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A USEFUL FORM OF THE VARIATIONAL PRINCIPLE OF IDEAL MHD FOR ONE-DIMENSIONAL NUMERICAL STABILITY CALCULATIONS

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Abstract

We treat the variational problem of ideal one-dimensional MHD stability by the method of Ritz-Galerkin [4] . Finite elements are taken as the basic functions. In order to find the best approximating elements we have performed a radius dependent transformation on the eigenvectors. The matrices of the transformed eigenvalue problem are given explicitely.

This report serves as an appendix to published papers [2, 5, 6].

Lausanne

1. Normal mode analysis

We study weak perturbation of a cylindrical ideal MHD equilibrium. The perturbation is described by the normal modes [1, 2, 3]

$$\vec{\xi}(t, r, \theta, z) = \vec{\xi}(r) \ \exp\left[i(\omega t + m\theta + kz)\right] \quad \text{with}$$

$$\vec{\xi}(r) = (\xi_r, \xi_\theta, \xi_z) = (\xi_1, -i\xi_2, -i\xi_3).$$

The linearized equations of motion can be brought into the variational form of Hamilton's principle

$$\omega^2 \delta k = \delta W_p \tag{1}$$

where the potential and the kinetic energy of the plasma are given by W_p and $\omega^2 K$ respectively. W_p and $\omega^2 K$ are quadratic forms of $\vec{\xi}(r)$. Takeda et al. [1] have shown how the problem (1) can be treated by the method of finite elements. In this approximation problem (1) reduces to an algebraic eigenvalue problem.

Takeda et al. [1] approximate the real physical displacement $\vec{\xi}(r)$ by linear admissible functions. However, it may be more appropriate to expand a transformed displacement

$$\vec{\eta}(r) = U(r)\vec{\xi}(r) \tag{2}$$

in a series of finite elements [2].

The transformation taken in [2] has given better results for the displacements near the axis. In [2] it was shown that the singularities induced by the cylindrical coordinates may give rise to numerical difficulties near the axis. Further it was shown that these difficulties may be removed by a special choice of U. Then there was a difficulty of a different character, which could not be removed at that time. One class of solutions was badly represented, because the internal condition (div $\vec{\xi}$ = 0) characterising this class, could not be exactly fulfilled with the linear elements. So we proposed to use constant elements [6] together with another transformation U which allows to satisfy the internal condition. Therefore we have chosen the transformation

$$\mathcal{U}(r) = \begin{pmatrix} r^{-\alpha} & 0 & 0 \\ br^{\beta} & \frac{1}{c}r^{-\gamma} & 0 \\ 0 & 0 & dr^{-\delta} \end{pmatrix} \tag{3}$$

where α, β, γ, δ, b, c and d are constants. Note that ξ_z is not mixed up with the other components, because the regularity conditions at the axis [2] have no influence on ξ_z. Because (1) contains only derivatives on ξ_r the form of U will introduce only derivatives on η_1. So U is a simple generalisation of the transformations used in [2] and [6]. The expansion of $\vec{\eta}(r)$ in finite elements is given by

$$\vec{\eta}(r) = \begin{pmatrix} \eta_1(r) \\ \eta_2(r) \\ \eta_3(r) \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{N_r} R_i\, r_i(r) \\ \sum_{i=1}^{N_\theta} \theta_i\, t_i(r) \\ \sum_{i=1}^{N_z} Z_i\, z_i(r) \end{pmatrix} \tag{4}$$

where the elements are denoted by r_i, t_i and z_i, the expansion coefficients by R_i, Q_i and Z_i. Note that the numbers N_r, N_θ, N_z may differ by 1, if a mixture of linear and constant finite elements [6] is used.

Let us define

$$\vec{X}^T = (R_1, \theta_1, Z_1, R_2, \theta_2, Z_2, \ldots) \equiv (x_1, \ldots, x_m) \tag{5}$$

where $m = N_r + N_\theta + N_z$. Then (1) can be written in the form

$$\omega^2 \sum_{k=1}^{m} \left(\int_0^1 R_{\ell k}(r)\,dr \right) x_k = \sum_{k=1}^{m} \left(\int_0^1 C_{\ell k}(r)\,dr \right) x_k, \quad \ell = 1, \ldots, m \tag{6}$$

Let us denote the total number of integration intervals by N and a specific interval by i. Then the matrices R and C may be written in terms of partial matrices ($\tilde{R}_i$) and ($\tilde{C}_i$) which are for U = $\mathbb{1}$ and linear finite elements the same as in [1]. They are given by

$$
(\tilde{C}_i) = \begin{bmatrix}
f_{11}a_{11}+2f_{14}a_{14}+f_{44}a_{44} & f_{12}a_{12}+f_{24}a_{42} & f_{13}a_{13}+f_{34}a_{43} & f_{11}a_{15}+f_{14}(a_{18}+a_{54})+f_{44}a_{48} & f_{12}a_{16}+f_{24}a_{46} & f_{13}a_{17}+f_{34}a_{47} \\
f_{12}a_{21}+f_{24}a_{24} & f_{22}a_{22} & f_{23}a_{23} & f_{12}a_{25}+f_{24}a_{28} & f_{22}a_{26} & f_{23}a_{27} \\
f_{13}a_{31}+f_{34}a_{34} & f_{23}a_{32} & f_{33}a_{33} & f_{13}a_{35}+f_{34}a_{38} & f_{23}a_{36} & f_{33}a_{37} \\
f_{11}a_{51}+f_{14}(a_{81}+a_{45})+f_{44}a_{84} & f_{12}a_{52}+f_{24}a_{82} & f_{13}a_{53}+f_{34}a_{83} & f_{11}a_{55}+2f_{14}a_{58}+f_{44}a_{88} & f_{12}a_{46}+f_{24}a_{86} & f_{13}a_{47}+f_{34}a_{87} \\
f_{12}a_{61}+f_{24}a_{64} & f_{22}a_{62} & f_{23}a_{63} & f_{12}a_{64}+f_{24}a_{68} & f_{22}a_{66} & f_{23}a_{67} \\
f_{13}a_{71}+f_{34}a_{74} & f_{23}a_{72} & f_{33}a_{73} & f_{13}a_{74}+f_{34}a_{78} & f_{23}a_{76} & f_{33}a_{77}
\end{bmatrix} \tag{7}
$$

and

$$
(\tilde{R}_i) = \begin{bmatrix}
g_{11}\varrho a_{11} & g_{12}\varrho a_{12} & 0 & g_{11}\varrho a_{15} & g_{12}\varrho a_{16} & 0 \\
g_{12}\varrho a_{21} & g_{22}\varrho a_{22} & 0 & g_{12}\varrho a_{25} & g_{22}\varrho a_{26} & 0 \\
0 & 0 & g_{33}\varrho a_{33} & 0 & 0 & g_{33}\varrho a_{37} \\
g_{11}\varrho a_{51} & g_{12}\varrho a_{52} & 0 & g_{11}\varrho a_{55} & g_{12}\varrho a_{56} & 0 \\
g_{12}\varrho a_{61} & g_{22}\varrho a_{62} & 0 & g_{12}\varrho a_{65} & g_{22}\varrho a_{66} & 0 \\
0 & 0 & g_{33}\varrho a_{73} & 0 & 0 & g_{33}\varrho a_{77}
\end{bmatrix} \tag{8}
$$

where:

$$f_{11} = r^{2\alpha+1}\left\{\frac{1}{r^2}\left(B_z^{\,2}+\Gamma p\right)\left[\alpha+1-bcmr^{\beta+\gamma}\right]^2+\left[(\alpha-1)\frac{B_\theta}{r}+bckB_z r^{\beta+\gamma}\right]^2+\right.$$

$$\left.+\left(kB_z+\frac{mB_\theta}{r}\right)^2-4\left(\frac{B_\theta}{r}\right)^2-2\,\frac{B_\theta}{r}\,r\,\frac{d}{dr}\left(\frac{B_\theta}{r}\right)\right\} \tag{9}$$

$$f_{12} = c\,r^{\gamma+\alpha+1}\left\{\frac{m}{r^2}\left(B_z^{\,2}+\Gamma p\right)\left[\alpha+1-bcmr^{\beta+\gamma}\right]-B_z k\left[(\alpha-1)\frac{B_\theta}{r}+bckB_z r^{\beta+\gamma}\right]\right\}$$

$$f_{13} = d\,r^{\alpha+\delta+2}\left\{\frac{1}{r^2}\left(\Gamma pk-\frac{B_\theta}{r}mB_z\right)\left[\alpha+1-bcmr^{\beta+\gamma}\right]+\frac{B_\theta}{r}k\left[(\alpha-1)\frac{B_\theta}{r}+bckB_z r^{\beta+\gamma}\right]\right\}$$

$$2f_{14} = r^{2\alpha+2}\left\{\frac{1}{r^2}\left(B_z^{\,2}+\Gamma p\right)\left[\alpha+1-bcmr^{\beta+\gamma}\right]+\frac{B_\theta}{r}\left[(\alpha-1)\frac{B_\theta}{r}+bckB_z r^{\beta+\gamma}\right]\right\}$$

$$f_{22} = c^2 r^{2\gamma-1}\left\{\Gamma p\,m^2+\left(k^2r^2+m^2\right)B_z^{\,2}\right\}$$

$$f_{23} = c\,d\,r^{\gamma+\delta}\left\{\Gamma p\,mk-\left(k^2r^2+m^2\right)\frac{B_\theta}{r}\,B_z\right\}$$

$$f_{24} = c\,r^{\gamma+\delta}\left\{\Gamma p\,m-\left(krB_\theta-mB_z\right)B_z\right\}$$

$$f_{33} = d^2 r^{2\delta+1}\left\{\Gamma p\,k^2+\left(k^2r^2+m^2\right)\left(\frac{B_\theta}{r}\right)^2\right\}$$

$$f_{34} = d\,r^{\alpha+\delta+1}\left\{\Gamma p\,k+\left(krB_\theta-mB_z\right)\frac{B_\theta}{r}\right\}$$

$$f_{44} = r^{2\alpha+1}\left\{B_z^{\,2}+B_\theta^{\,2}+\Gamma p\right\}$$

and

$$g_{11} = r^{2\alpha+1}\left\{1 + c^2 b^2 r^{2\beta+2\gamma}\right\}$$

$$g_{12} = -bc^2 r^{\alpha+\beta+2\gamma+1}$$

$$g_{22} = c^2 r^{2\gamma+1} \tag{10}$$

$$g_{33} = d^2 r^{2\delta+1}$$

The factors $a_{\nu\mu}$ are the contributions of the basic functions and are determined by:

$$a_{\nu\mu} = e_\nu e_\mu \quad , \quad \nu,\mu = 1,\ldots, 8 \tag{11}$$

where $\vec{e} = (r_i, t_i, z_i, r_i', r_{i+1}, t_{i+1}, z_{i+1}, r_{i+1}')$ and $\frac{d}{dr} = {}'$.

2. Boundary and regularity conditions

We want to solve the eigenvalue problem

$$(C - \omega^2 R)\cdot\vec{X} = 0 \tag{12}$$

with C, R symmetric and R positive definite.

ω^2 is the eigenvalue and $\vec{x}$ the corresponding eigenfunction.

The boundary and the regularity conditions are

$$\xi_r(r=1) = 0 \tag{13}$$

$$r\vec{\xi}\,(r=0)=0 \tag{14}$$

The regularity condition (14) is satisfied, if the matrix elements of (7) and (8) are finite at $r=0$. The transformation used in [2] corresponds to the parameters

$$b=c=d=1\;,\;\alpha=|m-1|\;,-\beta=\gamma=\alpha+2\delta_{m1}\;,\;\delta=|m| \tag{15}$$

where δ_{m1} is the Kronecker symbol.

This transformation was specifically chosen to describe the asymptotic behaviour of the displacements in a θ pinch at r = 0.

For the case $m>0$, we propose here a weaker condition to fulfil the regularity condition (14):

$$\beta+\gamma=0,\;\alpha=0\;,\;\delta=0\;,\;bcm=1 \tag{16}$$

In stability calculations $\operatorname{div}\vec{\xi}=\frac{1}{r}(r\xi_1)'+\frac{m}{r}\xi_2+k\xi_3$ is for some eigenmodes very small [5, 6] or even zero [2, 6]. In order to describe this fact we have to choose the elements properly. The parameters b = 1 and $c=\frac{1}{m}$ correspond to the transformation

$$\begin{aligned}\eta_1&=\xi_1\\ \eta_2&=\frac{\xi_1+m\xi_2}{r}\\ \eta_3&=\xi_3\end{aligned} \tag{17}$$

Taking for η_1 linear and for η_2 and η_3 constant elements, we are able to describe the divergence exactly.

It was this choice of transformation and elements which made it possible to calculate degenerated modes [6] , localised modes [5] and even singular modes with a continuous spectrum.

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