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A b s t r a c t

For the eigenvalue problem $A\vec{x} = \lambda B\vec{x}$, where the band matrices A, B are symmetric and B positive definite, we count the number of eigenvalues in a given interval $[\lambda_0, \lambda_1]$. The chosen method conserves the band form of the sparse matrices during all calculations.

Lausanne

1. Introduction

The numerical approach to some physical eigenvalue problems leads to the algebraic equation $A\vec{x} = \lambda B\vec{x}$, where the matrices are symmetric, have band form and B is positive definite. Because of limited memory space and calculation time the band structure of big matrices has to be preserved during any vector operation.

In our field of interest, i.e. the stability problem in ideal MHD, we have to know the number of unstable modes ($\lambda < 0$) in order not to miss anyone. We present here a band structure converving method to count eigenvalues in a given interval $[\lambda_0, \lambda_1]$.

First we show that the number of negative eigenvalues of the problem $A\vec{x} = \lambda B\vec{x}$ is equal to the number of negative terms of the diagonal matrix D in the decomposition $A = LDL^T$ where L is a regular left side matrix and L^T its transposed. The number of eigenvalues lying in $[\lambda_0, \lambda_1]$ then is given by the difference of the number of eigenvalues of the matrices $A - \lambda_1 B$ and $A - \lambda_0 B$. We then show that our method is especially powerful in combination with the simultaneous inverse vector iteration for finding eigenvectors.

2. Number of negative eigenvalues

Theorem : The number of negative eigenvalues of the problem

$$A\vec{x} = \lambda B\vec{x} \tag{1}$$

where $A = LDL^T$ and B are symmetric and B is positive definite, is equal to the number of negative terms of the diagonal matrix D. L is a regular left side matrix.

Proof : B symmetric and positive definite implies a unique decomposition

$$B = R^T R \tag{2}$$

where R is a right hand matrix with the same band structure as B. Multiplying the eigenvalue problem (1) by $(R^T)^{-1}$ from the left we obtain a new problem

$$\tilde{A}\vec{y} = \lambda E \vec{y} \tag{3}$$

where $\tilde{A} = (R^T)^{-1} A R^{-1}$, $\vec{y} = R\vec{x}$ and E is the unity matrix. The symmetry of A implies a decomposition

$$A = L D L^T \tag{4}$$

where L is a regular left side matrix and D is a pure diagonal matrix*. A similar decomposition can be done for $\tilde{A}$

$$\tilde{A} = \tilde{L} \tilde{D} \tilde{L}^T \quad . \tag{5}$$

Introducing (4) and (5) into (3)

$$\tilde{L} \tilde{D} \tilde{L}^T \vec{y} = (R^T)^{-1} L D L^T R^{-1} \vec{y} = \lambda E \vec{y} \tag{6}$$

and replacing $S = (R^T)^{-1} L$ we obtain the identities

$$\tilde{L} \tilde{D} \tilde{L}^T \vec{y} = S D S^T \vec{y} = \lambda E \vec{y} \tag{7}$$

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*In this transformation L has the same band structure as A. Note, that choosing unity diagonal elements in L, D contains the diagonal terms of the Gauss eliminated matrix A.

Theorem 1 is proofed when Sylvester's inertia theorem is applied on (3), (5) and (7). This theorem implies that the number of negative eigenvalues of a matrix A is unchanged after a transformation QAQ^T, where Q is regular. With Sylvester's theorem it is easily seen that A, $\tilde{A}$, $\tilde{D}$ and D have the same number of negative eigenvalues.

3. Connection to vector iteration

An eigenvalue problem (1) can be solved in different ways [1]. It can be put on form (3) and afterwards be solved by Jacobi's method. Bisection or QD algorithms are used after a Householder transformation, whenever the eigenvectors are not of interest. These three mentionned methods have all a big disadvantage, because $\tilde{A}$ is an entirely full matrix. In order to economize memory space and computing time it is more favourable to take a method which treats directly problem (1). In the following we describe the simultaneous inverse vector iteration (SIVI).

A certain number s of eigenvectors corresponding to the s absolute smallest eigenvalues of the problem (1) are found by the iterative process

$$AX_{k+1} = BX_k \tag{8}$$

where the matrix X_k contains the s orthonormal eigenvectors with n components and k denotes the iteration step. For starting the iteration a system of s orthonormal vectors are chosen arbitrarily. To solve the system of linear equations (8) we once decompose the matrix A in a left hand matrix L, containing unity diagonal elements and in a right hand matrix R with the same number of negative diagonal elements as A has negative eigenvalues.

To obtain the new iteration system

$$LRX_{k+1} = BX_k \tag{9}$$

we have to perform only n x m^2 operations, where m denotes the band width of L and R.

System (9) is solved in two steps. First we perform

$$LY_{k+1} = BX_k \tag{10}$$

and afterwards we solve

$$RX_{k+1} = Y_{k+1} \tag{11}$$

The number of operation in (10) and (11) is proportional to n·m·s. When i iterations have to be done for convergence, the total computing time CT is given by

$$CT = n * m * (a_1 * m + a_2 * s * i) \tag{12}$$

where a_1 and a_2 are constants, which define the computing time to perform one operation in the decomposition for A resp. to solve (10) and (11).

4. Discussion

The use of SIVI to solve (1) has the advantage that the band structure of A and B are preserved during all vector operations. Memory space and computing time are kept small. By the LR decomposition of A we know the number of negative eigenvalues of A. So we always know if we missed the most interesting, the most unstable mode or not. Also it is comfortable to calculate the eigenvalues with its corresponding modes in a given interval.

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