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TRANSVERSE DYNAMIC STABILIZATION OF A THETA PINCH

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Abstract

The stability of a straight plasma column confined by a static axial magnetic field and a small oscillating transverse magnetic field is discussed. Growth rates for the most important parametric instabilities are obtained. The influence of damping in providing a stabilizing mechanism against these instabilities is discussed and a set of stability criteria are obtained using a viscous fluid model for the plasma. The possibility of suppressing the fastest growing instability by "wobbling" the frequency of the applied field is also discussed. An order of magnitude calculation for dynamic stabilization of a toroidal plasma is presented.

Lausanne

I. Introduction

The problem of transforming the stable straight θ-pinch into an equilibrated, stable toroidal configuration by adding some suitable additional static field, is still unsolved. Various schemes which use additional oscillating magnetic fields to dynamically stabilize the confinement have been proposed[1-10]. In the present paper we return to the transverse oscillating field scheme proposed by Weibel[5,6], which consists in superposing an oscillating axial current to a θ-pinch configuration.

The original basic idea[6] is to oscillate the current at a very high frequency. The plasma then feels an average force due to the current and, providing certain geometrical conditions are met, this force is a positive restoring force. The marginally stable θ-pinch then becomes positively stable. The system could then be curved into a torus[11], the oscillating current providing a dynamic equilibrium; or, alternatively, a small amount of "bumpiness" or a small steady axial current could provide the toroidal equilibrium without destroying the stability.

The minimum frequency has not been determined precisely. Weibel[5] has shown, in the case of a high-β collisionless plasma, that a frequency larger than the transit frequency of an ion across the plasma column should be sufficient. For a collision dominated plasma, the frequency will always be higher than in the collisionless case, since the damping is smaller. The transit frequency is already very high and application of a frequency of this magnitude causes great technical difficulties, while leading to a prohibitive ohmic heating of the plasma. We want to reexamine this problem in order to investigate the possibility of using lower frequencies.

To study this problem we assume the plasma is field free (β=1), surrounded by vacuum. Because the dangerous modes all have long wavelengths (low frequency approximation) we describe the plasma behaviour by a fluid model.

We can summarize the results in the following manner. The oscillation increases the stability of the non-resonant modes which have $k \approx o$ for all $m \gtrsim 1$, while for $m = o$ it reduces only slightly the stability of the non-resonant modes. This stabilizing influence is proportional to B_θ^2. The most dangerous modes for stability are those which are in parametric resonance. Without dissipative terms, these modes are always unstable with a growth rate proportional to B_θ. If one includes damping there is a threshold on B_θ for the onset of parametric excitation which defines the limit of stability.

We find that a plasma of radius a, confined by a static axial magnetic field B_o and an oscillating azimuthal field $B_\theta \cos\omega t$ and surrounded by a concentric, perfectly conducting cylinder of radius b is stable if

$$\varepsilon < \frac{1}{4}\frac{\lambda}{a}\left(\frac{\omega}{\nu_p}\right)^2\left(\frac{\gamma}{\eta_1}\right)^{3/2} \tag{1}$$

where ε is defined through

$$B_\theta = \sqrt{2}\,\varepsilon\,\frac{a}{r}\,B_o \quad , \quad \varepsilon \ll 1$$

$$\eta_n = \frac{1 + (a/b)^{2|n|}}{1 - (a/b)^{2|n|}}$$

γ is the ratio of specific heats, ν_p is the ion transit frequency for the plasma ($\nu_p = u/a$, where u is the sound speed in the plasma) and λ is the effective ion collision length for the plasma. If (1) is violated, a parametric excitation of a long wavelength kink mode appears. Condition (1) was derived using a fluid model for the plasma, which remains valid only for $\lambda < a$. However, a calculation using the Vlasov equation[13] shows that (1) remains correct, with λ of the order of a.

If the straight pinch were curved to form a torus of major radius R, there is a minimum value of ε which would be needed to compensate for the outward toroidal drift and prevent the plasma from touching the wall. For an average displacement of the equilibrium position of the plasma column $\frac{b-a}{2}$, a new condition is obtained from (1),

$$\frac{R}{a+b} > 16\left(\frac{a^2}{\lambda^2}\right)\gamma^{-3}\left(\frac{\nu_p}{\omega}\right)^4 \tag{2}$$

This shows that for any reasonable R/b, ω would have to be near ν_p.

By "wobbling" the frequency ω by a small amount $\Delta\omega$, the condition (1) is weakened and becomes

$$\varepsilon^2 < \frac{1}{8}\left(\frac{\lambda}{a}\right)\left(\frac{\gamma}{\eta_1}\right)^2\left(\frac{\omega}{\nu_p}\right)^2\frac{\Delta\omega}{\nu_p} \tag{3}$$

which gives a new condition on R,

$$\frac{R}{a+b} > 32\left(\frac{a}{\lambda}\right)\gamma^{-2}\left(\frac{\nu_p}{\omega}\right)^2\left(\frac{\nu_p}{\Delta\omega}\right) \tag{4}$$

which is still very stringent.

It appears therefore that dynamic stabilization of a θ-pinch, using a frequency appreciably below the sound transit frequency, cannot provide a sufficient restoring force in a torus of reasonable dimensions, for equilibrium. Other possible ways to equilibrate the toroidal drift must be considered.

II. The Basic Equations

A. The Steady State

The magnetic pressure P at the plasma surface is

$$p = \frac{B_0^2}{2\mu_0}\left[1+\varepsilon^2+\varepsilon^2\cos 2\omega t\right] = p_0\left(1+\frac{\varepsilon^2}{1+\varepsilon^2}\cos 2\omega t\right),$$

where p_o is the average pressure. Under the influence of the small oscillating pressure, the surface will also oscillate. For small ω, using an adiabatic equation of state, we find (cf. Appendix B, eq.(B7))

$$a(t) = a\left(1-\frac{\varepsilon^2}{2\gamma}\cos 2\omega t\right)$$

where a is the average plasma radius.

To study the stability of the system, we consider small motions of the surface of the form

$$r = a(t) + \xi_n(k,t)\,e^{ikz+in\varphi}$$

To find the equation of motion for $\xi(t)$ we shall compute separately the perturbation of the magnetic pressure $p_m(t)$ and of the plasma pressure $p_G(t)$ in terms of $\xi(t)$ and then equate them.

B. Perturbation of the Magnetic Field

The surface deformation causes a change in the magnetic pressure p_m at the deformed surface, which has been computed by Weibel to be

$$p_m(n,k,t) = \frac{-B_0^2\,\xi_n(k,t)}{a(t)}\left[\frac{y_n(k,a(t))}{a(t)\,y_n'(k,a(t))}\left\{n\sqrt{2}\,\varepsilon\cos\omega t + a(t)k\right\}^2 + 2\varepsilon^2\cos^2\omega t\right] \qquad (5)$$

where

$$y_n(k,x) = K_{|n|}'(|k|b)\,I_{|n|}(|k|x) - I_{|n|}'(|k|b)\,K_{|n|}(|k|x),$$

$I_n(x)$ and $K_n(x)$ being the modified Bessel functions.

Expanding (5) in powers of ε and retaining terms up to order ε^2 we find

$$P_m(n,k,t) = \frac{P_0 \xi_n(k,t)}{a} \left\{ X_n(k) + A_n(k)\cos\omega t + B_n(k)\cos(2\omega t) \right\} \tag{6}$$

where

$$A_n(k) = \pm 4\varepsilon \left[2 f_n(k)\; h_n(k) \right]^{1/2} \tag{7}$$

$$B_n(k) = 2\varepsilon^2 \left[f_n(k) - 1 - \frac{a^2}{2\gamma}\frac{\partial}{\partial a}\left(\frac{h_n(k)}{a}\right)\right] \tag{8}$$

and

$$X_n(k) = \frac{2}{1+\varepsilon^2}\left[h_n(k) + \varepsilon^2 (f_n(k) - 1) \right] \tag{9}$$

where

$$f_n(k) = -\frac{n^2 y_n(k,a)}{a\, y_n'(k,a)} \tag{10}$$

$$h_n(k) = -\frac{a k^2 y_n(k,a)}{y_n'(k,a)} \tag{11}$$

The properties of the functions $f_n(k)$ and $h_n(k)$ have been studied by Weibel[5]. Both are positive definite, $h_n(k)$ being a monotonically increasing function of k, while $f_n(k)$ is a monotonically decreasing function of k. All other properties of these functions necessary to the development of this paper will be quoted in the text when required.

It may be noted that $A_n(k)$ is of order ϵ while $B_n(k)$ is of order ϵ^2. However, in discussing non resonant interactions, it will be shown that these coefficients appear in the relevant equations as $A_n^2(k)$. Hence, contrary to work reported by other authors[7], we retain terms of order ϵ^2 in $X_n(k)$ in order to remain consistent in our discussion of these interactions.

C. Perturbation of the Plasma Pressure

The surface perturbation ξ causes a change in the plasma pressure $\delta p(r,t)\, e^{in\theta + ikz}$. Assuming that $\xi(t) = o$ for $t \leq o$, the equation for the Laplace-transform of δp (to order ϵ^2), derived in Appendix B, is of the form

$$\nabla^2 \delta\tilde{p}_n(s) + (s^2/u^2)\,\delta\tilde{p}_n(s) = \epsilon^2 \left\{ M_n^+(s,\omega)\,\tilde{\xi}(s+2i\omega) + M_n^-(s,\omega)\,\tilde{\xi}(s-2i\omega) \right\} \tag{12}$$

where a superposed tilde has been used to represent the Laplace-transform, defined through

$$\tilde{\xi}(s) = \int_0^\infty dt\, e^{-st}\, \xi(t)$$

The right hand side of (12) represents the coupling between the plasma pressure and the forced motion, and since its dependence is on $\tilde{\xi}(s \pm 2i\omega)$ the effect of this term in the final dispersion relation will be to modify the function $B_n(k)$ in equation (6).

We have also shown in Appendix B that

$$M_0^\pm(s,\omega) \sim O\left(\frac{\omega^2}{\nu_p^2}\right)$$

$$M_n^\pm(s,\omega) \sim O\left(\frac{\omega^4}{\nu_p^4}\right) \quad , \quad n \neq 0$$

Thus the correction term in $\delta\tilde{p}$ due to the alternating field at most is of order $\epsilon^2 \dfrac{\omega^2}{\nu_p^2}$, while for the most dangerous mode (n=1) the correction is of order $\varepsilon^2 \frac{\omega^4}{\nu_p^4}$. However, from equations (6) to (9) we may observe that the terms of order ε^2 in the magnetic pressure response are independent of the applied frequency, and since we are interested here in the low frequency limit, we feel justified in ignoring the effects of the alternating field on the perturbed plasma pressure.

Neglecting the right-hand side of (12), we find

$$\delta\tilde{p}_n = -\tilde{R}(n,k,s)\,\tilde{\dot{\xi}}(k,s)$$

$$\tilde{R}(n,k,s) = p_0 \frac{s\,\gamma\,J_n(k_H a)}{u^2 k_H J_n'(k_H a)} \qquad (13)$$

where $k_H^2 = -k^2 - s^2/u^2$ and we have used the boundary condition on $\delta\tilde{v}_r$

$$\delta\tilde{v}_r = \tilde{\dot{\xi}} = -\frac{u^2}{\gamma s}\frac{\partial\delta\tilde{p}}{\partial r} \quad \text{at} \quad r = a .$$

Inverting back to time, the perturbed plasma pressure $p_G(n,k,t)$ may be written in the form

$$p_G(n,k,t) = -\int_0^t R(n,k,t-t')\,\dot{\xi}_n(k,t')\,dt' \qquad (14)$$

where $R(n,k,t)$ is defined through (13).

We are now in a position to discuss the dispersion relation for the system.

III. The Dispersion Relation in the limit $\varepsilon = 0$

Using equations (6) and (13) to equate the magnetic and gas pressures at the perturbed plasma surface in the absence of the alternating component of the magnetic field (i.e. $\varepsilon = 0$), one obtains the dispersion relation

$$\frac{2\,k_n(k)}{a} + \frac{s^2 \gamma\, J_n(k_H a)}{u^2 k_H\, J_n'(k_H a)} = 0. \tag{15}$$

For each value of n, the eigenmode solutions to equation (15) are an infinite set of purely imaginary functions of k. In Fig. 1 the two lowest frequency modes $\alpha = |s|$ are plotted as functions of k for each of n = 0, 1, 2. It can be seen that for each value of n there is a mode which starts at the origin and increases with increasing k. Except in the case where n = 0, the higher frequency modes at k = 0 are solutions to $J_n(\alpha k/u) = 0$, and the frequency of all modes again increases with increasing k. For n = 0 it can be seen that the higher frequency modes are slightly above the zero of the Bessel function for k = 0, but otherwise follow the same pattern as the higher n modes.

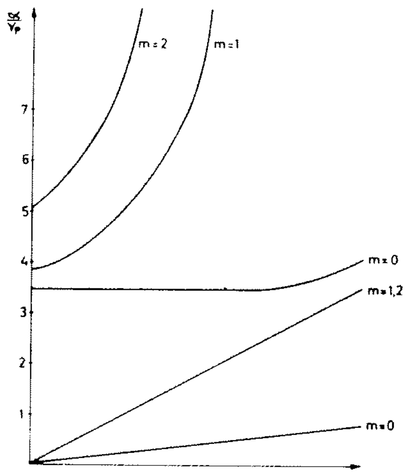

Fig. 1

Fig. 1 Schematic representation of the first natural frequencies of a θ-pinch for m = 0, 1, 2 and ak < 1.

We examine first the limit $k \to 0$, for which one has [5] for $n = 0$

$$h_0(k) \longrightarrow \frac{2}{(b/a)^2 - 1} \tag{16}$$

and for $n \neq 0$

$$h_n(k) \to \frac{\eta_n}{|n|}(ak)^2 \tag{17}$$

where

$$\eta_n = \frac{1 + (a/b)^{2|n|}}{1 - (a/b)^{2|n|}}$$

In the case $n = 0$, equation (15) has a solution in the limit of small k given by

$$\alpha = \left[1 + \frac{\gamma}{2}\left((b/a)^2 - 1\right)\right]^{-1/2} uk \tag{18}$$

together with a series of solutions of the form

$$\alpha = \alpha_0 + \frac{4\nu_p^2}{\alpha_0 \gamma}\left[(b/a)^2 - 1\right]^{-1} \tag{19}$$

where α_0 are the series of frequencies defined by

$$J_0(\alpha_0/\nu_p) = 0 .$$

For higher values of n, there is a mode which, for small k, is given by

$$\alpha = \left(\frac{2\eta_n}{\gamma}\right)^{1/2} ak\,\nu_p \tag{20}$$

There are also a series of modes described by

$$\alpha = \alpha_n + \frac{2\nu_p^2 \eta_n}{n\alpha_n \gamma}(ak)^2, \tag{21}$$

where α_n satisfy $J_n(\alpha_n/\nu_p) = 0$. From the form of (20) and (21) and the definition of η_n, one may note that for large values of n wall effects are negligible (i.e. $\eta_n \sim 1$).

As $k \to \infty$, $h_n(k) \to |ak|$ and the mode described by equation (18) becomes

$$\alpha = uk,$$

while all other modes approach asymptotically the value

$$\alpha = \frac{\sqrt{2}}{\gamma}\left[\sqrt{(1+\gamma^2)} - 1\right]^{1/2} uk$$

In the next section we shall discuss the effect of the oscillating field on the modes which have been described above.

IV. Influence of the Oscillating Field

For $\varepsilon \neq 0$, equating the magnetic and plasma pressures at the deformed interface given by equations (6) and (14) respectively, one has

$$\left(X_n(k) + A_n(k)\cos(\omega t - \phi) + B_n(k)\cos 2(\omega t - \phi)\right)\frac{\xi_n(k,t)}{a} + \int_0^t R(n,k,t-t')\,\dot{\xi}_n(k,t')\,dt' = 0, \tag{22}$$

where ϕ represents the phase difference between the oscillating field and the perturbation, which is arbitrary, since the perturbation may be created at any time.

We examine first the effect of the oscillating field on the non-oscillatory term, $X_n(k)$, of equation (22). If $\varepsilon \neq 0$, it can be seen from equation (9) that the oscillating field introduces a frequency shift on the modes described in the previous section. Since the correction to $X_n(k)$ is of order ε^2, it will be of importance only in the region where $h_n(k)$ is small. For n = 0, one has

$$f_o(k) = 0,$$

and hence the effect of the oscillating field is destabilising. However, since $h_o(k)$ is finite for all values of k, the correction term will never become important. For $n \neq 0$, discussion of the effect of ε may be restricted to the region where k is small, since $h_n(k) \rightarrow o$ as $k \rightarrow o$ (Cf. eq. (17)). In this region one has

$$f_n(k) = \eta_n |n|$$

where, as remarked earlier, $\eta_n \geqslant 1$. Thus, from equation (9) it may be noted that the effect of the oscillating field is a stabilising one, since near the origin

$$X_n(k) = \frac{\eta_n (ak)^2}{|n|} + \varepsilon^2 (\eta_n |n| - 1).$$

We now proceed to examine the effect of the oscillating terms on the stability of the system under consideration. As is well known, the most dangerous instabilities which may be parametrically excited in a system are those for which the applied frequency, ω, is exactly one half of the natural frequency, α, of the system (i.e. $\alpha = \omega/2$). Parametric excitations may also occur when α is an integer multiple of $\omega/2$, but these are of higher order in ε and are therefore less important. In the absence of dissipation, one does not expect non-resonant interactions to play an important role. We shall consider the regions of interest (i.e. those regions which may produce instabilities of order up to and including ε^2) separately.

Since we have assumed that the applied frequency ω is smaller than the plasma transit frequency, parametric interaction between the higher frequency modes described by equations (19) and (21) are of very high order in ε and may therefore be ignored.

(i) Non-Resonant Interactions

A purely imaginary mode $s = i\alpha$ will experience a frequency shift due to the presence of the oscillating field which, when α and ω are not in resonance, is given by (Cf. Appendix A eq. (A5)),

$$\delta s = \frac{A_n^2(k)}{4d'(s)}\left[g(s+i\omega) + g(s-i\omega)\right], \tag{23}$$

where $d(s) = X_n + s\tilde{R}(s) = g^{-1}(s)$. In the absence of dissipation therefore, δs is purely imaginary and is given by

$$\delta s \simeq \frac{i\varepsilon^2 \nu_p^2 \eta_n n^2 \alpha}{2\gamma}\left[\frac{1}{\alpha^2-(\alpha+\omega)^2} + \frac{1}{\alpha^2-(\alpha-\omega)^2}\right] \tag{24}$$

for $n \neq 0$. For $n = 0$, δs is of order ε^4, since $A_n(k) \equiv 0$, and these interactions may be neglected. Equation (23) is no longer valid in the limit $\alpha \ll \omega$ and we must use equation (A6), which is given, in the absence of damping by

$$\delta s = \frac{A_n^2(k)}{2d''(0)} \mathrm{Re}\, g'(i\omega) \pm \sqrt{-\alpha^2 + \frac{A_n^2(k)}{d''(0)} \mathrm{Re}\, g(i\omega)} \tag{25}$$

Once again, for $n = 0$, non-resonant interactions are of order ε^4. For $n \neq 0$ one has

$$d''(0) = 2\gamma a/u^2 n > 0 \qquad \text{and}$$

$$\mathrm{Re}\,g(i\omega) \simeq -\frac{nu^2}{a\gamma\omega^2} < 0 \text{ and}$$

$$\mathrm{Re}\,g'(i\omega) = 0$$

Hence

$$\delta s = \pm\sqrt{-\alpha^2 - \frac{2\varepsilon^2 n^2 \eta_n \alpha^2 \nu_p^2}{\gamma\omega^2}} \tag{26}$$

and we see that the oscillating field stabilises all modes for $n \neq 0$.

(ii) $\alpha = \omega/2$: In this region an instability is excited, and its growth rate is given by (eq. (A8))

$$\mathrm{Max}\,(\delta s) = \frac{A_n(k)}{2|d'(\alpha)|} \tag{27}$$

The width of the spectrum Δ over which the resonance is important is given by $\delta\omega = 2\delta s$.

Once again one may observe that for $n = 0$ no important instability is excited in this region. For $n \neq 0$ one finds, by substituting the appropriate values in (27), that the instability has a growth rate which is given, to order ε by

$$\delta s = \frac{\varepsilon n \eta_n^{1/2}}{\gamma^{1/2}} \nu_p \tag{28}$$

The validity of this calculation requires that

$$\delta s < \omega/2,$$

which yields the condition

$$\varepsilon \ll \frac{1}{n}\left(\frac{\gamma}{\eta_n}\right)^{1/2}\left(\frac{\omega}{2\nu_p}\right)$$

At first sight, it would appear that the most dangerous modes would be those for higher values of n, since the growth rates in (28) are unbounded functions of n. However, as will be demonstrated in the following section, the stability criterion obtained when damping is included is independent of the value of d'(s), and the n in equation (28) and its associated validity condition derive precisely from this term.

(iii) $\alpha \simeq \omega$: In this region parametric instabilities are also excited, with growth rate given by equation (A12), which becomes in the absence of damping

$$\mathrm{Max}\,(\delta s) = \frac{|A_n^2(k) - 2X_n(k)B_n(k)|}{2d'(s)X_n(k)} \tag{29}$$

Thus we obtain

$$\delta s = \left[\frac{\gamma a^2(b^2/a^2-1)^2 + a^2 + b^2}{8a^2(2+\gamma(b^2/a^2-1))}\right]\omega\varepsilon^2, \quad n = 0 \tag{30}$$

and

$$\delta s = \frac{(3\eta_n|n|+1)\,n\,\nu_p^2}{2\omega\gamma}\,\varepsilon^2, \quad n \neq 0$$

Summarizing, we see that in the absence of damping this system is always unstable, due to the parametric excitation of long wavelength modes, the maximum growth rates being of order ε. If one includes dissipation, these modes become stable provided ε is sufficiently small. We now derive the relation between damping and the maximum value of ε which can be used without destroying stability.

V. Influence of Damping

If one includes the effects of damping in the equations for the system, one finds the response function $\tilde{R}(n,k,s)$ is no longer purely imaginary, but contains a small real part, which displaces the resonant frequencies α away from the imaginary axis. Assuming the damping is small, the new eigenfrequency may be approximated by

$$s = i\alpha - \frac{\alpha \, \mathrm{Re}\, \tilde{R}(i\alpha)}{|d_0'(i\alpha)|} \tag{31}$$

where

$$d_0'(i\alpha) \equiv X_n - \mathrm{Im}\, \tilde{R}(i\alpha) \quad \text{and} \quad d_0(i\alpha) = 0 .$$

Clearly, if Re(s) in equation (31) is greater than the frequency shift due to parametric excitation the system will be stable. In order to stabilize the most dangerous mode, given by equation (28), one would require

$$\varepsilon < \frac{a\nu_p}{2\rho_0(\gamma\eta)^{1/2}} \, \mathrm{Re}\, \tilde{R}\left(\frac{i\omega}{2}\right), \tag{32}$$

while the higher order modes, described by equations (30), will be stabilized provided

$$\begin{aligned} \varepsilon^2 &< \frac{a\omega}{\rho_0}\left[1 + \frac{a^2(a^2+b^2)}{\gamma(b^2-a^2)}\right]^{-1} \mathrm{Re}\, \tilde{R}(i\omega), \quad n = 0, \\ \varepsilon^2 &< \frac{a\omega}{\rho_0(3\eta|n|+1)} \, \mathrm{Re}\, \tilde{R}(i\omega), \qquad n \neq 0. \end{aligned} \tag{33}$$

respectively.

Clearly condition (32) is the most stringent condition on ε, since the other conditions are on ε^2 and Re $\tilde{R}(i\omega)$ is an increasing function of ω.

These relations are useful, since Re $\tilde{R}(i\omega)$ is an experimentally measurable property of the static system.

A realistic assessment of the effect of damping is, as usual, much more difficult to obtain than the (model independent) spectrum of oscillations for the system. We shall endeavour to give an idea of the characteristic dependence of Re $\tilde{R}(i\alpha)$ by using a viscous fluid model to describe the plasma. In view of the uncertainties involved, we shall use the values of Re $\tilde{R}(i\alpha)$ already derived for a slab model[12] in the limit of small k, the Im $\tilde{R}(i\alpha)$ being essentially unaffected. One finds for symmetric modes,

$$\mathrm{Re}\,\tilde{R}(i\alpha) = \frac{4 Z_0 \lambda}{a(k^2u^2-\alpha^2)^2}\left\{\frac{\alpha^4}{3} + k^2u^2(k^2\mu^2-\alpha^2)\right\}, \tag{34}$$

while for antisymmetric modes

$$\mathrm{Re}\,\tilde{R}(i\alpha) = 4\, Z_0\, \lambda\, a k^2, \tag{35}$$

where $Z_o = \rho_o u$, ρ_o is the fluid density and $\lambda = \nu/u$, ν being the kinematic viscosity. One may note that λ is effectively the ion collision length for the plasma.

In cylindrical geometry, the symmetric and antisymmetric modes in the slab model may be used to represent the modes n = 0 and n = 1 respectively. For these cases, the stability conditions (32) reduces to condition (1)

$$\varepsilon < \frac{1}{4}\,\frac{\lambda}{a}\,\frac{\omega^2}{\nu_p^2}\left(\frac{\gamma}{\eta_1}\right)^{3/2}, \qquad n = 1 \tag{1}$$

while conditions (33) become

$$\varepsilon^2 < \frac{8\lambda\omega\left[8+3\gamma(b^2/a^2-1)\right]}{3a\nu_p\left[\gamma(b^2/a^2-1)^2+1+b^2/a^2\right]}\ , \qquad n = 0$$
(36)
$$\varepsilon^2 < \frac{2\lambda\gamma^2\omega^3}{a\nu_p^3\,\eta_1(3\eta_1+1)}\ , \qquad n = 1\ .$$

The modes $n > 1$ have an effective wavelength given by $k \sim n/a$ and since damping is strongest for shorter wavelengths we would expect the stability condition on ε to be much less severe for these modes.

Finally, for non-resonant interactions, we note that if the pressure response function $\tilde{R}(s)$ is written in the form

$$\tilde{R}(s) = R_1 + R_0 s + R_2 s^2 + R_3 s^3\ ,$$

then the real part of the frequency shift due to both damping and the oscillating field is given, from equation (A6), by

$$\mathrm{Re}\,(\delta s) = x\left\{1 - \frac{2R_3 A_n^2}{\omega^2 R_0^3}\right\} - \frac{2R_2}{\omega^2 R_0^3}\ , \qquad (37)$$

where x represents the damping in the absence of the oscillating field ($x < 0$). In the present problem

$$R_0 = \rho_0 a\ , \quad R_2 = 0\ , \qquad R_3 = \frac{-\rho_0 a}{4\nu_p^2}$$

and hence the effect of the oscillating field is small but stabilizing. Substituting the appropriate values in (37), one has

$$\mathrm{Re}\,(\delta s) = x\left\{1 + \frac{16\,\varepsilon^2\eta_n^2\nu_p^2 a^2 k^2}{\omega^2\gamma^2}\right\} \qquad (38)$$

We have shown, therefore, that there is always a value of ε below which the system is stable to parametric excitations. But the maximum size of ε , say ε_{max}, goes as $(\frac{\omega}{\nu_p})^2$. The stabilizing effect on the long wavelength modes is of order ε^2 and thus goes as $(\frac{\omega}{\nu_p})^4$. The coefficient is of the order $\frac{\lambda}{a}$, which as remarked in the introduction, is at most of the order of 1. The conclusion is then, that ω has to be chosen very near ν_p in order to have a practically interesting stabilizing effect.

As emphasized previously the damping is model dependent. It may be that other dissipation mechanisms are more effective than viscosity. But the essential difficulty lies in the fact that the stability condition is linear in ε while the stabilizing effect is in ε^2 only. If it were possible to suppress the first order parametric resonance responsible for the condition (1) we would be left with the less stringent conditions (36) which involve ε^2. If it were possible to suppress the second resonance the next condition would be still better, involving ε^3, and so on, but in no case can one hope to go much below $\varepsilon \approx \left(\frac{\omega}{\nu_p}\right)$. We now examine possible ways to alleviate the most dangerous parametric instabilities.

VI. Suppression of Parametric Resonances

We have already observed that the mode n = 1 is the most susceptible to parametric excitations. In the absence of dissipation effects, the unstable region is given by

$$\left(\frac{\gamma}{2\eta_1}\right)^{1/2}\frac{\omega}{2\nu_p}\left[1-4\varepsilon\left(\frac{\eta_1}{\gamma}\right)^{1/2}\frac{\nu_p}{\omega}\right] < ak < \left(\frac{\gamma}{2\eta_1}\right)^{1/2}\frac{\omega}{2\nu_p}\left[1+4\varepsilon\left(\frac{\eta_1}{\gamma}\right)^{1/2}\frac{\nu_p}{\omega}\right]$$

Since the most dangerous modes are of long wavelength, they are not sensitive to finite skin effects. The only possible method of suppressing, or at least weakening this instability, is by "wobbling" the frequency ω. To estimate this effect, let us assume the frequency spectrum of the oscillating field has a width $\pm\,\Delta\omega$ around ω, and that $\Delta\omega \gg \varepsilon\,\nu_p$. The maximum growth rate in the region is reduced to

$$\delta s \simeq \varepsilon \left(\frac{\eta}{\gamma}\right)^{1/2} \nu_p \left(\frac{2\varepsilon\nu_p}{\Delta\omega}\left(\frac{\eta_1}{\gamma}\right)^{1/2}\right) \tag{39}$$

since a given mode is at most unstable during a fraction $2\delta s/\Delta\omega = \frac{2\varepsilon\nu_p \eta^{1/2}}{\gamma^{1/2}\Delta\omega}$ of the time. This result is precisely what we are seeking - the growth rate has been reduced to order ε^2 instead of order ε. This "wobbling" will be even more effective on the higher resonances, which have a smaller width. Using (35) and (39), we arrive at condition (3), quoted in the introduction,

$$\varepsilon^2 < \frac{1}{8}\left(\frac{\lambda}{a}\right)\left(\frac{\gamma}{\eta_1}\right)^2\left(\frac{\omega}{\nu_p}\right)^2 \frac{\Delta\omega}{\nu_p} .$$

Thus we conclude that by wobbling the applied frequency it is possible to suppress the most dangerous parametric resonances, provided $\varepsilon \ll \frac{\omega}{\nu_p}$. The inclusion of dissipation may permit values of $\varepsilon \sim \frac{\omega}{\nu_p}$, but this would appear to be the most optimistic value of ε which this system could tolerate.

VII. Dynamic Stabilization in a Torus

If the straight θ-pinch were to be curved into a torus, the plasma would experience a net outward force due to the axial field. To first order in the inverse aspect ratio, this force can be represented by a forcing term in the n = 1 component of the magnetic pressure, given by

$$\delta p_1 = -\frac{a}{2R} B_0^2 = -\frac{a}{R} p_0 , \tag{40}$$

where R is the major radius of the torus. The average restoring force due to the alternating magnetic field may be represented, to lowest order in ε by

$$\delta \bar{p}_1 = X_1(0)\,\bar{\xi}(k=0) = \varepsilon^2 p_0 (\eta_1 - 1)\,\frac{\bar{\xi}(k=0)}{a} . \tag{41}$$

The oscillating component of the restoring force causes an oscillation of the plasma column about the average position $\bar{\xi}$ (k = 0), but provided $\varepsilon \cdot \lesssim \frac{\omega}{\nu_p}$, this amplitude is negligible. Equating (40) and (41) we find for the mean displacement

$$\bar{\xi}(k=0) = \frac{a^2}{\varepsilon^2 R(\eta_1 - 1)}$$

In order to prevent the plasma column from touching the wall we require $\bar{\xi}$ (k = 0) $<$ (b-a)/2, hence

$$\varepsilon^2 > R/(a+b) \tag{42}$$

However, in order not to excite any parametric instabilities, either condition (1) or (3) must be satisfied, depending on whether the applied frequency is "wobbled" or not. Using these conditions we derive the new conditions (2) and (4),

$$\frac{R}{a+b} > 16\left(\frac{a^2}{\lambda^2}\right)\gamma^{-3}\left(\frac{\nu_p}{\omega}\right)^4 ,$$

$$\frac{R}{a+b} > 32\left(\frac{a}{\lambda}\right)\gamma^{-2}\left(\frac{\nu_p}{\omega}\right)^2\left(\frac{\nu_p}{\Delta\omega}\right)$$

which are both very stringent. Thus we conclude that the outlook for dynamic stabilization in a torus is rather pessimistic.

APPENDIX A

Consider the equation

$$\left[X + A\cos(\omega t - \phi) + B\cos(2\omega t - 2\phi)\right]\xi(t) + \int_0^t R(t-t')\dot{\xi}(t')\,dt' = 0,$$

ϕ being an arbitrary phase. By taking the Laplace transform of this equation we find the difference equation for $\tilde{\xi}(s)$ (a superposed tilde is used to denote transformed quantities):

$$\tilde{\xi}(s) = -\frac{1}{2} A g(s)\left\{e^{-i\phi}\tilde{\xi}(s+i\omega) + e^{i\phi}\tilde{\xi}(s-i\omega)\right\}$$
$$-\frac{1}{2} B g(s)\left\{e^{-2i\phi}\tilde{\xi}(s+2i\omega) + e^{2i\phi}\tilde{\xi}(s-2i\omega)\right\}$$
$$+\xi(0)\tilde{R}(s)\,g(s) \qquad \text{(A1)}$$
$$g(s) = \frac{1}{X + s\tilde{R}(s)}$$

The poles of g(s) are the natural frequencies of the static system (A = B = 0).

If A and B are non zero, the poles of $\tilde{\xi}(s)$ are given by the zeroes of the infinite determinant D(s)[12]

$$D(s) = \left\| \delta_{\ell,m} + \frac{1}{2} A g(s+i\ell\omega)\,\delta_{\ell,m\pm 1} + \frac{1}{2} B g(s+i\ell\omega)\,\delta_{\ell,m\pm 2} \right\| \qquad \text{(A2)}$$

where ℓ, m take all values between $-\infty$ and $+\infty$. Because of the symmetry relations

$$D(s^*) = D^*(s) \quad ,$$
$$D(s+i\omega) = D(s) \quad ,$$

the zeroes of D(s) can be divided into subsets of the form

$$\left\{ z_i + i\ell\omega,\ z_i^* + i\ell\omega;\ -\infty < \ell < \infty \right\},$$

which we shall denote for convenience by $\left\{ z_i \right\}$. Whenever Im $z_i = 0 (\mathrm{mod}\ \frac{\omega}{2})$ the points $z_i^* + i\ell\omega$ coincide with the points $z_i + i\ell\omega$ and the "density" of zeroes in the set $\left\{ z_i \right\}$ is then half of the density when Im $z_i \neq 0 (\mathrm{mod}\ \frac{\omega}{2})$.

We classify the zeroes in the following way. A column of zeroes $\left\{ z_i \right\}$ is said to be

class-M	if	Im $z \neq 0$ (mod $\frac{\omega}{2}$)
class-Ω	if	Im $z = \frac{\omega}{2}$ (mod ω)
class-0	if	Im $z = 0$ (mod ω)

D(s) is an entire function of A and B and can therefore be expanded in powers of these coefficients. We are interested only in the first two parametric instabilities and hence we keep terms of order A, A^2, A^4, B, B^2, A^2B in the expansion (i.e. order ε^4), and find

$$D(s) = 1 - \left(\frac{A}{2}\right)^2 \sum_{\ell=-\infty}^{\infty} G_1(s+i\ell\omega) + \left(\frac{A}{2}\right)^4 \sum_{\substack{\ell_1,\ell_2 \\ \ell_1 \leq \ell_2 - 2}}^{\infty} G_1(s+i\ell_1\omega)\, G_1(s+i\ell_2\omega)$$
$$- \left(\frac{B}{2}\right)^2 \sum_{-\infty}^{\infty} G_2(s+i\ell\omega) + \frac{1}{4} A^2 B \sum_{\ell=-\infty}^{\infty} G_2(s+i\ell\omega)\, g(s+i\ell\omega) \tag{A3}$$

where

$$G_1(s) = g(s)\, g(s+i\omega), \quad G_2(s) = g(s-i\omega)\, g(s+i\omega)$$

In the limit A,B $\rightarrow$ 0 the zeroes coincide (modulo ω) with the poles of g(s), namely with the natural frequencies of the static system. We denote these zeroes $s_i (s_i^*)$. For A and B $\neq$ 0 but small the zeroes will remain in the vicinity of the poles (the poles are fixed). We consider successively the various possibilities.

a) <u>non-resonant case</u>:

We follow the zero which for $A,B \to 0$ coalesces with the pole s_i. We assume that $g(s+i\ell\omega) \simeq O(1)$ for $\ell \neq 0$. This implies in particular that $\text{Im } s_i \neq 0$ (mod $\frac{\omega}{2}$). D(s) can then be rewritten, keeping the leading term

$$D(s) = 1 - \frac{1}{4} A^2 g(s)\left[g(s+i\omega) + g(s-i\omega)\right] \tag{A4}$$

In the vicinity of s_i we can expand

$$d(s) \equiv g^{-1}(s) \equiv x + s\tilde{R}(s) \approx d'(s_i)(s-s_i)$$

Replacing for g(s) in (A4) we find for the zero located in the vicinity of s_i :

$$s = s_i + \frac{A^2}{4d'(s_i)}\left[g(s_i+i\omega) + g(s_i-i\omega)\right] \tag{A5}$$

These zeroes are of class-M.

When $\text{Im } s_i$ is small, the approximation for d(s) becomes inadequate, because of the presence of the other zero at s_i^*. In this case we approximate d(s) by

$$d(s) = \frac{1}{2} d''(x)\left[(s-x)^2 + \Delta^2\right] \qquad (d''(x) \text{ real})$$

where $\Delta = \text{Im } s_i$ and $x = \text{Re } s_i$. Substituting in (A4) we find for the two zeroes in the vicinity of s_i and s_i^* :

$$s = x + \frac{A^2 \operatorname{Re} g'(x+i\omega)}{2d''(x)} \pm \sqrt{-\Delta^2 + \frac{A^2}{d''(x)} \operatorname{Re} g(x+i\omega)} \tag{A6}$$

When $\operatorname{Re} g(x + i\omega)/d''(x) > 0$, the two zeroes which are of class-M move towards each other as A increases, coalesce on the real axis, then separate again giving two real zeroes which are now of class-0.

b) <u>The first resonance case:</u>

When $\operatorname{Im} s_i \simeq \frac{\omega}{2}$, the terms in $g(s - i\omega)$ are very large. Retaining the leading terms in equation (A3) gives

$$D(s) = 1 - \frac{1}{4} A^2 g(s) g(s - i\omega) \approx 1 - \frac{A^2}{4|d'{s_i}|^2 (s - s_i)(s - s_i + i\delta\omega)}$$

where $\delta\omega = \omega - \operatorname{Im} s_i$. $D(s) = 0$ has the two roots

$$s = s_i - \frac{i\delta\omega}{2} \pm \frac{1}{2} \sqrt{\frac{A^2}{|d'(s_i)|^2} - \delta\omega^2} \tag{A7}$$

For $A = 0$ we recover the roots s_i and $s_i^* + i\omega$. As A increases these two roots move together with $\operatorname{Re} s$ = constant and coalesce when $\delta\omega = \frac{A}{|d'(s_i)|}$. For $A > \delta\omega\,|d'(s_i)|$ the two roots are on the line $\operatorname{Im} s = \frac{i\omega}{2}$ and are thus of class-Ω and the square-root being real, one of the roots is destabilized. The maximum destabilizing effect occurs at $\delta\omega = 0$ for which we have

$$(\operatorname{Re} s)_{max} = \operatorname{Re} s_i + \frac{A}{2|d'(s_i)|} \tag{A8}$$

For a given A the half-width of the resonance (corresponding to the square root being real) is

$$\delta\omega = \frac{A}{|d'(s_i)|} \ .$$

3) The second resonance case:

When Im $s_i \approx \omega$, the terms in $g(s - 2i\omega)$ are very large in the vicinity of s_i. We must therefore retain these terms in equ. (A3). Bearing in mind that $B \approx O(A^2)$, we approximate D(s) by

$$
\begin{aligned}
D(s) = 1 &- \left(\frac{A}{2}\right)^2 g(s)\left[g(s+i\omega) + g(s-i\omega)\right] \\
&- \left(\frac{A}{2}\right)^2 g(s-2i\omega)\left[g(s-i\omega) + g(s-3i\omega)\right] \\
&+ \left(\frac{A}{2}\right)^4 g(s)g(s-2i\omega)\Big[g(s-i\omega)\left\{g(s-3i\omega)+g(s+i\omega)\right\} \\
&+ g(s-3i\omega)g(s+i\omega)\Big] - \left(\frac{B}{2}\right)^2 g(s)\, g(s-2i\omega) \\
&+ \frac{1}{4} A^2 B\, g(s)\, g(s-2i\omega)\, g(s-i\omega)
\end{aligned}
\tag{A9}
$$

We write $s = x + i\omega + \delta s$, $s_i = x + i\omega + i\delta\omega$ (x, $\delta\omega$ real). We have for small $\delta\omega$ and δs

$$
\begin{aligned}
d(s) &\approx d'(s_i)(\delta s - i\delta\omega) \\
d(s-2i\omega) &= d'^*(s_i)(\delta s + i\delta\omega)
\end{aligned}
$$

Replacing these values in D(s) and returning only the leading terms we find the equation for δs

$$
\begin{aligned}
&|d'(s_i)|^2 \delta s^2 - \frac{1}{2} A^2 \delta s \,\mathrm{Re}\left\{d'^*(s_i)\left[g(x)+g(x+2i\omega)\right]\right\} + \\
&+\left(\frac{A}{2}\right)^4\left[|g(x+2i\omega)|^2 + 2g(x)\,\mathrm{Re}\, g(x+2i\omega)\right] - \frac{1}{4}B^2 + \frac{1}{4}A^2 B\, g(x) + \\
&+|d'(s_i)|^2 \delta\omega^2 - \frac{1}{2}A^2 \delta\omega\, \mathrm{Im}\left\{d'^*(s_i)\left[g(x)+g(x+2i\omega)\right]\right\} = 0
\end{aligned}
\tag{A10}
$$

which has the two solutions

$$S = x + i\omega + \frac{\mathcal{R}e\,\alpha}{|d'(s_i)|^2}\left(\frac{A}{2}\right)^2 \pm \sqrt{\frac{(A^2 g(m) - 2B)^2}{4|d'(s_i)|^2} - \left(\delta\omega - \frac{\mathcal{I}m\,\alpha}{|d'(s_i)|^2}\left(\frac{A}{2}\right)^2\right)^2} \qquad \text{(A11)}$$

where

$$\alpha = d'^{*}(s_i)\left[g(x) + g(x + 2i\omega)\right]$$

For A = B = 0 the two zeroes are in s_i and $s_i + 2i\omega$, which are two elements of a row of zeroes of class-M. As long as the square root is pure imaginary, the real shift is proportional to A^2 with a coefficient Re k, which is small if damping is small. There is resonance when the square root becomes real, namely

$$-\frac{|A^2 g(x) - 2B|}{2|d'(s_i)|} \leq \left|\delta\omega - \left(\frac{A}{2}\right)^2 \frac{\mathcal{I}m\,\alpha}{|d'(s_i)|^2}\right| \leq \frac{|A^2 g(x) - 2B|}{2|d'(s_i)|}$$

We now have two rows of zeroes of class-O.

The maximum destabilizing contribution of the resonance is given by

$$\left(\mathcal{R}e\ s\right)_{max} = x + \frac{|A^2 g(x) - 2B|}{2|d'(s_i)|} \qquad \text{(A12)}$$

Note that the half width of the resonance region is equal to $(\mathrm{Re}\ \delta s)_{max}$.

APPENDIX B

Forced motion

We wish to investigate the effect of the oscillating magnetic field on the plasma pressure. For the system under consideration, the unperturbed magnetic pressure at the plasma surface r = a is

$$p = p_0 \left\{ 1 + \frac{\varepsilon^2}{1+\varepsilon^2} \cos 2\omega t \right\} \tag{B1}$$

The oscillating term in (B1) forces an oscillatory motion onto the plasma surface. The effects on the plasma quantities due to this forced motion may be obtained, to order ε^2, by equating the plasma and magnetic pressures at the unperturbed plasma surface r = a. We assume that the plasma may be described by an ideal fluid model. The fluid equations, assuming a scalar pressure and zero viscosity, may be written

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \underline{v} = 0$$

$$\frac{\partial}{\partial t} (\rho \underline{v}) + \nabla \cdot (p \underline{\underline{I}} + \rho \underline{v}\, \underline{v}) = 0 \tag{B2}$$

and

$$\frac{\partial p}{\partial t} + \underline{v} \cdot \nabla p + \gamma p \nabla \cdot \underline{v} = 0$$

where p is the plasma pressure, ρ is the density, $\underline{v}$ is the fluid velocity and γ the ratio of specific heats.

To examine the effects of the forced motion on the plasma, we expand the field quantities in the form

$$\rho = \rho_0 (1 + \varepsilon^2 \rho_F)$$

$$\underline{v} = \varepsilon^2 v_F \; \underset{\sim}{e}_r$$

$$\varsigma = \varsigma_0 (1 + \varepsilon^2 \varsigma_F)$$

Equations (B2) now reduce to

$$\frac{1}{u^2} \frac{\partial^2}{\partial t^2} p_F + \frac{1}{r} \frac{\partial}{\partial r} \left\{ r \frac{\partial p_F}{\partial r} \right\} = 0 \qquad \text{(B3)}$$

Applying the boundary condition (B1) at r = a, we find that, to order ε^2, (B3) has a solution

$$p_F = \frac{J_0 \left(\frac{2\omega}{u} r \right)}{J_0 \left(\frac{2\omega}{\nu_p} \right)} \cos(2\omega t) \qquad \text{(B4)}$$

Since we wish to examine the limit $\omega \ll \nu_p$, we may approximate (B4) by

$$p_F = \left(1 - \frac{\omega^2 (r^2 - a^2)}{u^2} \right) \cos(2\omega t) \qquad \text{(B5)}$$

From equations (B2) we obtain

$$v_F = \frac{\omega r}{\gamma} \sin(2\omega t) \qquad \text{(B6)}$$

and

$$\varsigma_F = p_F / \gamma$$

The amplitude of the surface oscillation may be written $\varepsilon^2 \xi_F$, where (for $\omega \ll \nu_p$),

$$\xi_F = -\frac{a}{2\gamma} \qquad \text{(B7)}$$

Perturbed motion

We next consider the effect of a small surface perturbation of the form

$$r = a(t) + \xi_n(k,t)\, e^{ikz + in\theta}$$

on the plasma pressure. We expand the plasma parameters in the form

$$\begin{aligned} p &= p_0 (1 + \varepsilon^2 p_F + \delta p) \\ \rho &= \rho_0 (1 + \varepsilon^2 \rho_F + \delta\rho) \\ \underline{v} &= \varepsilon^2 v_F\, \underset{\sim}{e}_r + \delta \underline{v} \end{aligned} \tag{B8}$$

Substituting (B8) into (B2) and discarding terms p_F^2, ρ_F^2, $v_F^2 (\sim \varepsilon^4)$ and δp^2, $\delta\rho^2$, $\delta \underset{\sim}{v}^2$, whilst retaining the cross terms $p_F\, \delta p$ etc., we obtain

$$\begin{aligned} \frac{\partial \delta\rho}{\partial t} + \nabla\cdot\delta\underset{\sim}{v} &= -\varepsilon^2 \left\{ \delta v_r \frac{\partial \rho_F}{\partial r} + v_F \frac{\partial \delta\rho}{\partial r} + \rho_F \nabla\cdot\delta\underset{\sim}{v} + \delta\rho\, \nabla\cdot\underset{\sim}{v}_F \right\} \\ \frac{\partial \delta\underset{\sim}{v}}{\partial t} + \frac{u^2}{\gamma}\nabla\delta p &= \varepsilon^2 \left\{ \frac{u^2}{\gamma}\left[\rho_F \nabla\delta p + \delta\rho\, \nabla p_F\right] \right. \\ &\qquad \left. - (\delta\underset{\sim}{v}\cdot\nabla)\underset{\sim}{v}_F - (\underset{\sim}{v}_F\cdot\nabla)\delta\underset{\sim}{v} \right\} \\ \frac{\partial \delta p}{\partial t} + \gamma\nabla\cdot\delta\underset{\sim}{v} &= -\varepsilon^2 \left\{ \delta v_r \frac{\partial p_F}{\partial r} + v_F \frac{\partial \delta p}{\partial r} + \gamma\left[p_F \nabla\cdot\delta\underset{\sim}{v} + \delta p\, \nabla\cdot\underset{\sim}{v}_F\right]\right\} \end{aligned} \tag{B9}$$

We now take the Laplace transform of (B9), introducing the rotation

$$f_F = \hat{f}_F \begin{matrix} \cos\omega t \\ \sin\omega t \end{matrix}$$

and obtain

$$\begin{aligned} s\delta\tilde{\rho}(s) + \nabla\cdot\delta\underset{\sim}{\tilde{v}}(s) = -\tfrac{1}{2}\varepsilon^2 \Big\{ &\left[(\nabla\hat{\rho}_F)\cdot + \hat{\rho}_F \nabla\cdot\right]\left[\delta\underset{\sim}{\tilde{v}}(s+2i\omega) + \delta\underset{\sim}{\tilde{v}}(s-2i\omega)\right] \\ &+ i\left[\hat{v}_F \frac{\partial}{\partial r} + \nabla\cdot\underset{\sim}{\hat{v}}_F\right]\left[\delta\tilde{\rho}(s+2i\omega) - \delta\tilde{\rho}(s-2i\omega)\right]\Big\} \end{aligned}$$

$$s\delta\tilde{\underset{\sim}{v}}(s) + \frac{u^2}{\gamma}\nabla\delta\tilde{p}(s) = \frac{1}{2}\varepsilon^2\Big\{\frac{u^2}{\gamma}\hat{\xi}_F\nabla(\delta\tilde{p}(s+2i\omega)+\delta\tilde{p}(s-2i\omega)$$
$$+\frac{u^2}{\gamma}\nabla\hat{\rho}_F(\delta\tilde{\xi}(s+2i\omega)+\delta\tilde{\xi}(s-2i\omega))$$
$$-i\Big[(\delta\tilde{\underset{\sim}{v}}(s+2i\omega)-\delta\tilde{\underset{\sim}{v}}(s-2i\omega))\cdot\nabla\Big]\hat{v}_F$$
$$-i(\underset{\sim}{v}_F\cdot\nabla)(\delta\tilde{\underset{\sim}{v}}(s+2i\omega)-\delta\hat{\underset{\sim}{v}}(s-2i\omega))\Big\}$$
$$s\delta\tilde{p}+\gamma\nabla\cdot\delta\tilde{\underset{\sim}{v}} = -\frac{1}{2}\varepsilon^2\Big\{\Big[(\nabla\hat{p}_F)\cdot+\gamma\hat{p}_F\nabla\cdot\Big]\Big[\delta\tilde{\underset{\sim}{v}}(s+2i\omega)+\delta\hat{\underset{\sim}{v}}(s-2i\omega)\Big] \qquad \text{(B10)}$$
$$+i\Big[\hat{v}_F\frac{\partial}{\partial r}+\gamma\nabla\cdot\hat{\underset{\sim}{v}}_F\Big]\Big[\delta p(s+2i\omega)-\delta p(s-2i\omega)\Big]\Big\}$$

Henceforth we shall use $S^{\pm}$ to denote $s \pm 2i\omega$. We may solve equations (B10) to order ε^2 by replacing quantities $\delta p(s^{\pm})$, $\delta\xi(s^{\pm})$, $\delta\underset{\sim}{v}(s^{\pm})$ by the solutions to the equations

$$s^{\pm}\delta\tilde{p}(s^{\pm}) + \gamma\nabla\cdot\delta\tilde{\underset{\sim}{v}}(s^{\pm}) = 0$$
$$s^{\pm}\delta\tilde{\underset{\sim}{v}}(s^{\pm}) + \frac{u^2}{\gamma}\nabla\delta p(s^{\pm}) = 0 \qquad \text{(B11)}$$
$$\delta\tilde{\xi}(s^{\pm}) = \delta\tilde{p}(s^{\pm})/\gamma$$

together with the boundary condition at r = a

$$(s^{\pm})^2\,\xi(s^{\pm}) = -\frac{u^2}{\gamma}\frac{\partial}{\partial r}\delta\tilde{p}(s^{\pm}) \qquad \text{(B12)}$$

Equations (B10) and (B11) may be reduced to a single equation

$$\nabla^2\delta\hat{p}(s)+(s^2/u^2)\delta\tilde{p}(s) = \frac{u^2\varepsilon^2}{2\gamma}\Big\{F_+(\delta\tilde{p}(s^+))+F_-(\delta\tilde{p}(s^-))\Big\} \qquad \text{(B13)}$$

where

$$F_{\pm}(\delta\tilde{p}(s^{\pm})) = \left[\frac{\hat{\rho}_F}{u^2}\left(s^{\pm 2} - 4\omega^2 - \frac{s^2}{\gamma}\right) \pm \frac{i}{s^{\pm}}\left\{\frac{2\partial\hat{v}_F}{\partial r}\left(\frac{n^2}{r^2} + k^2 + \frac{s^{\pm 2}}{u^2}\right) - \frac{2n^2}{r^3}\hat{v}_F\right\}\right]\delta\tilde{p}(s^{\pm})$$

$$+\left[\frac{\partial\hat{\rho}_F}{\partial r}\left(2 - \frac{s}{s^{\pm}}\right) \pm \frac{i}{s^{\pm}}\left\{\frac{\partial^2\hat{v}_F}{\partial r^2} - \frac{1}{r}\frac{\partial\hat{v}_F}{\partial r} + \left(k^2 + \frac{s^{\pm 2}}{u^2}\right)\hat{v}_F + \frac{\hat{v}_F}{r^2} + \frac{\hat{v}_F s s^{\pm}}{u^2}\right\}\right]\frac{\partial\delta\hat{p}}{\partial r}(s^{\pm}) \qquad \text{(B14)}$$

Equations (B11) together with the boundary condition (B12) have the solution

$$\delta\rho_n(s^{\pm}) = -\frac{s^{\pm 2}\gamma J_n(k_H^{\pm} r)}{u^2 k_H^{\pm} J_n'(k_H^{\pm} a)}\,\xi_n(s^{\pm}) \qquad \text{(B15)}$$

where $k_H^{\pm 2} = -k^2 - s^{\pm 2}/u^2$. Substituting the values of $\hat{v}_F$, $\hat{\rho}_F$ and $\delta\tilde{p}(s^{\pm})$ in equation (B14), we find to lowest order in $\frac{\omega}{\nu_p}$, assuming s is at most of order ω and in the limit $k \to 0$,

$$F_{\pm}(\delta\tilde{p}_0(s^{\pm})) = \frac{2s^{\pm 2}}{k_H^{\pm 2} a u^2}\left[\frac{s^{\pm 2} - 4\omega^2 - s^2/\gamma}{u^2} \mp \frac{2i\omega}{s^{\pm}} k_H^{\pm 2}\right]\tilde{\xi}_0(s^{\pm}),\ n=0$$

and for $n \neq 0$

$$F_{\pm}(\delta\tilde{p}_n(s^{\pm})) = \left\{\frac{a}{n}\left[\frac{s^{\pm 2} - 4\omega^2 - s^2/\gamma}{u^2} \mp \frac{2i\omega}{s^{\pm}}\left(\frac{n^2\omega^2}{a^2\nu_p^2} + k_H^{\pm 2}\right)\right]\right.$$

$$\left.\pm \frac{2i\omega^3}{a s^{\pm}\nu_p^2} - \frac{2\omega^2}{a\nu_p^2}\left(2 - \frac{s}{s^{\pm}}\right) \pm \frac{i\omega}{\nu_p s^{\pm}}\left(\frac{s s^{\pm}}{u} - k_H^{\pm 2} u - \frac{4\omega^2}{u^2}\right)\right\}\frac{s^{\pm 2}}{u^2}\,\xi_n(s^{\pm}) \qquad \text{(B16)}$$

Thus the RHS of (B13) is of order $\varepsilon^2 \frac{\omega^2}{\nu_p^2}$ for n = 0 and $\varepsilon^2 \frac{\omega^4}{\nu_p^4}$ for $n \neq 0$.

References

1. F.A. Haas and J.A. Wesson , Phys. Rev. Letters 19, 833 (1967)

2. J.A. Wesson and F.A. Haas, Phys. Fluids 12, 1271 (1969)

3. G.H. Wolf and G. Berge, Phys. Rev. Letters 22, 1096 (1969)

4. G. Berge, Phys. Fluids 13, 1031 (1970)

5. E.S. Weibel, Phys. Fluids 3, 946 (1960)

6. E.S. Weibel, in Proceedings of the Second International Conference on the Peaceful Uses of Atomic Energy, (United Nations, Geneva, 1958) Vol. 32, p. 166

7. F.L. Ribe and W.B. Riesenfeld, Phys. Fluids 11, 2035 (1968)

8. N.A. Bobyrev, Zh. Tech. Fiz. 36, 427 (1966) (Sov. Phys. Tech. Phys. 11, 316 (1966))

9. P.R. Forman, A. Haberstich, H.J. Karr, J.A. Phillips and A.E. Schofield, Phys. Fluids 14, 684 (1971)

10. D.V. Orlinskii, S.M. Osovets and V.I. Sinitsyn, in Plasma Physics and Controlled Nuclear Fusion Research (I.A.E.A., Vienna, 1969) Vol. II, p. 313

11. I.R. Jones, Phys. Rev. Letters 28, 135 (1972)

12. F.S. Troyon, Phys. Fluids 10, 2660 (1967)

13. F.S. Troyon in American Institute of Physics Conference Proceedings No 1, Feedback and Dynamic Control of Plasmas, (AIP, New York, 1970) p. 231