July 1972

LRP 53/72

CENTRE DE RECHERCHES EN PHYSIQUE DES PLASMAS

FINANCÉ PAR LE FONDS NATIONAL SUISSE DE LA RECHERCHE SCIENTIFIQUE

ON THE STABILITY OF A SYSTEM IN WHICH ONE PARAMETER OSCILLATES SINUSOIDALLY WITH SMALL AMPLITUDE

G.G. Lister

F.S. Troyon

LAUSANNE

July 1972 LRP 53/72

ON THE STABILITY OF A SYSTEM IN WHICH ONE PARAMETER OSCILLATES SINUSOIDALLY WITH SMALL AMPLITUDE

G.G. Lister

F.S. Troyon

Abstract

The study of the dynamic stabilization of certain plasma confinement and fluid systems leads to equations of the form

$$(X + 2\ A \cos(\omega t - \emptyset))\ y(t) + \int_{o}^{t} R(t - t')\ \dot{y}(t')\ dt' = 0\ ,$$

where X and A are constants, $\emptyset$ an arbitrary constant and R(t) a singular response function. This equation is solved for all values of A and the general stability properties of the system are discussed . General formulae are given which relate the perturbed spectrum of the system to the unperturbed system. For the case where the static system has well separated eigenfrequencies, a perturbation expansion is derived.

Lausanne

I. Introduction

The study of the dynamic stabilization of certain plasma confinement [1, 2] and fluid [3] systems reduces to the analysis of an equation of the form

$$\left\{X + 2\,A\cos(\omega t - \emptyset)\right\}\,y(t) + \int_{0}^{t} R(t - t')\,\dot{y}(t')\,dt' = 0 \qquad (1)$$

The function R(t) is a singular response function which is characteristic of the system, X is a constant, y(t) is a displacement about the equilibrium position for the system and $\emptyset$ is an arbitrary phase which represents the fact that the perturbation may be excited at any time.

In this paper we wish to examine the stability of the solutions to equation (1) when A is small, by developing a perturbation theory about A = 0.

II. Properties of the System

We introduce the Laplace transform $\tilde{R}(s)$ of the response function R(t)

$$\tilde{R}(s) = \int_{0}^{\infty} e^{-st}\,R(t)\,dt, \qquad (2)$$

where a superposed tilde is used to distinguish the Laplace transform.

We restrict this discussion to include only those classes of systems for which the following properties of $\tilde{R}(s)$ apply:

a) $\tilde{R}(s)$ is analytic in Re $s > 0$ and meromorphic in

$$\text{Re } s \leq 0 \tag{3}$$

b) Re $\tilde{R}(s) > 0$ in Re $s > 0$

c) $\lim\limits_{|s|\to\infty} |\tilde{R}(s)| \geq M > 0$ for $\lim\limits_{|s|\to\infty} |\text{Arg } s| \leq \frac{\Pi}{2}$

The fact that $R(t)$ is real implies the symmetry relation

$$\tilde{R}(s^*) = \tilde{R}^*(s)$$

The assumptions (3) are sufficiently general for most applications.

III. Stability of the System in the Absence of the Oscillating Term (A=0)

If $A = 0$, equation (1) may be resolved immediately by applying the Laplace transform, giving

$$\tilde{y}(s) = g(s)\, \tilde{R}(s)\, y(o) \tag{4}$$

where

$$g(s) = \frac{1}{X + s\, \tilde{R}(s)}$$

The symmetry of $\tilde{R}(s)$ implies

$$g(s^*) = g^*(s) \tag{5}$$

$g(s)$, like $\tilde{R}(s)$, is a meromorphic function. We denote the complex poles of $g(s)$ by s_i, s_i^* and the real poles by p_i.

If $X > 0$, one has $\mathrm{Re}\, s_i \leq 0$, $p_i < 0$ and the system is stable.

If $X < 0$, there is one and only one zero in $\mathrm{Re}\,(s) > 0$. It is a simple pole on the real axis. The system is thus unstable.

IV. Stability of the System with the Oscillating Term

The Laplace transform of equation (1) may be written

$$\tilde{y}(s) = -\,A g(s) \left\{ e^{-i\phi}\, \tilde{y}(s+i\omega) + e^{i\phi}\, \tilde{y}(s-i\omega) \right\} + g(s)\, \tilde{R}(s)\, y(0) \tag{6}$$

This equation is resolved in the Appendix and gives

$$\tilde{y}(s) = y(0) \sum_{k=-\infty}^{\infty} \frac{N_k(s)}{D(s)}\, \tilde{R}(s + ik\omega)\, e^{-ik\phi}$$

where

$$D(s) = 1 + \sum_{\ell=1}^{\infty} (-1)^{\ell}\, S_{\ell}^{\infty}(s)\, A^{2\ell}$$

$$S_{\ell}^{\infty}(s) = \sum_{\substack{k_1, k_2 \ldots k_{\ell} = -\infty \\ k_i \leq k_{i+1} - 2}}^{\infty} G(s + ik_1\omega)\, G(s + ik_2\omega) \ldots G(s + ik_{\ell}\omega) \tag{7}$$

$$G(s) = g(s)\, g(s + i\omega)$$

$$N_k(s) = A^k\, D\left[-\,k, 0/s\right] \prod_{\ell=o}^{k} g(s - i\ell\omega)$$

and

$$N_{-k}(s) = A^k D\left[0,\ k/s\right] \prod_{\ell=o}^{k} g(s + i\ell\omega)$$

The minor $D\left[i,k/s\right]$ is defined in the Appendix. From equations (5) and (7) one may deduce that

$$D(s^*) = D^*(s)$$

and (8)

$$D(s + i\omega) = D(s)$$

We denote the set of points $s + i\ell\omega$, $s^* + i\ell\omega$ (where ℓ is integral and $-\infty \leq \ell \leq \infty$) by $\left\{s\right\}$. $D(s)$ and $N_k(s)$ are entire functions of A, provided $s \neq \left\{s_i\right\}$, $\left\{p_i\right\}$ and they are meromorphic on s. The poles of $D(s)$ and $N_k(s)$ are in $\left\{s_i\right\} \cup \left\{p_i\right\}$. The zeroes of $D(s)$ are denoted by $\left\{r_i\right\}$. The identity

$$\frac{D'(s)}{D(s)} = -A^2 \sum_{\ell=-\infty}^{+\infty} G'(s + i\ell\omega) \frac{D\left[0,0/s + i\ell\omega\right]}{D(s + i\ell\omega)} \tag{9}$$

shows that $N_o(s) = \dfrac{D\left[0,0/s\right]}{D(s)}$ has at least one pole in each row $\left\{r_i\right\}$. Thus, for almost all values of $\emptyset$ (which is arbitrary) $\tilde{y}(s)$ has also at least one pole in the row $\left\{r_i\right\}$. Since $N_k(s)$ and $D_k(s)$ have, in general, the same poles, $\tilde{y}(s)$ is regular in $\left\{s_i\right\}$, $\left\{p_i\right\}$. However, it can happen that a row $\left\{s_i\right\}$ or $\left\{p_i\right\}$ is not a pole of $D(s)$ because of a cancellation between the terms in (7). Since $D(s)$ is analytic in A, a pole can only decrease in multiplicity (or disappear) if a zero coalesces with a pole. As it is impossible to have a simultaneous coalescence in $N_o(s)$ of all the poles in a row, $\tilde{y}(s)$ has at least one pole for almost every value of $\emptyset$ in the row considered. Using the convention that a zero which coalesces with a pole is still counted as a zero, one can write a single stability condition

$$\text{Re } r_i \le 0 \tag{10}$$

Classification of the zeroes of D(s)

In a row $\{r_i\}$ there is always one and only one value r such that $0 \le \text{Im } r \le \omega/2$. To simplify the terminology, we introduce the following nomenclature. A row is said to be

$$\begin{array}{lll} \text{Class-M} & \text{if} & 0 \neq \text{Im } r \neq \omega/2 \\ \text{Class-}\Omega & \text{if} & \text{Im } r = \omega/2 \\ \text{Class-0} & \text{if} & \text{Im } r = 0 \end{array} \tag{11}$$

The analyticity in A implies the conservation of the number of zeroes less the number of poles (counted with their degrees of multiplicity) in the interior of any closed contour in the s-plane, as long as no pole or zero cuts the contour. Since the number of poles is constant, the number of zeroes is consequently constant. Consider the half-band defined by $\text{Re } s \ge -c (c \ge 0)$, $0 \le \text{Im } s \le \omega/2$. $D(+\infty) = 1$ and no zero can penetrate in the domain from $+\infty$. The symmetry relations (8) imply that a zero never leaves the domain by the boundaries $\text{Im } s = 0$ and $\text{Im } s = \omega/2$. When a zero reaches one of the boundaries, another zero, the image of the first, arrives equally at the same point giving a zero of double multiplicity. The two zeroes can separate by staying either on the boundary or by separating, one to the interior and one to the exterior of the domain. Thus, if one uses the convention of dividing the multiplicity of zeroes on the boundary by 2 (which can then have a semi-entire multiplicity) the number of zeroes in the band can only change when a zero crosses the boundary $\text{Re } s \ge -c$. On making $c \to -\infty$ one deduces that the number of zeroes in the band $0 \leq \mathrm{Im}\, s \leq \omega/2$ can only change by the appearance or disappearance of zeroes at $s = -\infty$.

When $A \to 0$, $D(s) = 1$. This implies that the zeroes of D(s) coincide with the poles of D(s). One has therefore

$$\{r_i\} \equiv \{s_i\} \cup \{p_i\} , \qquad (12)$$

with the same degree of multiplicity. We wish to follow the displacement of the zeroes when A is small. To study the stability of the system, it is sufficient to follow the zeroes which are either in the half plane $\mathrm{Re}\, s > 0$, or in the neighbourhood of the imaginary axis.

Displacement of the zeroes of D(s)

We limit ourselves to the cases where g(s) has either 1 pole (necessarily on the real axis) or 2 poles on the real axis, or 2 poles which are complex conjugates. The more complicated cases may reduce to one of these cases if the poles are sufficiently separated from each other (i.e. having sufficiently different real parts). The other cases, which give rise to other possible parametric excitations are treated in the same manner but the results are not as simple.

A: g(s) has 1 pole in p_o

For $A \to 0$, D(s) has a row of single poles $\{p_o\}$ of class-0. By virtue of the conservation of zeroes, this row of zeroes cannot change its class unless it coalesces with another row of the same class. One can write

$$D(s) = D\left[0,0/s\right] - A^2\, g(s)\, D_T(s) \qquad (13)$$

where $D[0,0/s]$ and $D_T(s)$ are regular at $s = p_o$. The zeroes of D(s) are given by the equation

$$d(s) \equiv X + s\,\tilde{R}(s) = A^2 \frac{D_T(s)}{D[0,0/s]} \tag{14}$$

Since the RHS is regular at $s = p_o$, it can be developed in powers of A^2 about $s = p_o$, which gives

$$d(s) = A^2 \left[g(s + i\omega) + g(s - i\omega)\right] + 0(A^4) \tag{15}$$

Thus we find, in lowest order, for the zero of D(s) which reduces to p_o as $A \to 0$

$$p = p_o + \frac{2\,A^2}{d'(p_o)}\ \mathrm{Re}\ g(p_o + i\omega) \tag{16}$$

For small A, D(s) has as zeroes the row $\{p\}$ of class-0. If p_o is the only pole, one must have $d'(p_o) > 0$. The sign of $p - p_o$ is then the sign of $\mathrm{Re}\ g(i\omega + p_o)$.

When $X < 0$ one has $p_o > 0$. If $\mathrm{Re}\ g(x + i\omega) < 0$ for $0 \leq x \leq p_o$, the row of zeroes $\{p\}$ is displaced towards the imaginary axis as A increases and for

$$A^2 > \frac{X}{2\ \mathrm{Re}\ g(i\omega)} \tag{17}$$

one has $\mathrm{Re}\ \{p\} < 0$ and the system becomes stable. This formula is only valid if X is small. If $\mathrm{Re}\ g(i\omega) > 0$ the row of zeroes cannot cross the imaginary axis and the system remains unstable.

B. g(s) has 2 poles on the real axis

We designate the position of these poles by $m + \Delta$ and $m - \Delta$. When $A \to 0$, D(s) has 2 rows of single zeroes of class-0 at $\{m \pm \Delta\}$.

For small A, each one of these rows is displaced according to equation (16), but one has $d'(m + \Delta) > 0$ and $d'(m - \Delta) < 0$. If $\mathrm{Re}\, g(p + i\omega)$ retains the same sign in the range $m - \Delta < p < m + \Delta$, the rows of zeroes will be displaced in opposite directions, approaching each other when $\mathrm{Re}\, g(p + i\omega) < 0$ and separating if $\mathrm{Re}\, g(p + i\omega) > 0$. Equation (16) becomes invalid when the displacement becomes of the same order as Δ . One may obtain an interesting formula if one supposes that Δ is small, such that one may write $d(s) \simeq \frac{d''(m)}{2}(s-m)^2$ $(d''(m) > 0)$. By substitution into equation (15) one obtains for the position of the two zeroes which reduce to $m \pm \Delta$ when $A \to 0$,

$$s_\pm = m + \frac{2A^2}{d''(m)} \mathrm{Re}\, g'(m + i\omega) \pm \sqrt{\Delta^2 + \frac{4A^2}{d''(m)} \mathrm{Re}\, g(m + i\omega)} \tag{18}$$

The new interesting case is that for which $\mathrm{Re}\, g(m + i\omega) < 0$. For $A \ll A_o$, where

$$A_o^2 = -\frac{\Delta^2 d''(m)}{4\, \mathrm{Re}\, g(m + i\omega)} , \tag{19}$$

equation (18) reduces to (16) and D(s) has 2 rows of single zeroes of class-0 $\{s_\pm\}$. If $A = A_o$ the two rows coalesce into one row of double zeroes and if $A > A_o$ we have only one row of single zeroes of class-M, $\{s_+\}$. This is another illustration of the conservation of zeroes. A row M has twice as many zeroes as a row 0. For $A \gg A_o$

$$\mathrm{Re}\left\{s\right\} = m - \frac{\Delta^2}{2} \frac{\mathrm{Re}\, g'(m + i\omega)}{\mathrm{Re}\, g(m + i\omega)} \left(\frac{A}{A_o}\right)^2 \tag{20}$$

The sign of m is linked to the behaviour of $\tilde{R}(s)$ in the neighbourhood of $s = 0$:

$$\tilde{R}(s) \approx R_o + R_1 s + R_2 s^2 \ldots \;;\quad R_1 > 0 \tag{21}$$

When $R_o > 0$, $m \doteq - \frac{R_o}{2R_1} < 0$; but if $R_o = 0$ (implying that $R_2 < 0$, by virtue of (3b) $m \approx \frac{XR_2}{2R_1} > 0$. The sign of Re $g'(m + i\omega)$ is undefined and hence the sign of Re $\left\{s\right\}$ is equally undefined.

C. g(s) has 2 poles on the imaginary axis s_1, s_1^*

It is convenient to divide the section into separate cases.

a) Im $s_1 \neq \frac{n\omega}{2}$

This is the non-resonant case. When $A \to 0$, D(s) has a row of single zeroes at $\left\{s_1\right\}$ of class-M. To follow the displacement of the zero of D(s) which coincides with s_1 when $A \to 0$, one may once again use equation (15), the term $O\,(A^4)$ being regular in s_1. One finds that, to lowest order, the position of the zero is

$$s = s_1 + \frac{A^2}{d'(s_1)} \left[g(s_1 + i\omega) + g(s_1 - i\omega)\right] \tag{22}$$

The zeroes of D(s) are given by $\left\{s\right\}$. In this ordering, the row cannot change class. Note that the sign of $\mathrm{Re}(s-s_1)$ is undefined. As Im s_1 tends to $n\omega/2$, the range of validity of this equation tends to zero and in this region a better approximation for the displacement of the roots must be found.

b) Im $s_1 \approx 0$

The 2 zeroes $s_1 = m + i\Delta$, $s_1^* = m - i\Delta (m < 0, \Delta > 0)$ of $d(s)$ are sufficiently close to enable $d(s)$ to be represented by

$$d(s) \approx \frac{d''(m)}{2} \left[(s-m)^2 + \Delta^2\right]$$

For $A > 0$, the 2 zeroes of $D(s)$ which reduce to s_1, s_1^* when $A \to 0$ are once again given by equation (18) in which one replaces Δ^2 by $-\Delta^2$, i.e.

$$s_\pm = m + \frac{2A^2}{d''(m)} \operatorname{Re} g'(m + i\omega) \pm \sqrt{-\Delta^2 + \frac{4A^2}{d''(m)} \operatorname{Re} g(m + i\omega)} \qquad (23)$$

For small A the zeroes in (23) reduce to the zero in (22) and its complex conjugate. As long as the discriminant under the square root is negative, expression (23) differs from (22) by an imaginary part, which corresponds to a frequency shift. This shift is of no importance to the stability, provided it does not invalidate the approximation made. In this case $D(s)$ has a row of zeroes $\{s_+\}$, which is again of class-M.

When the discriminant is zero, s_+ and $s_- = s_+^*$ coalesce and one obtains a row of double zeroes of $D(s)$ of class-O. If the discriminant becomes positive, then s_+ and s_- are real and $D(s)$ has two rows of single zeroes $\{s_+\}$, $\{s_-\}$ of class-O. This can only occur if $\operatorname{Re} g(m + i\omega)/d''(m) > 0$.

c) Im $s_1 \approx \omega/2$

When Im $s_1 = \omega/2$, there is a parametric resonance. If $A = 0$, $D(s)$ has a row of double zeroes $\{s_1\}$ of class-Ω. If either $A \neq 0$ or ω is different from the resonance frequency $\omega_o = 2$ Im s_1, the row splits into either 2 rows of single zeroes of class-Ω, or into 1 row of single zeroes of class-M.

In order to study this resonance, we write

$$D(s) = D\left[-1,0/s\right] - A^2 D_T^1(s)\, g(s) - A^2 D_T^2(s)\, g(s - i\omega)$$
$$- A^2 D_T^3(s)\, g(s)\, g(s - i\omega) \tag{24}$$

where $D\left[-1, 0/s\right]$ and the $D_T^j(s)$ are analytic at $s = s_1$ and $s = i\omega + s_1^*$. Thus in the neighbourhood of these points one can develop these functions in powers of A. The equation $D(s) = 0$ may then be written

$$d(s)\, d(s - i\omega) = A^2 + A^2 \left[d(s)\, g(s - 2i\omega) + d(s - i\omega)\, g(s + i\omega)\right]$$
$$+ O\,(A^4) \tag{25}$$

We write s_1, s in the form $s_1 = m + i\Delta + \frac{i\omega}{2}$, $s = m + \frac{i\omega}{2} + \delta s$. By developing $d(s)$, $d(s - i\omega)$, $g(s - 2i\omega)$, $g(s + i\omega)$ about $\Omega = m + i\omega/2$, equation (25) becomes

$$|d'(\Omega)|^2(\delta s^2 + \Delta^2) = A^2 + 2\,A^2\,\delta s\ \mathrm{Re}\left[d'(\Omega)\, g^*(\Omega + i\omega)\right]$$
$$+ 2\,A^2\,\Delta\ \mathrm{Im}\left[d'(\Omega)\, g^*(\Omega + i\omega)\right]. \tag{26}$$

The solutions are, to lowest order

$$\delta s = \frac{A^2 \text{ Re } \alpha}{|d'(\Omega)|^2} \pm \sqrt{-\Delta^2 + \frac{A^2}{|d'(\Omega)|^2} + \frac{2 A^2 \Delta}{|d'(\Omega)|^2} \text{ Im } \alpha} \qquad (27)$$

where $\alpha = d'(\Omega)\ g^*(\Omega + i\omega)$.

When $A = 0$, $\delta s = \pm i\Delta$ and one recovers the two zeroes s_1 and $i\omega + s_1^*$, neighbouring elements of the row of zeroes $\{s_1\}$ of class-M. When the discriminant is negative, the square root is purely imaginary and Re s given by (27) coincides with the Re s calculated for the non-resonant interactions (22). When A increases from zero the two roots (27) approach each other symmetrically with respect to Im $s = \frac{i\omega}{2}$ and for $A \simeq \Delta|d'(\Omega)|$ they coalesce to give one double zero. At this point D(s) has a row of double zeroes of class-Ω. If A still increases the square root becomes real and the two zeroes are displaced along the line Im $s = \frac{i\omega}{2}$. Thus D(s) has two rows of single zeroes $\{s_+\}$, $\{s_-\}$ of class-Ω. If $A \gg \Delta|d'(\Omega)|$ the second term under the square root becomes dominant and $\delta s \simeq \pm \frac{A}{|d'(\Omega)|}$. This expression is real and linear in A and hence corresponds to a strong destabilizing effect. The half width of the resonance band is

$$\Delta_\ell = \frac{A}{|d'(\Omega)|}$$

and is equal to δs_{max}.

d) Im $s_1 \simeq \omega$

When Im $s_1 = \omega$, there is again parametric resonance. For $A = 0$, D(s) has a row of double zeroes $\{s_1\}$ of class-0. If either $A \neq 0$ or ω is not in resonance, the row splits into either 2 rows of single zeroes of class-0 or one row of single zeroes of class-M.

To study the effect of this resonance on the stability, we write

$$D(s) = D\left[-2,\ 0/s\right] - A^2\ D_T^1(s)\ g(s) - A^2\ D_T^2(s)\ g(s - 2i\omega)$$

$$+ A^4\ D_T^3(s)\ g(s)\ g(s - 2i\omega), \tag{28}$$

where $D\left[-2,\ 0/s\right]$ and the $D_T^j(s)$ are analytic in $s = s_1$ and $s = 2i\omega - s_1^*$. The equation $D(s) = 0$ can be written

$$d(s)\ d(s - 2i\omega) = \frac{-A^4\ D_T^3(s) + A^2\ d(s-2i\omega)\ D_T^1(s) + A^2\ d(s)\ D_T^2(s)}{D\left[-\ 2,\ 0/s\right]} \tag{29}$$

The RHS is regular in $s = s_1$ and $s = 2i\omega + s_1^*$ (for small A) and can be developed in powers of A, which gives

$$d(s)\ d(s-2i\omega) = -A^4\left\{g(s-i\omega)\left[g(s-3i\omega) + g(s+i\omega)\right] + g(s+i\omega)\ g(s-3i\omega)\right\}$$

$$+ A^2\ d(s-2i\omega)\left[g(s-i\omega) + g(s+i\omega)\right] + A^2\ d(s)\left[g(s-i\omega) + g(s-3i\omega)\right]$$

$$+ 0\ (A^6) \tag{30}$$

We write $s_1 = m + i\omega + i\Delta (m,\Delta$ real) and $s = m + i\omega + \delta s$.

Developing (30) in powers of Δ and δs about $\Omega = m + i\omega$, we obtain

$$\delta s^2 - \frac{2\ A^2\ \mathrm{Re}\alpha}{|d'(\Omega)|^2}\,\delta s + \frac{A^4}{|d'(\Omega)|^2}\left[2\ g(m)\ \mathrm{Re}\ g(\Omega+i\omega) + |g(\Omega+i\omega)|^2\right]$$

$$- \frac{2\ A^2\ \mathrm{Im}\alpha}{|d'(\Omega)|^2}\ \Delta + \Delta^2 = 0. \tag{31}$$

where

$$\alpha = d'(\Omega) \left[g(m) + g(\Omega + i\omega) \right] \quad .$$

Hence

$$\delta s = \frac{A^2 \ \mathrm{Re}\ \alpha}{|d'(\Omega)|^2} \pm \sqrt{\frac{A^4 \ g^2(m)}{|d'(\Omega)|^2} - \left(\Delta - \frac{A^2 \ \mathrm{Im}\ \alpha}{|d'(\Omega)|^2} \right)^2} \tag{32}$$

When $A = 0$, $\delta s = \pm\, i|\Delta|$ and we recover the two zeroes of $D(s)$ in s_1 and $2i\omega + s_1^*$, neighbouring elements of the row of zeroes $\{s_1\}$ of class-M. If the discriminant is negative, the square root is purely imaginary and $\mathrm{Re}\left\{s_i + \delta s\right\}$ given by (32) coincides with the value calculated in the non-resonant approximation (22), as for the first resonance.

When the discriminant is positive, i.e. when

$$\frac{\mathrm{Im}\ \alpha}{|d'(\Omega)|^2} - \frac{|g(m)|}{|d'(\Omega)|} < \frac{\Delta}{A^2} < \frac{\mathrm{Im}\ \alpha}{|d'(\Omega)|^2} + \frac{|g(m)|}{|d'(\Omega)|} \quad , \tag{33}$$

δs is real and the two roots are on the line $\mathrm{Im}\ s = \omega$. Thus $D(s)$ has two rows of single zeroes $\left\{m + \delta s_+\right\}$, $\left\{m + \delta s_-\right\}$, of class-0. This destabilizing resonant effect is maximum for

$$\Delta = \frac{\mathrm{Im}\ \alpha}{|d'(\Omega)|^2} A^2 \ ,$$

in which case the supplementary contribution δs_{max} to $\mathrm{Re}\ \delta s$ is

$$\delta s_{max} = \frac{A^2 |g(m)|}{|d'(\Omega)|^2} \tag{34}$$

This contribution is thus of order A^2, as is the non-resonant interaction. The non-resonant contribution, however, is often very small or null (generally Re α << Im α). The half width of the frequency band over which there is a contribution from the resonant term is given by

$$\Delta_\ell = \frac{A^2 |g(m)|}{|d'(\Omega)|^2}$$

and is equal to δs_{max}, as in the case of the first resonance.

d) Im s $\simeq \frac{n\omega}{2}$ (n $\geq$ 3)

Generalizing the preceeding results, one can see that for Im $s_1 = \frac{n\omega}{2}$ and $A \to 0$, D(s) has a row of double zeroes $\{s_1\}$ of class-Ω or 0, depending on whether n is odd or even respectively. If $A \neq 0$ and (or) $\Delta = \text{Im } s_1 - \frac{n\omega}{2} \neq 0$ the row of double zeroes separates either into two rows of single zeroes of class-0 or Ω, or 1 row of class-M.Off resonance, the row is of class-M and Re s is given correctly by equation (22). There is a resonance when the row of class-M becomes 2 rows of classes Ω or 0. This occurs when

$$- \beta A^n + \alpha A^2 \leq \Delta \leq \alpha A^2 + \beta A^n , \qquad (36)$$

where α and β are independent of A. The width of the resonance region varies as $\sim A^n$. The additional contribution to Re δs in this region also varies as A^n. It is thus of higher order then the non-resonant contribution. However, when the system is non dissipative, the non-resonant contribution is identically zero. When the system is weakly dissipative (Re $\tilde{R}(iy)$ << Im $\tilde{R}(iy)$), some resonances can make a non-negligible contribution for $n \geq 3$, but for sufficiently large n the non-resonant effects will dominate in any case.

V. Summary

For the reader's convenience we present here a recipe for the use of the results which we have derived in the main text of this paper. One proceeds as follows:

1) determine the unperturbed spectrum (A = 0); s_i, s_i^* and p_i, poles of $g(s)$;

2) complete the spectrum by adding all the points modulo $(i\omega)$, thus forming the row spectrums $\{s_i\}$ and $\{p_i\}$;

3) if the zeroes $\{s_i\}$ and $\{p_i\}$ are simple and

$$\left|\{s_i\} - \{s_j\}\right| \gg O(A) \ , \qquad \left|\{p_i\} - \{p_j\}\right| \gg O(A^2) \ ,$$

the rows do not interact with each other in leading order (in A) and we find for the modified spectrum:

a) $\{\bar{p}_i\}$, where

$$\bar{p}_i = p_i + \frac{2\,A^2}{d'(p_i)} \,\mathrm{Re}\; g(p_i + i\omega) \ .$$

b) $\{\bar{s}_i\}$, where

$$\bar{s}_i = s_i + \frac{A^2}{d'(s_i)} \left[g(s_i + i\omega) + g(s_i - i\omega)\right]$$

This result requires $|\bar{s}_i - s_i| \ll |s_i - s_i^*|$. mod (ω)

c) When $|\bar{s}_i - s_i| \approx \mathrm{Im}\, s_i$ and d(s) can be approximated by

$$d(s) = \frac{d''(m)}{2} \left[(s - m)^2 + \Delta^2\right]$$

valid in the range $|s - m| \simeq \Delta$, where $m = \mathrm{Re}\, s_i$ and $\Delta = \mathrm{Im}\, s_i$, the spectrum becomes either $\{\bar{s}_i\}$, where

$$\bar{s}_i = m + \frac{2\, A^2}{d''(m)} \;\mathrm{Re}\; g'(m + i\omega) + i \sqrt{- D}\;,$$

$$D = - \Delta^2 + \frac{4\, A^2}{d''(m)} \;\mathrm{Re}\; g(m + i\omega) \quad ,$$

in the case where $D < 0$,

or $\{\bar{p}_+\}$, $\{\bar{p}_-\}$, where

$$\bar{p}_\pm = m + \frac{2\, A^2}{d''(m)} \;\mathrm{Re}\; g'(m + i\omega) \pm \sqrt{D}$$

when $D > 0$.

d) When $|\bar{s}_i - s_i| \simeq |s_i - s_i^* + i\omega|$ (the first parametric resonance), the spectrum is given by either $\{\bar{s}_i\}$, where

$$\bar{s}_i = s_i - i\Delta + \frac{A^2 \;\mathrm{Re}\;\alpha}{|d'(\Omega)|^2} + i \sqrt{- D}$$

$$D = -\Delta^2 + \frac{A^2}{|d'(\Omega)|^2} + \frac{2A^2\Delta}{|d'(\Omega)|^2} \text{ Im } \alpha \quad ,$$

$\alpha = d'(\Omega)\ g^*(\Omega + i\omega)$ and $\Omega = m + \frac{i\omega}{2}$, when $D < 0$,

or by $\{s_{i+}\}$; $\{s_{i-}\}$, where

$$\bar{s}_{i\pm} = s_i - i\Delta + \frac{A^2 \text{ Re } \alpha}{|d'(\Omega)|^2} \pm \sqrt{D}$$

when $D > 0$.

e) When $|\bar{s}_i - s_i| \sim |s_i - s_i^* + 2i\omega|$ (the second parametric resonance), the spectrum is given either $\{\bar{s}_i\}$, where

$$\bar{s}_i = s_i - i\Delta + \frac{A^2 \text{ Re}}{|d'(\Omega)|^2} + i\sqrt{-D} \quad ,$$

$$D = \frac{A^4 g^2(m)}{|d'(\Omega)|^2} - \left(\Delta - \frac{A^2 \text{ Im } \alpha}{|d'(\Omega)|^2}\right)^2 \quad ,$$

$\alpha = d'(\Omega)\ \left[g(m) + g(\Omega + i\omega)\right]$, $\Omega = m + i\omega$,

when $D < 0$, or by $\{\bar{s}_{i+}\}$, $\{\bar{s}_{i-}\}$, where

$$\bar{s}_{i\pm} = s_i - i\Delta + \frac{A^2 \text{ Re } \alpha}{|d'(\Omega)|^2} \pm \sqrt{D}$$

when $D > 0$.

f) When $|\bar{s}_i - s_i| \approx |s_i - s_i^* + ni\omega|$, $n \geq 3$, corresponding to higher order parametric resonances, the resonance effects are of order A^n. They are of higher order than the non-resonant shift b) and are only important if the non-resonant contribution is very small (system very weakly dissipative).

4) If there are two interacting rows $\{p_1\}$ and $\{p_2\}$, where $p_{1,2} = m \pm \Delta$, and $d(s)$ may again be represented by $d(s) = \frac{d''(m)}{2} \left[(s - m)^2 + \Delta^2\right]$ in the range $|s - m| \approx |p_1 - p_2|$ then the modified spectrum becomes either $\{\bar{s}_i\}$, where

$$\bar{s}_i = m + \frac{2A^2}{|d''(m)|} \operatorname{Re} g'(m + i\omega) \pm i \sqrt{-D},$$

$$D = \Delta^2 + \frac{4A^2}{d''(m)} \operatorname{Re} g(m + i\omega) \text{ , when } D < 0$$

or $\{\bar{p}_+\}$, $\{\bar{p}_-\}$ where

$$\bar{p}_\pm = m + \frac{2A^2}{d''(m)} \operatorname{Re} g'(m + i\omega) \pm \sqrt{D} \qquad ,$$

when $D > 0$.

5) If the rows $\{s_i\}$ are not well separated there may be interactions every time the displacement of the roots calculated separately for each row by 3) becomes of the same order as the separation of the rows. In this case, our results are not applicable, although the method used here may again be applied.

APPENDIX A

In order to resolve equation (6) one follows exactly the same procedure used in resolving the Fredholm equation.

By successive replacement of s by s + inω in equation (6), one obtains an infinite system of difference equations for $\tilde{y}(s + in\omega)$,

$$\tilde{y}(s + in\omega) = - Ag(s + in\omega) \left\{ \tilde{y}(s + i\omega + in\omega)e^{-i\emptyset} + \tilde{y}(s + in\omega - i\omega)e^{i\emptyset} \right\} + y(0)\ g(s + in\omega)\ \tilde{R}(s + in\omega), \tag{A1}$$

n taking all values from $-\infty$ to ∞ .

Making the change of function

$$\tilde{y}(s + in\omega) = e^{in\emptyset}\ Y(n),$$

$$Y(n) + Ag(s + in\omega) \left\{ Y(n + 1) + Y(n - 1) \right\} = y(0)e^{-in\emptyset}\ \tilde{R}(s + in\omega)\ g(s + in\omega)$$

Resolving this linear system by Kramers rule,

$$Y(n) = \frac{N(n,s)}{D(s)}\ y(0)$$

where D(s) is the infinite tridiagonal determinant

$$D(s) = \left| \delta_{\ell,m} + Ag(s + i\ell\omega)\ \delta_{\ell,m\pm1} \right| \qquad (A2)$$

and where the indices ℓ,m take all values from $-\infty$ to $+\infty$. $N(n,s)$ is a periodic function of $\emptyset$. Writing

$$N(n,s) = \sum_{k=-\infty}^{\infty} N_k(n,s)\ \tilde{R}(s + ik\omega)\ e^{-ik\emptyset} ,$$

one has the relation

$$N_k(n,s) = N_{k+n}(0,s + in\omega)$$

$N_k(0,s)$ may be expressed as a function of the minors of $D(s)$, $D\left[i,k/s\right]$ defined in the following manner: the minor $D\left[i,k/s\right]$ is equal to the infinite determinant (A2), from which the lines and columns $i \leq \ell \leq k$, $i \leq m \leq k$ have been suppressed. One has $(k > 0)$:

$$N_k(0,s) = A^k\ D\left[0,k/s\right] \prod_{\ell=o}^{k} g(s + i\ell\omega)$$

$$N_{-k}(0,s) = A^k\ D\left[-k,0/s\right] \prod_{\ell=o}^{k} g(s + i\ell\omega)$$

<u>Development of D(s)</u>

Introducing the truncated determinants D_k, where

$$D_k = D\left[k, +\infty/s\right],$$