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THE KORTEWEG-DE VRIES EQUATION FOR A TWO COMPONENT PLASMA

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A b s t r a c t

A Korteweg-de Vries equation for a two component-plasma with cold ions ($T_i = 0$ for both species) and isothermal electrons has been derived. The amplitude of the solitons is reduced drastically by a few percent of light ions and an initial perturbation is breaking up into more solitons than in a one component plasma.

Lausanne

Finite amplitude ion waves in a cold ion and isothermal electron plasma are described by the Korteweg-de Vries (KdV) equation (Washimi and Taniuti, 1966). Following Gardner et al. (1967), this equation can be solved analytically as an initial value problem and the amplitude and number of the solitons are predictable. In this report we are going to derive a KdV equation for a two component plasma and deduce some general conclusions about amplitude and number of solitons.

In the system of cold ions ($T_i = 0$ for both species) and isothermal electrons the dispersion relation is

$$\omega^2 = k^2 \left[\frac{k_B T_e}{m_2} \left(1 - \alpha + \alpha \mu \right) \right] \left[1 + k^2 \lambda^2_{De} \right]^{-1} ,$$

where μ is the mass ratio ($\mu = m_2/m_1 > 1$), α the concentration of light ions ($\alpha = N_1/N_1+N_2 = N_1/N_e$) and λ_{De} the electron Debye length.

The basic equations used to obtain the KdV equation are:

- Continuity and momentum equation for the two ion-species ($T_i = 0$)

$$\frac{\partial n_i}{\partial t} + \frac{\partial (n_i u_i)}{\partial x} = 0 \qquad i = 1,2 \tag{1}$$

$$(1 - \alpha + \alpha\mu) \left[\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} \right] = \frac{m_2}{m_i} E \tag{2}$$

- Electron momentum equation neglecting electron inertia

$$\frac{\partial n_e}{\partial x} = - n_e E \tag{3}$$

- Poisson's equation

$$\frac{\partial E}{\partial x} = + \sum_{i=1}^{2} n_i - n_e \quad . \tag{4}$$

In these equations, distances are normalized to λ_{De}, time to the inverse of ion plasma frequency ω_{pi}^{-1} $\left(\omega_{pi}^2 = \frac{N_e e^2}{\varepsilon_0 m_2}(1-\alpha+\alpha\mu)\right)$, density to unperturbed electron density N_e, velocity to ion sound velocity C_s $(C_s = \lambda_{De}\,\omega_{pi})$ and electric potential to $k_B T_e/e$.

Following Washimi and Taniuti (1966) we make the transformation*

$$\xi = \varepsilon^{1/2}(x-t)$$
$$\eta = \varepsilon^{3/2}\, x$$

and (1) to (4) become

$$-\frac{\partial n_i}{\partial \xi} + \frac{\partial (n_i u_i)}{\partial \xi} + \varepsilon \frac{\partial (n_i u_i)}{\partial \eta} = 0 \qquad (5)$$

$$i = 1,2$$

$$A^2\left[-\frac{\partial u_i}{\partial \xi} + u_i \frac{\partial u_i}{\partial \xi} + \varepsilon u_i \frac{\partial u_i}{\partial \eta}\right] = \frac{m_2}{m_i}\tilde{E} \qquad (6)$$

$$\frac{\partial n_e}{\partial \xi} + \varepsilon \frac{\partial n_e}{\partial \eta} = -n_e \tilde{E} \qquad (7)$$

$$\varepsilon \frac{\partial \tilde{E}}{\partial \xi} + \varepsilon^2 \frac{\partial \tilde{E}}{\partial \eta} = \sum_{i=1}^{2} n_i - n_e \qquad (8)$$

where $\tilde{E} = \varepsilon^{-1/2} E$ and $A^2 = 1-\alpha+\alpha\mu$.

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*As pointed out by these authors, the transformation $\xi = \varepsilon^{1/2}(x-t)$, $\eta = \varepsilon^{3/2} t$ gives the same result.

Developing all quantities in powers of ε up to second order

$$n_e = 1 + \varepsilon n_e^{(1)} + \varepsilon^2 n_e^{(2)} + \ldots\ldots$$

$$n_1 = \alpha + \varepsilon n_1^{(1)} + \varepsilon^2 n_1^{(2)} + \ldots\ldots$$

$$n_2 = (1-\alpha) + \varepsilon n_2^{(1)} + \varepsilon^2 n_2^{(2)} + \ldots\ldots$$

$$u_i = \varepsilon u_i^{(1)} + \varepsilon^2 u_i^{(2)} + \ldots\ldots \qquad i = 1,2$$

$$\tilde{E} = \varepsilon \tilde{E}^{(1)} + \varepsilon^2 \tilde{E}^{(2)} \quad ,$$

(5) to (8) separate into first and second order equations, respectively,

$$\frac{\partial n_e^{(1)}}{\partial \xi} = \frac{A^2}{\mu} \frac{\partial u_1^{(1)}}{\partial \xi} = A^2 \frac{\partial u_2^{(1)}}{\partial \xi} = \frac{A^2}{1-\alpha} \frac{\partial n_2^{(1)}}{\partial \xi} = \frac{A^2}{\mu\alpha} \frac{\partial n_1^{(1)}}{\partial \xi} = -\tilde{E}^{(1)} \tag{9}$$

and

$$-\frac{\partial n_1^{(2)}}{\partial \xi} + \alpha \frac{\partial u_1^{(2)}}{\partial \xi} + \frac{\partial \left(n_1^{(1)} u_1^{(1)}\right)}{\partial \xi} + \alpha \frac{\partial u_1^{(1)}}{\partial \eta} = 0 \tag{10}$$

$$-\frac{\partial n_2^{(2)}}{\partial \xi} + (1-\alpha) \frac{\partial u_2^{(2)}}{\partial \xi} + \frac{\partial \left(n_2^{(1)} u_2^{(1)}\right)}{\partial \xi} + (1-\alpha) \frac{\partial u_2^{(1)}}{\partial \eta} = 0 \tag{11}$$

$$A^2 \left[-\frac{\partial u_1^{(2)}}{\partial \xi} + u_1^{(1)} \frac{\partial u_1^{(1)}}{\partial \xi} \right] = \mu \tilde{E}^{(2)} \tag{12}$$

$$A^2 \left[-\frac{\partial u_2^{(2)}}{\partial \xi} + u_1^{(1)} \frac{\partial u_2^{(1)}}{\partial \xi} \right] = \tilde{E}^{(2)} \tag{13}$$

$$\frac{\partial n_e^{(2)}}{\partial \xi} + \frac{\partial n_e^{(1)}}{\partial \eta} = -\tilde{E}^{(2)} - n_e^{(1)} \tilde{E}^{(1)} \tag{14}$$

$$\frac{\partial \tilde{E}^{(1)}}{\partial \xi} = n_1^{(2)} + n_2^{(2)} - n_e^{(2)} \quad . \tag{15}$$

Integrating (9) with the boundary conditions at $\xi = \pm \infty$

$$u_i^{(1)} = n_i^{(1)} = 0 \qquad i = 1,2$$

$$n_e = 1 \quad , \; n_1 = \alpha \; , \; n_2 = 1-\alpha \; , \; u_i = 0$$

we obtain

$$n_e^{(1)} = \frac{A^2}{\mu} u_1^{(1)} = A^2 u_2^{(1)} = \frac{A^2}{1-\alpha} n_2^{(1)} = \frac{A^2}{\mu\alpha} n_1^{(1)} . \tag{16}$$

Eliminating second order terms in (10) to (15) with (9) and (16) we get the following KdV equation

$$2 \frac{\partial n_e^{(1)}}{\partial \eta} + n_e^{(1)} \frac{\partial n_e^{(1)}}{\partial \xi} \left(\frac{3(\mu^2\alpha + 1-\alpha)}{(1-\alpha+\alpha\mu)^2} - 1 \right) + \frac{\partial^3 n_e^{(1)}}{\partial \xi^3} = 0 \; . \tag{17}$$

(17) reduces to the familiar KdV equation for $\alpha = 0$ or 1.

A stationary solution of (17) in a frame moving with velocity M is

$$n_e^{(1)} = \frac{6\,a}{M\left[\frac{3(\mu^2\alpha+1-\alpha)}{(1-\alpha+\mu\alpha)^2} - 1\right]} \operatorname{sech}^2 \left[\left(\frac{a}{2M^3}\right)^{1/2} \varepsilon^{1/2} \left(x - x_0 - M(t-t_0)\right)\right] , \tag{18}$$

where a is defined by $M = 1 + \varepsilon a$, and x_0 and t_0 are integration constants (in the lowest order, M = 1). (18) shows that for fixed M the amplitude of the soliton varies with α and μ while its width remains unchanged. The figure is illustrating the dependence of the solitons'amplitude on α for fixed μ. The striking feature is the reduction of the solitons'height due to a small concentration of light ion impurities: 2 % of Hydrogen in an Argon plasma reduces the amplitude from 3 a to 0.2 a. Experimentally it is well known that light ion impurities prevent the formation of solitons (Ikezi, 1973). Consequently, a very high vacuum is required for working with a heavy ion plasma. On the other hand, using a light ion plasma (Hydrogen or Helium), contamination occurs only with heavy ions. α is very close to 1, the amplitude is not much reduced, and solitons can be easily observed.

Let us consider now the number of solitons obtained from a pulse perturbation at $\eta = 0$

$$\delta n_e = \begin{cases} \delta n_e(t') & 0 < t' \leq \Delta t' \\ 0 & t' > \Delta t' \end{cases}$$

where t' and $\Delta t'$ are real, unnormalized times.

Using

$$\xi \rightarrow \xi' = 2^{1/3}\,\xi$$
$$n_e^{(1)} \rightarrow n_e^{(1)'} = -\frac{2^{1/3}}{12}\left[\frac{3(\mu^2\alpha + 1 - \alpha)}{(1-\alpha+\mu\alpha)^2} - 1\right] n_e^{(1)} \tag{19}$$

and following Gardner et al. (1967), (17) is transformed into

$$\frac{\partial n_e^{(1)'}}{\partial \eta} - 6 n_e^{(1)'} \frac{\partial n_e^{(1)'}}{\partial \xi'} + \frac{\partial^3 n_e^{(1)'}}{\partial \xi'^3} = 0 . \tag{20}$$

The number of solitons is given by the number of eigenvalues of the Schroedinger equation (Gardner et al., 1967)

$$\frac{\partial^2 \psi}{\partial \xi^2} + (\epsilon - \tilde{\delta} n_e)\psi = 0 \quad .$$

$\tilde{\delta} n_e$ is the transform of δn_e using (19) and depends now on the normalized time Δt.

As the amplitude of $\tilde{\delta} n_e$ and its normalized duration $\Delta t = \omega_{p_1} \Delta t'$ are increased for α different from 1 or 0 (see Fig.), the "potential" $\tilde{\delta} n_e$ becomes deeper and wider in a two component plasma: therefore the number of bound states is increased. Ikezi (1973) has shown experimentally that in a one component plasma the number of solitons is increased by increasing the amplitude and duration of the pulse.

Concluding, small but finite amplitude ion waves in a two component plasma can be described by the Korteweg-de Vries equation. From its coefficients, the amplitude of the solitons is drastically reduced by a few percent of light ions, while an initial perturbation is breaking up into more solitons.

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References

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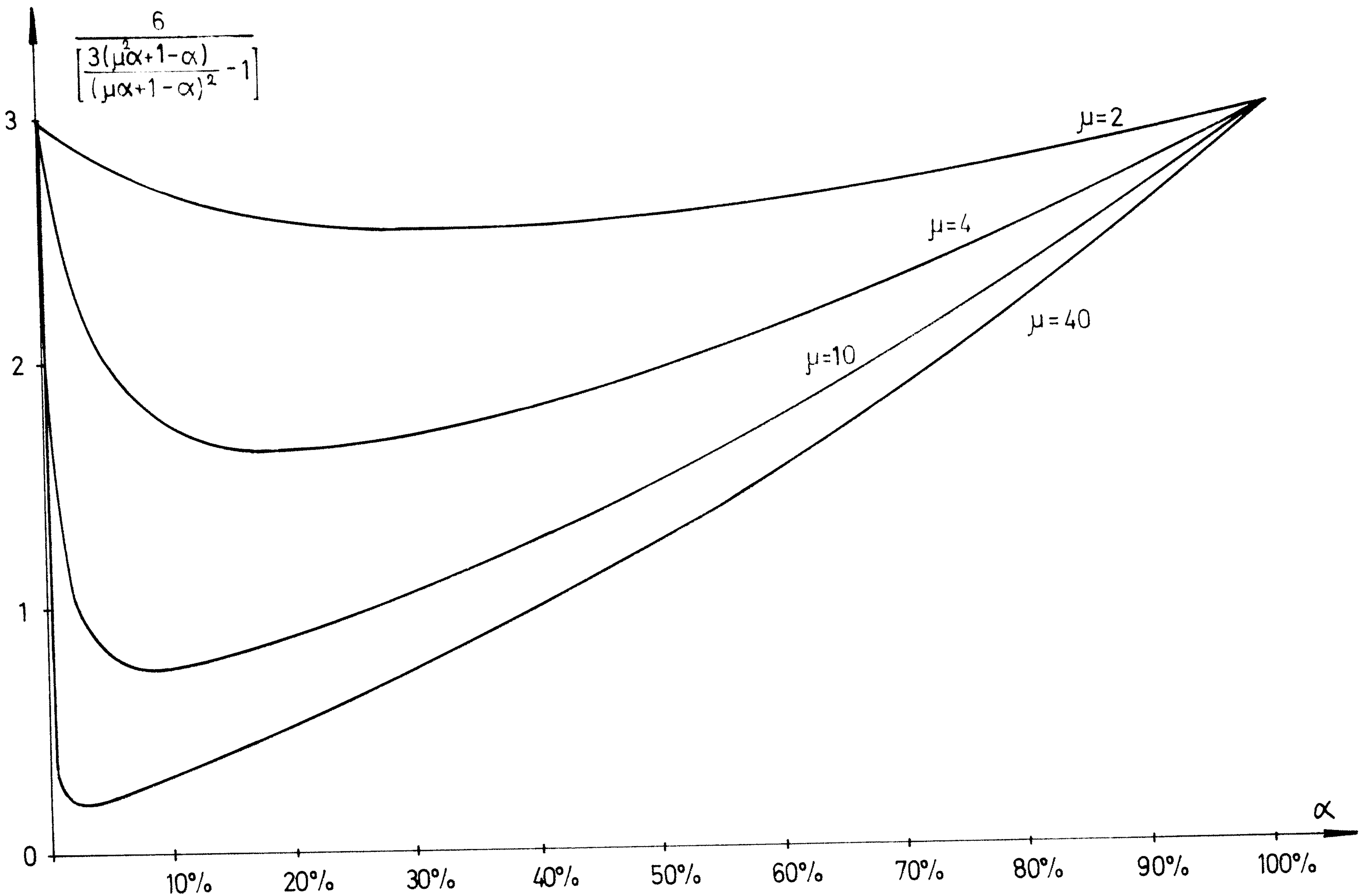
2) Ikezi H. 1973, Research Report, Nagoya, IPPJ-149.

3) Washimi M., Taniuti T. 1966, Phys.Rev.Lett., 17, 996.

Figure Caption

Variation of the solitons'amplitude with the concentration α of light ions. The parameter is the mass ratio μ , and the Mach number is fixed.



6
3(μ²α+1−α)
(μα+1−α)² −1
3
2
1
0
μ=2
μ=4
μ=10
μ=40
α
10%
20%
30%
40%
50%
60%
70%
80%
90%
100%