October 1973

LRP 71/73

PARAMETRIC EXCITATION OF ALFVEN AND ACOUSTIC WAVES

Nguyen The Hung

Centre de Recherches en Physique des Plasmas

ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE

October 1973

LRP 71/73

PARAMETRIC EXCITATION OF ALFVEN AND ACOUSTIC WAVES

Nguyen The Hung

Abstract

The non-linear interaction between two Alfven waves and a sound wave is studied, using the normal mode approach. This leads, in a simple way, to a set of coupled equations, and consequently to a dispersion relation for the waves under consideration. It is shown that a large amplitude Alfven wave can give rise to two distinct types of parametric instabilities, namely the oscillating and the purely growing waves. In each case, the expressions for the threshold pump intensity, the frequency shift, and the growth rate of the excited waves are obtained. In particular, the results for a propagating pump under perfect frequency matching conditions are compared with those of Sagdeev and Galeev.

Lausanne

1. INTRODUCTION

It is well known that an electromagnetic wave, above a certain threshold intensity, can excite other plasma waves through the parametric coupling mechanism. These effects, presumably responsible for various observed phenomena (e.g. anomalous reflection and enhanced absorption of laser radiation), have raised some recent interest, both in the domain of laser fusion and in ionosphere research (DuBois 1972).

In this paper we study the interaction of two Alfven waves and a sound wave, using the coupled mode theory where both the frequency mismatch and the finite wavelengths of the waves are taken into account. One of the interests of this problem lies in the fact that a large amplitude Alfven wave is an exact solution of the non-linear MHD equations: Resonant harmonic generation being absent, the main non-linear mechanism responsible for the decay of this wave should be its coupling with another Alfven and a sound wave. This coupling process can be studied by various well-known methods (Kadomtsev 1965, Sagdeev and Galeev 1969). Here, the normal mode approach will lead, in a simple way, to a set of first order differential equations.

In Section II, we shall obtain the normal modes of the Alfven and sound waves within a linear analysis of the MHD equations. Section III is devoted to the derivation of a coupled set of equations for the three-wave interaction. In Section IV, we shall consider the case of a large amplitude Alfven wave acting as a pump to excite another Alfven and a sound wave: It will be shown that, according to the nature of the pump wave (standing or propagating), there can exist two distinct types of parametric instabilities.

II. THE LINEAR EQUATIONS

Let us consider the case of Alfven and sound waves both propagating along an externally-imposed magnetic field, in a uniform, unbounded plasma. The equilibrium state of the plasma is characterized by a density ρ_0 , a scalar pressure p_o, a zero drift velocity ($\vec{V}_o = 0$), and a steady magnetic field $\vec{H}_o = H_o \hat{z}$.

In an ideally conducting, compressible plasma, the Alfven wave, as well as the acoustic wave, can be appropriately described by the following MHD equations (Van Kampen and Fenderhof 1967):

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{V} = 0 , \tag{1a}$$

$$\rho \frac{\partial \vec{V}}{\partial t} + \rho (\vec{V} \cdot \vec{\nabla}) \vec{V} = -\vec{\nabla} p - \frac{1}{4\pi} \vec{H} \wedge (\vec{\nabla} \wedge \vec{H}) , \tag{1b}$$

$$\frac{\partial \vec{H}}{\partial t} = \vec{\nabla} \wedge (\vec{V} \wedge \vec{H}) , \tag{1c}$$

together with an equation of state

$$p = f(\rho) . \tag{1d}$$

To obtain some approximate solution of this highly non-linear coupled system, we shall first study its normal modes within a linear analysis.

Let

$$\begin{aligned} \vec{H} &= \vec{H}_o + \vec{H}_\perp , \\ \vec{V} &= \vec{V}_\parallel + \vec{V}_\perp , \\ \rho &= \rho_o + \tilde{\rho} , \end{aligned} \tag{2}$$

where the subscripts $\parallel$ and $\perp$ refer to the components parallel and perpendicular to the static magnetic field. The linearized form of Eq. (1) can be expressed as follows

$$\frac{\partial \tilde{\rho}}{\partial t} + \rho_o \frac{\partial V_\parallel}{\partial z} = 0 , \tag{3a}$$

$$\rho_0 \frac{\partial V_{\parallel}}{\partial t} + c_S^2 \frac{\partial \tilde{\rho}}{\partial z} = 0 \quad , \tag{3b}$$

$$\frac{\partial H_{\perp}}{\partial t} - H_0 \frac{\partial V_{\perp}}{\partial z} = 0 \quad , \tag{3c}$$

$$\frac{\partial V_{\perp}}{\partial t} - \frac{c_A^2}{H_0} \frac{\partial H_{\perp}}{\partial z} = 0 \quad , \tag{3d}$$

where we have defined

$$c_S^2 = \left. \frac{df(\rho)}{d\rho} \right|_{\rho = \rho_0} \quad , \tag{3e}$$

and

$$c_A^2 = \frac{H_0^2}{4\pi \rho_0} \quad . \tag{3f}$$

This system can easily be solved by introducing the normal modes of the Alfven and sound waves, defined as follows

$$a = \tilde{\rho} + c_1 V_{\parallel} \quad , \tag{4a}$$

$$b = H_{\perp} + c_2 V_{\perp} \quad , \tag{4b}$$

with the coefficients c_1 and c_2 chosen such that

$$\frac{\partial a}{\partial t} = -i\omega_S a \quad , \tag{4c}$$

$$\frac{\partial b}{\partial t} = -i\omega_A b \quad , \tag{4d}$$

ω_S and ω_A being the frequencies of the linear waves under consideration.

By taking a linear combination of Eqs. (3a) and (3b), one immediately gets

$$\frac{\partial a}{\partial t} + \rho_0 \frac{\partial V_{\parallel}}{\partial z} + \frac{c_1 c_s^2}{\rho_0} \frac{\partial \tilde{\rho}}{\partial z} = 0 . \tag{5}$$

Assuming a spatial dependence of the form $\exp(ik_s z)$, and using Eq. (4c), one then obtains

$$\omega_s^2 = c_s^2 k_s^2 , \qquad \text{(the dispersion law)} \tag{6a}$$

and

$$c_1 = \pm \rho_0 \frac{k_s}{\omega_s} . \tag{6b}$$

Thus, for a given wave vector $\vec{k}_s$, there exist two normal modes of the sound wave:

$$a^+ = \tilde{\rho}^+ + \frac{k_s}{\omega_s} \rho_0 V_{\parallel}^+ \sim e^{ik_s z - i\omega_s t} , \tag{7a}$$

and

$$a^- = \tilde{\rho}^- - \frac{k_s}{\omega_s} \rho_0 V_{\parallel}^- \sim e^{ik_s z + i\omega_s t} . \tag{7b}$$

Clearly, a^+ and a^- correspond to the forward and backward propagating waves with respect to the wave vector $\vec{k}_s$ (of course, ω_s is here defined as a real positive quantity).

In a quite analogous way, one obtains the following normal modes of the Alfven wave:

$$b^+ = H_\perp^+ - \frac{k_A}{\omega_A} H_0 V_\perp^+ \sim e^{ik_A z - i\omega_A t} , \tag{8a}$$

$$b^- = H_\perp^- + \frac{k_A}{\omega_A} H_0 V_\perp^- \sim e^{ik_A z + i\omega_A t} , \tag{8b}$$

with

$$\omega_A^2 = c_A^2 \, k_A^2 \quad , \qquad \text{(the dispersion law)} \quad , \tag{8c}$$

and

$$c_2 = \mp H_0 \frac{k_A}{\omega_A} \quad . \tag{8d}$$

Before going to a non-linear analysis, let us express now the perturbed quantities $\tilde{\rho}$, $V_{\parallel}$, $V_{\perp}$ and $H_{\perp}$ in terms of the normal modes a and b. This is readily done by means of Eqs. (3) and (6)-(8):

$$\tilde{\rho}^{\pm} = \frac{1}{2} \, a^{\pm} \quad , \tag{9a}$$

$$V_{\parallel}^{\pm} = \pm \frac{\omega_s}{2 \rho_0 k_s} \, a^{\pm} \quad , \tag{9b}$$

$$V_{\perp}^{\pm} = \mp \frac{\omega_A}{2 H_0 k_A} \, b^{\pm} \quad , \tag{9c}$$

$$H_{\perp}^{\pm} = \frac{1}{2} \, b^{\pm} \quad . \tag{9d}$$

III. THE COUPLED EQUATIONS

We now consider the interaction between a sound wave and two Alfven waves (denoted by indices 1 and 2). Here, non-linear terms corresponding to the harmonic generation of these waves will be neglected. Eq. (1) then reduces to

$$\frac{\partial \tilde{\rho}}{\partial t} + \rho_0 \frac{\partial V_{\parallel}}{\partial z} = 0 \quad , \tag{10a}$$

$$\frac{\partial V_{\parallel}}{\partial t} + \frac{c_s^2}{\rho_0} \frac{\partial \tilde{\rho}}{\partial z} = - \frac{1}{4\pi \rho_0} \frac{\partial}{\partial z} (H_1 \cdot H_2) \quad , \tag{10b}$$

$$\frac{\partial H_1}{\partial t} - H_0 \frac{\partial V_1}{\partial z} = - \frac{\partial}{\partial z} (V_{\parallel} \cdot H_2) , \tag{10c}$$

$$\frac{\partial V_1}{\partial t} - \frac{c_A^2}{H_0} \frac{\partial H_1}{\partial z} = - V_{\parallel} \cdot \frac{\partial V_2}{\partial z} - \frac{c_A^2}{\rho_0 H_0} \tilde{\rho} \cdot \frac{\partial H_2}{\partial z} , \tag{10d}$$

$$\frac{\partial H_2}{\partial t} - H_0 \frac{\partial V_2}{\partial z} = - \frac{\partial}{\partial z} (V_{\parallel} \cdot H_1) , \tag{10e}$$

$$\frac{\partial V_2}{\partial t} - \frac{c_A^2}{H_0} \frac{\partial H_2}{\partial z} = - V_{\parallel} \cdot \frac{\partial V_1}{\partial z} - \frac{c_A^2}{\rho_0 H_0} \tilde{\rho} \cdot \frac{\partial H_1}{\partial z} , \tag{10f}$$

where the sign $\perp$, referring to the transverse components of the Alfven waves, has been omitted.

In a low-beta plasma ($c_S^2 \ll c_A^2$), Eqs. (10c)-(10f) can be further simplified by neglecting the first terms on the right hand sides, because of their smallness compared to the second terms.

Taking the same linear combination of Eq. (10) as we did to obtain Eq. (5), we now have

$$\frac{\partial a^{\pm}}{\partial t} \pm c_s \frac{\partial a^{\pm}}{\partial z} = \mp \frac{k_s}{4\pi \omega_s} \frac{\partial}{\partial z} (H_1 \cdot H_2) , \tag{11a}$$

$$\frac{\partial b_1^{\pm}}{\partial t} \pm c_A \frac{\partial b_1^{\pm}}{\partial z} = \pm \frac{c_A^2 k_1}{\rho_0 \omega_1} \tilde{\rho} \cdot \frac{\partial H_2}{\partial z} , \tag{11b}$$

$$\frac{\partial b_2^{\pm}}{\partial t} \pm c_A \frac{\partial b_2^{\pm}}{\partial z} = \pm \frac{c_A^2 k_2}{\rho_0 \omega_2} \tilde{\rho} \cdot \frac{\partial H_1}{\partial z} . \tag{11c}$$

As such, this system is still intractable, except for the case of weak coupling. Then, one can look for solutions in the form of plane waves with "slowly" varying amplitudes, i.e.

$$a^{\pm} = A^{\pm}(t) \exp(ik_s z \mp i\omega_s t) + \text{complex conjugate} , \tag{12a}$$

$$b^{\pm} = B^{\pm}(t)\exp(ik_A z \mp i\omega_A t) + \text{complex conjugate} , \tag{12b}$$

with

$$\left|\frac{d}{dt}\ln A^{\pm}\right| \quad , \quad \left|\frac{d}{dt}\ln B^{\pm}\right| \qquad \ll \quad \omega_A \quad . \tag{12c}$$

On substituting these solutions into Eq. (11a), and using Eq. (9) to express $\tilde{\rho}^{\pm}$, $H^{\pm}_{1,2}$ in terms of $a^{\pm}$ and $b^{\pm}_{1,2}$, we readily obtain*:

$$(\omega_s , k_s)\frac{\partial A^+}{\partial t} + (-\omega_s , -k_s)\frac{\partial \bar{A}^+}{\partial t} = \tag{13}$$

$$\frac{-k_s}{16\pi\omega_s}\frac{\partial}{\partial z}\Big\{\Big[B_1^+ B_2^+ (\omega_1+\omega_2 , k_1+k_2) + B_1^+ B_2^- (\omega_1-\omega_2 , k_1+k_2) +$$

$$B_1^- B_2^+ (-\omega_1+\omega_2 , k_1+k_2) + B_1^- B_2^- (-\omega_1-\omega_2 , k_1+k_2) +$$

$$B_1^+ \bar{B}_2^+ (\omega_1-\omega_2 , k_1-k_2) + B_1^+ \bar{B}_2^- (\omega_1+\omega_2 , k_1-k_2) +$$

$$B_1^- \bar{B}_2^+ (-\omega_1-\omega_2 , k_1-k_2) + B_1^- \bar{B}_2^- (-\omega_1+\omega_2 , k_1-k_2)\Big] +$$

$$\text{complex conjugate}\Big\} ,$$

---

*Notice that, in calculating the coupling terms on the R.H.S. of Eq. (11), one must allow for the influence of both the forward and backward modes.

where a barred quantity stands for its complex conjugate, and the abbreviation (ω, k) stands for $\exp(ikz - i\omega t)$.

Let us now consider only the interaction between three waves satisfying the following conditions:

$$\vec{k}_1 - \vec{k}_2 = \vec{k}_s \quad ,$$
$$\omega_1 - \omega_2 = \delta \approx \omega_s \quad . \tag{14}$$

On substituting (14) into (13), and keeping only terms with approximately the same oscillating exponentials on both sides of Eq. (13), this will reduce to

$$\frac{\partial A^+}{\partial t} + (-2\omega_s, -2k_s)\frac{\partial \bar{A}^+}{\partial t} = \frac{-ik_s^2}{16\pi\omega_s}\Big[B_1^+\bar{B}_2^+(\delta-\omega_s, 0) + B_1^-\bar{B}_2^-(-\delta-\omega_s, 0) - \bar{B}_1^+B_2^+(-\delta-\omega_s, -2k_s) - \bar{B}_1^-B_2^-(\delta-\omega_s, -2k_s)\Big] . \tag{15}$$

Noting that the mode amplitudes are only time-dependent, we can therefore take a space average of the above equation and obtain

$$\frac{\partial A^+}{\partial t} = \frac{-ik_s^2}{16\pi\omega_s}\Big[B_1^+\bar{B}_2^+ e^{-i(\delta-\omega_s)t} + B_1^-\bar{B}_2^- e^{i(\delta+\omega_s)t}\Big] . \tag{16a}$$

In a quite similar way, one gets

$$\frac{\partial A^-}{\partial t} = \frac{ik_s^2}{16\pi\omega_s}\Big[B_1^+\bar{B}_2^+ e^{-i(\delta+\omega_s)t} + B_1^-\bar{B}_2^- e^{i(\delta-\omega_s)t}\Big] , \tag{16b}$$

$$\frac{\partial B_1^\pm}{\partial t} = \frac{\pm ik_1k_2c_A^2}{4\rho_0\omega_1}\Big[A^+B_2^\pm e^{\pm i(\delta\mp\omega_s)t} + A^-B_2^\pm e^{\pm i(\delta\pm\omega_s)t}\Big] , \tag{16c}$$

$$\frac{\partial B_2^\pm}{\partial t} = \frac{\pm ik_1k_2c_A^2}{4\rho_0\omega_2}\Big[\bar{A}^+B_1^\pm e^{\mp i(\delta\mp\omega_s)t} + \bar{A}^-B_1^\pm e^{\mp i(\delta\pm\omega_s)t}\Big] . \tag{16d}$$

This set of equations represents the non-linear coupling between 6 normal modes treated on the same basis. In the next section, we shall consider the case of parametric coupling in which one of the waves has an amplitude much larger than that of the others, and can be treated as constant.

IV. THE LINEARIZED COUPLED EQUATIONS

In various situations, one deals with a strong, externally-imposed electromagnetic field in a plasma. The field will change the behavior of the plasma with respect to small disturbances: The problem then is to know how these disturbances will develop in the medium, and whether they can be driven unstable.

Here, we consider the case of a large amplitude Alfven wave (b_1) acting as a pump of constant power. Using the coupled equations derived in the last section, we shall determine the conditions under which some initially small perturbation will grow in the form of an Alfven and a sound wave. Within this approximation (constant pump power), Eq. (16) reduces to a simple set of four linear equations describing the parametric coupling of $A^{\pm}$ and $B_2^{\pm}$:

$$\frac{\partial A^{\pm}}{\partial t} = \mp \frac{i k_s^2}{16\pi\omega_s}\left[B_1^+ \bar{B}_2^+ e^{-i(\delta \mp \omega_s)t} + B_1^- \bar{B}_2^- e^{i(\delta \pm \omega_s)t} \right], \tag{17a}$$

$$\frac{\partial B_2^{\pm}}{\partial t} = \pm \frac{i k_1 k_2 c_A^2}{4\rho_0 \omega_2}\left[B_1^{\pm} \bar{A}^+ e^{\mp i(\delta \mp \omega_s)t} + B_1^{\pm} \bar{A}^- e^{\mp i(\delta \pm \omega_s)t} \right], \tag{17b}$$

$$B_1^{\pm} = \text{Constant} . \tag{17c}$$

Eq. (17) can be readily solved using the following transformation:

$$
\begin{aligned}
X^+ &= A^+ \,, \\
X^- &= A^- e^{2i\omega_s t} \,, \\
Y^+ &= \bar{B}_2^+ e^{-i(\delta-\omega_s)t} \,, \\
Y^- &= \bar{B}_2^- e^{i(\delta+\omega_s)t} \,.
\end{aligned}
\tag{18}
$$

On subtituting these new variables in Eq. (17), and assuming solutions of the form $exp\ (-i\omega t)$, we obtain

$$
\begin{pmatrix}
\omega & 0 & -\dfrac{k_s^2 B_1^+}{16\pi\omega_s} & -\dfrac{k_s^2 B_1^-}{16\pi\omega_s} \\[2ex]
0 & \omega + 2\omega_s & \dfrac{k_s^2 B_1^+}{16\pi\omega_s} & \dfrac{k_s^2 B_1^-}{16\pi\omega_s} \\[2ex]
-\dfrac{k_1 k_2 c_A^2 \bar{B}_1^+}{4\rho_0\omega_2} & -\dfrac{k_1 k_2 c_A^2 \bar{B}_1^+}{4\rho_0\omega_2} & \omega + \omega_s - \delta & 0 \\[2ex]
\dfrac{k_1 k_2 c_A^2 \bar{B}_1^-}{4\rho_0\omega_2} & \dfrac{k_1 k_2 c_A^2 \bar{B}_1^-}{4\rho_0\omega_2} & 0 & \omega + \omega_s + \delta
\end{pmatrix}
\begin{pmatrix}
X^+ \\ X^- \\ Y^+ \\ Y^-
\end{pmatrix}
= 0
\tag{19}
$$

which yields the following dispersion law:

$$
\omega(\omega + 2\omega_s)(\omega + \omega_s + \delta)(\omega + \omega_s - \delta) = \frac{k_1 k_2 k_s^2 c_A^2}{32\pi\rho_0\omega_2}\left[|B_1^+|^2(\omega + \omega_s + \delta) - |B_1^-|^2(\omega + \omega_s - \delta)\right] .
\tag{20}
$$

A first view of this equation suggests us to consider separately the two cases of standing and propagating pump.

A- Standing pump

In this case, we have

$$|B_1^+| = |B_1^-| \quad , \tag{21}$$

and Eq. (20) takes the form

$$(\Omega_s^2 - \omega_s^2)(\Omega_s^2 - \delta^2) - K\delta = 0 \quad , \tag{22a}$$

where

$$\Omega_s = \omega + \omega_s \quad , \tag{22b}$$

$$K = \frac{k_1 k_2 k_s^2 c_A^2 |B_1^+|^2}{16\pi \rho_0 \omega_2} \quad . \tag{22c}$$

It is worth noting that if we separate the real and imaginary parts of Ω_s , and write

$$\Omega_s = \Omega_r + i\gamma \quad , \tag{23}$$

then, Ω_r and γ represent the frequency and growth ($\gamma > 0$), or damping ($\gamma < 0$) rate of the sound wave, respectively. Physically, one should distinguish the two cases: *i)* $\Omega_r \neq 0$, $\gamma > 0$, corresponding to an oscillating instability, and *ii)* $\Omega_r = 0$, $\gamma > 0$, corresponding to a purely growing instability.

From its structure, it is easily shown that Eq. (22a) can admit either purely growing or oscillating solutions, depending on the sign of $K\delta$:

1. $K\delta > 0$:

In this case, there can only exist purely growing solutions given by

$$\Omega_r = 0 \quad , \tag{24a}$$

$$\gamma^2 = \frac{1}{2}\left[\sqrt{(\omega_s^2-\delta^2)^2 + 4K\delta} - (\omega_s^2+\delta^2)\right] . \tag{24b}$$

The threshold pump intensity for excitation of this wave is

$$K_m = \omega_s^2\,\delta \quad , \tag{25a}$$

or, in terms of the pump field H_1^+,

$$\frac{|H_1^+|_m^2}{4\pi\rho_0} = \left|\frac{\delta}{k_1}\right| \left(\frac{c_s}{c_A}\right)^2 c_A \quad . \tag{25b}$$

For a given pump intensity, the maximum growth rate is attained at

$$\delta_M = \left(\frac{K}{4} + \sqrt{\frac{K^2}{16} + \frac{\omega_s^6}{27}}\right)^{1/3} + \left(\frac{K}{4} - \sqrt{\frac{K^2}{16} + \frac{\omega_s^6}{27}}\right)^{1/3} , \tag{26a}$$

which reduces to

$$\delta_M = \left(\frac{K}{2}\right)^{1/3} \tag{26b}$$

in the case of a "strong" pump ($|K| \gg \omega_s^3$). For a "weak" pump ($|K| \ll \omega_s^3$) one has

$$\delta_M = \frac{K}{2\omega_s^2} \quad . \tag{26c}$$

The maximum growth rates are given by

$$\gamma_M = \left(\frac{|K|}{2} \right)^{1/3} \qquad \text{(strong pump)} \quad , \tag{27a}$$

and

$$\gamma_M = \frac{|K|}{2\omega_s^2} \qquad \text{(weak pump)} \quad . \tag{27b}$$

From these equations, one can notice that the maximum growth rate increases much more sensitively with the pump power in the weak pump regime than it does for strong pumps.

2. $K\delta < 0$:

Contrary to the previous case, Eq. (22a) now has only oscillating solutions. The frequency and growth rate of the sound wave are given by

$$\Omega_r^2 = \gamma^2 + \frac{1}{2}\left(\omega_s^2 + \delta^2\right) \quad , \tag{28a}$$

and

$$\gamma^2 = -\frac{\left(\omega_s^2 + \delta^2\right)}{4} + \frac{1}{2}\left(\omega_s^2\delta^2 - K\delta\right)^{1/2} \quad . \tag{28b}$$

The threshold power for excitation of this wave is

$$K_m = -\frac{\left(\omega_s^2 - \delta^2\right)^2}{4\delta} \quad . \tag{29}$$

In the weak pump regime, the growth rate attains its maximum value

$$\gamma_M = \left(\frac{|K|}{4\omega_s} \right)^{1/2} \tag{30a}$$

at the perfect matching frequency

$$\delta_M = \omega_s \quad . \tag{30b}$$

Then, the frequency of the sound wave becomes

$$\Omega_r = \left(1 + \frac{|K|}{4\omega_s^3}\right)^{1/2} \omega_s \quad . \tag{30c}$$

For a strong pump, one obtains

$$\gamma_M = \frac{\sqrt{3}}{2} \left(\frac{|K|}{4} \right)^{1/3} \quad , \tag{31a}$$

$$\delta_M = - \left(\frac{K}{4} \right)^{1/3} \quad , \tag{31b}$$

and

$$\Omega_r = \frac{\sqrt{5}}{2} \left(\frac{|K|}{4} \right)^{1/3} \quad . \tag{31c}$$

Let us notice that in both cases ($K\delta > 0$, and $K\delta < 0$), K is allowed to take either positive or negative values, corresponding to the forward or backward scattering of the Alfven wave, respectively.

B- Propagating Pump

This is the case where

$$B_1^- = 0 \quad . \tag{32}$$

From Eqs. (17b) and (20), one sees that the backward mode b_2^- is not excited while the three modes a^+, a^-, and b_2^+ are coupled according to the following dispersion relation:

$$\omega(\omega + 2\omega_s)(\omega + \omega_s - \delta) - \frac{K}{2} = 0 \quad , \tag{33a}$$

or

$$\Omega_s^3 - \delta\Omega_s^2 - \omega_s^2 \Omega_s + \delta\omega_s^2 - \frac{K}{2} = 0 \quad , \tag{33b}$$

which can admit unstable solutions if the pump intensity exceeds the following threshold value:

$$K_m = \frac{4}{27}\left[-\delta^3 + 9\delta\omega_s^2 \mp (3\omega_s^2 + \delta^2)^{3/2}\right] . \tag{34}$$

For $|K| > |K_m|$, the frequency and growth rate of the (unstable) sound wave are given by

$$\Omega_r = \frac{\delta}{3} - \frac{x+y}{2} , \tag{35a}$$

$$\gamma = \frac{\sqrt{3}}{2} \left| x - y \right| , \tag{35b}$$

where

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3}\left\{\left(-\delta^3 + 9\delta\omega_s^2 - \frac{27}{4}K\right) \pm \left[\left(-\delta^3 + 9\delta\omega_s^2 - \frac{27}{4}K\right)^2 - (3\omega_s^2+\delta^2)^3\right]^{1/2}\right\}^{1/3} . \tag{35c}$$

In the threshold region, Eq. (35) yields the following values:

$$\Omega_r = \frac{\delta}{3} + \frac{1}{3}(9\delta\omega_s^2 - \delta^3)^{1/3}\left(1 - \frac{9K}{4(9\delta\omega_s^2 - \delta^3)}\right) , \tag{36a}$$

and

$$\gamma = \frac{\left[2\delta^2\omega_s^4 - \delta^4\omega_s^2 - \omega_s^6 - \frac{K}{2}(9\delta\omega_s^2 - \delta^3)\right]^{1/2}}{(9\delta\omega_s^2 - \delta^3)^{1/3}} . \tag{36b}$$

Putting $\delta = \omega_s$ in Eq. (36), one obtains

$$\Omega_r = \omega_s - \frac{3K}{16\omega_s^2} , \tag{37a}$$

and

$$\gamma = \frac{1}{2}\left(\frac{-K}{\omega_s}\right)^{1/2} . \qquad (37b)$$

This can be compared with the results obtained by Sagdeev and Galeev (1969) in the same situation (i.e. propagating pump under perfect matching conditions), written in our notation:

$$\Omega_r = \omega_s \quad , \qquad (38a)$$

$$\gamma = \left(\frac{-K}{\omega_s}\right)^{1/2} , \qquad (38b)$$

where one notices that the frequency shift is zero, and the growth rate is twice that given by Eq. (37b). While the first discrepancy stems from the space and time averaging process used in their theory, the second is due to the fact that they have not accounted for the "slow-time" dependence of the fluid density in the mass conservation equation.

One can also note that their theory, using the space and time averaging, does not allow for any frequency mismatch, neither does it allow for any influence of the backward sound wave.

V. CONCLUSION

From the foregoing analysis, it appears that

1. a standing pump can excite both oscillating and non-oscillating sound waves;

2. a propagating pump can excite only propagating waves;

3. due to its low frequency (compared with the Alfven wave), the backward sound wave can couple efficiently with the Alfven waves;

4. the growth rates of the excited waves depend on the pump intensity in a characteristic way ($K^{1/2}$ or $K^{1/3}$), according to the pump regime (weak or strong).

Finally, note that in Section IV we have assumed the pump amplitude to remain constant during the coupling process. Although this is a very good approximation in the case of parametric excitation, it can be easily avoided: In fact, solutions to the non-linear system (16) can be expressed exactly in terms of elliptic integrals (Amstrong et al. 1962).

ACKNOWLEDGEMENT

The author wishes to thank Dr. F. Hofmann and Professor E.S. Weibel for the reading of the manuscript and comments on this paper.

This work was performed under the auspices of the Fonds National Suisse de la Recherche Scientifique.

REFERENCES

Amstrong, J.A. et al. 1962 Phys.Rev. 127, 1918

DuBois, D.F. 1972 Interaction of Intense Radiation with Plasmas. Lectures given at the University of Colorado, Dept. of Astro-Geophysics

Kadomtsev, B.B. 1965 Plasma Turbulence, Academic Press, Inc., London

Sagdeev, R.Z. and Galeev, A.A. 1969 Nonlinear Plasma Theory, W.A. Benjamin, Inc., New-York

Van Kampen, N.G. and Felderhof, B.U. 1967 Theoretical Methods in Plasma Physics, North-Holland Publishing Company, Amsterdam.