - \varkappa^K(R) X_2 + \text{------} \quad (24$$

The integration over X_1, X_2 can be carried out if we remember that $\varkappa^M X_1$, varies much faster than $e^M(R-X_1)$ or $E^M(R-X_1)$. Since the term under discussion is already a correction to the linearized equation, we may ignore these variations altogether. Thus

$$\frac{\partial j_p^{(2)}(R)}{\partial t} = \sum_{K\Pi} \hat{Q}_{pqr}(R, \varkappa^\Pi, \varkappa^K) e_q^\Pi(R) e_r^K(R) E^\Pi(R) E^K(R) \cdot$$
$$\cdot \exp i \left[S^\Pi(R) + S^K(R) \right] \qquad (25)$$

We substitute this expression into the right hand side of equation (22). Then we multiply the resulting equation by $e_p^N {}^*(R)\exp\left[-iS^N(R)\right]$, where upon it takes the form

$$i \sum_M \exp i\left[S^M - S^N\right]\left\{\Gamma^{N\Pi} + \Delta^{NM}\frac{\partial}{\partial R_\alpha}\right\} E^M(R)$$

$$= \sum_{K\Pi} Q^{N\Pi K}\exp i\left[S^K + S^\Pi - S^N\right] E^\Pi(R) E^K(R) \tag{26}$$

Higher order terms on the right hand side could easily be included. In equation (26) we have used the following abbreviations:

$$D^{N\Pi}(R,x) = e_p^{N*}(R) H_{pq}(R,x) e_q^\Pi(R) \tag{27}$$

$$\Gamma^{N\Pi}(R) = \left[e_p^{N*}(R) a_{pq}(R,x) e_q^\Pi(R) + e_p^{N*}(R)\frac{\partial H_{pq}(R,x)}{\partial x_\alpha}\frac{\partial e_q^\Pi(R)}{\partial R_\alpha}\right.$$

$$\left. + \frac{1}{2}\frac{\partial^2 D^{N\Pi}(R,x)}{\partial x_\alpha \partial x_\beta}\frac{\partial x_\alpha}{\partial R_\beta}\right]_{x\,=\,\partial S^\Pi(R)/\partial R} \tag{28}$$

and

$$\Delta_\alpha^{N\Pi}(R) = \left[\frac{\partial D^{N\Pi}(R,x)}{\partial x_\alpha}\right]_{x\,=\,\partial S^\Pi(R)/\partial R} \tag{29}$$

$$Q^{N\Pi K}(R) = \hat{Q}_{pqr}(R,x^\Pi, x^K,) e_p^{N*}(R) e_q^\Pi(R) e_r^K(R) \tag{30}$$

The dispersion relation for the Nth WKB mode can be written as

$$D^{NN}(R, \varkappa) = 0 \tag{31}$$

In practice, of course, this is only possible once we have constructed the eikonal and the polarization field belonging to the mode N. Therefore we could have taken $D^{NN}(R, \varkappa)$ instead of $D(R, \varkappa)$ as far as the Nth WKB mode is concerned thus we may write

$$\Delta_\alpha^{NN}(R) = \left.\frac{\partial D}{\partial \varkappa}\right|_{\varkappa = \partial S^N/\partial R} = \frac{dR}{d\tau}$$

Thus the derivative term in (26) is really a derivative along the ray trajectory. To express this fact in terms of r,t explicitely we solve (31) for ω

$$\omega^N = \Omega^N(R; r, t) \tag{32}$$

Introducing the group velocities

$$\underline{V}^N(r, t) = \frac{\partial \Omega^N}{\partial \underline{R}} = -\frac{\partial D^{NN}/\partial \underline{R}}{\partial D^{NN}/\partial \omega} \tag{33}$$

we can express the diagonal elements of the operator on the left of (26) in the form

$$\Gamma^{NN} + \Delta_\alpha^{NN}\frac{\partial}{\partial R_\alpha} = \frac{\partial D^{NN}}{\partial \omega}\left[\frac{\partial}{\partial t} - \underline{V}^N\frac{\partial}{\partial r} + \gamma^N\right] \tag{34}$$

This system of equations (26) determines the amplitude variations of the WKB modes as functions of time and space. These amplitudes are now coupled, both linearly and non-linearly. However, since by hypothesis the amplitudes are slowly varying functions, we must neglect the terms which contain the rapidly

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varying exponentials. Hence linear coupling between two modes M, N can occur only in a region of space-time where $\mathscr{x}^N \sim \mathscr{x}^\Pi$, while non-linear coupling requires that $\mathscr{x}^K + \mathscr{x}^M \sim \mathscr{x}^N$.

In the special case of non-linear three wave coupling between modes 1,2,3 the system (26) reduces to

$$ i\left[\Gamma'' + Q''_\alpha \frac{\partial}{\partial R_\alpha}\right] E^1 = S^{1,2,3}(R)\exp i\left[S^3 + S^2 - S^1\right] E^2 E^3 $$

$$ i\left[\Gamma^{22} + Q^{22}_\alpha \frac{\partial}{\partial R_\alpha}\right] E^2 = S^{2,-3,1}(R)\exp i\left[S^1 - S^3 - S^2\right] E^{3*} E^1 \qquad (35) $$

$$ i\left[\Gamma^{33} + Q^{33}_\alpha \frac{\partial}{\partial R_\alpha}\right] E^3 = S^{3,1,-2}(R)\exp i\left[-S^2 + S^1 - S^3\right] E^1 E^{2*} $$

These equations represent a generalization of those investigated by Rosenbluth[3].

R e f e r e n c e s

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2) S. Weinberg, Eikonal method in MHD, Phys.Rev. 126, 1899 (1962).

3) M.N. Rosenbluth, Parametric instabilities in inhomogeneous media, Phys.Rev.Lett. 29, 565 (1972).