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WIDE BAND MEASUREMENT OF LARGE CURRENTS

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A b s t r a c t

A method is described for the measurement of large currents with an accuracy of better than one percent from zero frequency to 47 Mc which is specially suited for currents above 100 A.

Lausanne

I. Introduction

In plasma physics experiments one is always faced with the problem of measuring large, rapidly varying currents, typically 1-100 kA and .01-10 Mc. This is usually done by magnetic pick-up coils, currents transformers with or without cores (Rogowski loops) or by means of low resistance shunts[1,2]. All of these methods are subject to errors due to stray magnetic fields and the frequency response of all such devices is often severely limited by the skin effect and parasitic capacity.

To overcome these difficulties two simple devices for the measurement of currents have been constructed and their performance has been carefully analysed and checked by experiments. The frequency intervals in which these devices can be used are expressed in terms of the maximum allowable relative error.

Each of these instruments gives a voltage

$$V_1 = L \frac{dI}{dt} \qquad (1)$$

in a certain frequency range. To obtain a signal proportional to the current I, V_1 must be integrated

$$V = \frac{1}{\tau} \int_0^t V_1 dt = \frac{L}{\tau} I \qquad (2)$$

In what follows we ignore possible deficiencies of integrators of which there exist many types, active and passive. A very accurate wide band integrator has been described by R. Keller[3]. What concerns us here is the accuracy with which the relation (1) can be realized in practice. The problems one encounters are due to the skin effect which gives rise to a frequency dependent impedance Z(d/dt) rather than simply L d/dt.

The first of the two devices is called a current probe: it satisfies (1) from zero frequency to 10 megacycles, with a maximum error of 1 % at the upper frequency limit. But the factor L cannot be determined from its construction.

The second device, called a calibrator satisfies (1) from .75 Mc to 47 Mc with the same precision of better than 1 %. In addition its factor L can be calculated from its geometrical dimensions with great accuracy (.1 %). This instrument is used to calibrate the current probe by comparing their outputs for a current whose spectrum lies in the frequency range common to both devices .75 Mc - 10 Mc. Both devices are immune to even the largest stray-fields encountered in a plasma physics laboratory.

II. The current probe

It is shown in Fig. 1. The current is introduced into the central conductor, flows around the walls of a cylindrical cavity and returns coaxially on the outer shell of the device. The voltage V_1 is induced in the loop placed into the cavity. The thick cavity walls serve to shield the loop from any stray fields while the long coaxial input guarantees a current distribution in the cavity which is independent of how the external leads are connected to the device. Such a measuring cavity can be built directly into an experiment as an integral part of the equipment and may have a different and even asymmetrical shape.

Let the radius of the wire be a, its length ℓ , and the area enclosed by the loop A (Fig. 2). If there were no loop, the field B within the cavity would be everywhere strictly proportional to the current I and independent of frequency. But the finite conductivity of the loop allows currents to flow in it which modify the field. These currents can be considered as the superposition of those which are present in the loop even if the terminals are open (eddies) and those which flow in the loop when an external load is connected. Consider first the effect of the eddy currents which is examined in the Appendix: the open circuit voltage can be expanded into a power series of $i\omega$ of which the first term equals

$$V_1 = i\omega \int_{(\Gamma,\gamma)} B_o \, dA = i\omega L I \qquad (3)$$

where B_o is the field which would be present without loop and the integral is over a surface bounded by the curves Γ and γ , Fig. 2. The next higher term, which is proportional to $(i\omega)^2$ is smaller than the first by the ratio

$$\mu_o \sigma \omega a^3 \ell \,/\, A.$$

where σ is the conductivity of the wire material. This term must be negligible so that one must require

$$\mu_o \sigma \omega a^3 \ell \,/\, A \leq \varepsilon$$

where ε is some preassigned accuracy. Thus the frequency is limited from above by

$$f \leq f_1 = \frac{\varepsilon A}{2\pi \mu_o \sigma a^3 \ell} \qquad (4)$$

If the voltage V_1 could be integrated at infinite impedance then (4) would be the only limitation of the frequency response. In reality there is always a stray capacity C connecting the terminals. Also one is forced to connect a cable of some characteristic impedance Z to the output. Now a net current flows through the loop, causing a voltage drop across its terminals. Hence the voltage which is transmitted through the cable, integrated and observed is not V_1 but V_2, Fig. 3. The impedance of the loop consists of two parts: an inductive part $L_1\omega$ due to the magnetic field external to the wire and a part Z_w due to the finite resistance and the magnetic field inside the wire. The theory of the skin effect in wires gives

$$Z_w = R \frac{\eta J_o(\eta)}{2 J_1(\eta)}$$

where

$$R = \ell / (\pi\sigma a^2) \quad , \quad \eta^2 = -\mu\sigma a^2 i\omega$$

and J_n are Bessel functions. At low frequency ($|\eta| << 1$) Z_w can be approximated by expanding the Bessel functions up to the second term in their power series

$$Z_w = R + \mu\ell\, i\omega / 8\pi = R + L_2 i\omega$$

Thus the inductance L_o of the equivalent circuit (Fig. 3) equals $L_1 + L_2$. Examination of the equivalent circuit yields V_2 in terms of V_1.

$$V_2 = V_1 \frac{Z}{Z+R} \cdot \left[1 + \frac{L_o + ZRC}{Z+R}(i\omega) - \frac{ZL_oC}{Z+R}\omega^2\right]^{-1}$$

If the ratio V_2 to V_1 is to be independent of frequency within an error ε the following two conditions must be satisfied

$$f \leq f_2 = \frac{\varepsilon}{2\pi} \frac{Z + R}{L_1 + L_2 + ZRC} \qquad (5)$$

$$f \leq f_3 = \frac{\sqrt{\varepsilon}}{2\pi} \sqrt{\frac{Z + R}{Z(L_1 + L_2)C}} \qquad (6)$$

A probe was built with the following characteristics: $a = .01$ mm, $\ell = 10$ cm, $A = .8$ cm^2. As conductor for the loop Molybdenum was chosen, because it is less fragile than copper. The resistance R was measured to be $R = 18.1\ \Omega$. The probe is used with a $Z = 75\ \Omega$ cable. The inductance L_o was calculated approximately to be $L_o = 1.4\ 10^{-8}$ Hy. The stray capacity C is roughly 10^{-12} F. Hence one obtains for the three limiting frequencies in cycles per second

$$f_1 = 1.5\ 10^{10}\ \varepsilon\ , \quad f_2 = 10^9\ \varepsilon\ , \quad f_3 = 1.4\ 10^{10} \sqrt{\varepsilon}$$

The second bound is the most stringent one. If one requires an accuracy of $\varepsilon = 10^{-2}$ one obtains as upper frequency limit

$$f = 10 \text{ Megacycles.}$$

III. The current calibrator

It is shown in Fig. 4. It is similar to the current probe, except that there is no loop and the voltage V_1 is tapped off the cavity itself. The parts are machined with a precision of a few microns.

The voltage developed at the terminal of the device equals

$$V_1 = Z(i\omega)\ I$$

where Z consists of an inductance due to the field within the cavity and a frequency dependent part due to the skin effect:

$$Z(i\omega) = Li\omega + \lambda\sqrt{i\omega} + \lambda_1 + \lambda_2(i\omega)^{-1/2} + \lambda_3(i\omega)^{-1} + \ldots\ .$$

The first two terms suffice for our purpose and are easily calculated.

$$L = \frac{\mu_o}{2\pi}\left\{h\ \lg(b/a) + h_1\ \lg(b/a)\right\}\ , \tag{7}$$

$$\lambda = \frac{1}{2\pi}\sqrt{\frac{\mu_o}{\sigma}}\left\{\frac{h+h_1}{b} + \frac{h}{a} + \frac{h_1}{a_1} + 2\ \lg\ b/a - \lg\ b/a_1\right\}. \tag{8}$$

For the dimensions shown in Fig. 4 the values of L and λ are

$$L = 2.00\ 10^{-9}\ \text{H},$$

$$\lambda = 4.33\ 10^{-8}\ \Omega s^{1/2}.$$

If one wishes to keep the corrective term $\lambda\sqrt{i\omega}$ down to a fraction ε of the principal term $Li\omega$ one obtains the frequency condition

$$f \geq f_4\ \frac{\lambda^2}{2\pi\varepsilon^2 L^2} = .75\ \text{Mc}.$$

The distortion of the wave due to the term $\lambda\sqrt{i\omega}$ can be exhibited by examining the response of the calibrator to a step current $I = 0$, $t < 0$, $I = 1$, $t > 0$:

$$V = \frac{1}{\tau}\int_0^t V_1 dt = \frac{L}{\tau}\left[1 + \frac{2}{\pi}\frac{\lambda}{L}\sqrt{t}\right].$$

The second term in the parenthesis represents the distortion due to the skin effect. It is negligible for very short times. If this distortion is to be a fraction ε then $t \leq \frac{\pi}{4} (\varepsilon \frac{L}{\lambda})^2$. Therefore a square wave with a period of .33 μs is reproduced accurately to 1 %. If 5 % distortion can be tolerated the period of the square wave can be 8.4 μs[4].

At a fixed single frequency the calibrator can be used down to lower frequencies if one takes into account the term $\lambda\sqrt{i\omega}$.

$$V_1 = \left[i \left(L\omega + \lambda\sqrt{\frac{\omega}{2}}\right) + \lambda \frac{\omega}{2} \right] I$$

or for the calibrator described above

$$V_1 = i\omega L \left[1 + .0061\ (1 + i)\ f^{-1/2} \right] I$$

where f is in megacycles and L = 2.00 nH.

As mentioned in the introduction this current calibrator is used primarily to calibrate the current probe by connecting the two in series, passing a current whose frequency spectrum lies between .75 Mc and 10 Mc, and comparing outputs.

The upper frequency limit is given by the self resonance of the cavity.

For low frequencies up to and including its lowest resonance, the cavity can be approximated by the equivalent circuit of Fig. 5 where $L + L_1$ is the total inductance of the device as seen from its input. C is the capacity between the central rod and disc and the surrounding return conductor. Thus

$$V_1 = i\omega L \left[1 - (L + L_1)\ C\omega^2 \right]^{-1} I$$

To keep the response proportional to $i\omega$ within an error of ε the frequency must be limited from above

$$f \leq f_5 = \frac{\sqrt{\varepsilon}}{2\pi} \left[(L + L_1) C \right]^{-1/2}$$

For the calibrator described above one finds $L + L_1 = 4$nH, $C = 29$ pF and

$$f_5 = \sqrt{\varepsilon} \quad 4.7 \quad 10^8 \quad \text{sec}^{-1}$$

If the error tolerated is 1 %, the frequency must lie below 47 Mc. Thus the calibrator is accurate to 1 % from .75 Mc to 47 Mc.

IV. Optimum design of the calibrator

The dimensions of the cavity should be chosen such as to minimize the ratio λ/L. In practice the value of L is determined by the magnitude of the currents to be measured, their frequency content and the characteristics of the integrator. Also one wishes to keep the calibrator down to a reasonable size. Hence one wants to minimize λ while keeping L and b fixed at predetermined values. This can be done by using the method of Lagrange for minimizing λ/L while keeping L constant. If one sets

$$\tilde{L} = \frac{4\pi}{\mu_o} L \qquad \text{and} \qquad a = b\, e^{-x}$$

one obtains the equation

$$\left(\frac{4b}{\tilde{L}} x^2 - 1\right) e^{-x} = 1 - x,$$

which must be solved for x. With x determined one finds a and h:

$$a = be^{-x} \qquad\qquad h = \tilde{L}/2x$$

Fixing b = 5 cm, L = 2nH that is $\tilde{L}$ = 2 cm one obtains x = .432, a = 3.25 cm and h = 2.32 cm. These are the dimensions used in the calibrator shown in Fig. 4. The value of λ is:

$$\lambda = 4.33\ 10^{-5}\ \mathrm{VA}^{-1}(\mu s)^{1/2}.$$

V. Comparison of Probe and Calibrator

In the frequency interval from 1 to 10 Mc the probe and the calibrator have both the same frequency response (namely proportional to ω). This can be verified by passing the same current through both devices and recording their output. This has been done and the results are plotted in Fig. 6. At frequencies below .75 Mc the calibrator gives too high an output which can be corrected according to equation (10). With this correction the ratio of the output of the calibrator and of the probe stays constant within one percent between 100 kc and 15 Mc. At higher frequencies the probe gives too low a voltage due to the drop across the inductance L.

Together the two instruments cover the frequency range from zero to 47 Mc with an overlap between 1 and 10 Mc.

APPENDIX

Consider the loop shown in Fig. 3. As before let the radius of the wire be a, its length ℓ and the area enclosed A. An exact solution of the problem would require the solution of Maxwell's equations without displacement current (because the wavelength shall always be much greater than the size of the loops)

$$\begin{aligned} \operatorname{curl}\,(\underline{B}_o + \underline{B}) &= \mu_o \sigma \underline{E} \\ \operatorname{curl}\,\underline{E} &= -i\omega \underline{B} \\ \operatorname{div}\,\underline{B} &= 0 \\ \operatorname{div}\,\underline{E} &= 0 \end{aligned} \qquad \text{(A 1)}$$

Here $\underline{B}_o$ is the magnetic field that would be present in the cavity without loop:

$$B_{o\varphi} = \mu_o I \,/\, 2\pi r.$$

The boundary conditions at the surface of the conductors are as usual:

$\underline{B}$ continuous	(no surface currents)
$\underline{n} \mathrm{x} \underline{E}$ continuous	(dB/dt finite)
$\underline{n} \cdot \underline{E}_c = 0$	(no current leaves conductor)

where $\underline{n}$ is the vector normal to the surface of the conductor and the subscript c denotes the field within the conductor.

The purpose of this appendix is not to solve these equations, which would be very difficult and not necessary, but to make estimates of the frequency dependence of the solution. To this end we expand the solution into a power series in the parameter

$$\alpha = i\mu_o \sigma\omega$$

by writing

$$\underline{B} = \sum_{1}^{\infty} \underline{B}_n \, \alpha^n$$

$$\underline{E} = i\omega \sum_{o}^{\infty} \underline{E}_n \, \alpha^n \qquad \text{(A 2)}$$

This leads to a system of equations ($n \geq o$)

$$\text{curl } \underline{E}_n = - \underline{B}_n \qquad \text{div } \underline{E}_n = 0$$

$$\text{curl } \underline{B}_{n+1} = \underline{E}_n \qquad \text{div } \underline{B}_{n+1} = 0$$

which could, in principle, be solved successively for $\underline{E}_o$, $\underline{B}_1$, $\underline{E}_1$ and so on. The fields $\underline{E}_n$, $\underline{B}_n$ ($n \geq 1$) are due to the currents within the loop and are therefore concentrated in and around this conductor.Thus one has the following relation for the orders of magnitude

$$\left[\text{curl } \underline{E}_n\right] \approx \frac{1}{a} \left[\underline{E}_n\right] \qquad n \geq 1$$

$$\left[\text{curl } \underline{B}_n\right] \approx \frac{1}{a} \left[\underline{B}_n\right] \quad , \qquad n \geq 1$$

and hence

$$\left[\underline{E}_n\right] \approx a^{2n-1} \left[B_o\right] \qquad n \geq 1$$

$$\left[\underline{B}_n\right] \approx a^{2n} \left[B_o\right] \qquad n \geq 1 \qquad \text{(A 3)}$$

where the brakets indicate "order of magnitude". Let us now integrate equation (A 1) over surface bounded by two curves Γ and γ , indicated in Fig. 2.

$$\frac{1}{i\omega}\int_{\gamma} \underline{E} \cdot \underline{ds} = \int_{(\Gamma,\gamma)} \underline{B} \cdot d\underline{A} + \frac{1}{i\omega}\int_{\Gamma} \underline{E} \cdot \underline{ds}$$

The line integral along γ equals the voltage V_1 which appears at the terminals of the loop. The surface integral and the line integral along Γ can be written in terms of the expansions (A 2)

$$\frac{V_1}{i\omega} = \int_{(\Gamma,\gamma)} (\underline{B}_o + i\mu\sigma\omega\underline{B}_1 + \ldots) \cdot d\underline{A} + \int_{\Gamma} (\underline{E}_o + i\mu\sigma\omega E_1 + \ldots) \cdot \underline{ds} \qquad \text{(A 4)}$$

To estimate the order of magnitude of the four terms on the right side of (A 4) we observe that $\underline{B}_o$ extends over the entire loop, while B_1 is concentrated in and around the wire of the loop. This is so because B_1 is the field due to the eddy currents in the wire which form two anti-parallel streams. Therefore $\underline{B}_1$ falls off as the inverse square of the distance from the wire axis. Thus the effective area of integration for B_1 is of the order of $a\ell$ while for the field B_o it is A. It is now possible to write down the orders of magnitude of the first four terms of (A 4):

$$B_oA \quad : \quad \mu_o\sigma\omega B_1 \ell a \quad : \quad E_o\ell \quad : \quad \mu_o\sigma\omega E_1\ell$$

Using (A 3) a d dividing through by B_oA one obtains the relative orders of magnitude of these terms:

$$1 \quad : \quad \mu_o\sigma\omega \frac{a^3\ell}{A} \quad : \quad \frac{a\ell}{A} \quad : \quad \mu_o\sigma\omega \frac{a^3\ell}{A}$$

Only the second and the fourth terms are frequency dependent and their ratio to the constant dominant term is

$$\mu_o \sigma \omega \ \frac{a^3 \ell}{A}$$

which is the desired result.

The integral over E_o disappears if the path of integration τ is chosen such that it passes only through points at which E is zero. This is possible because $E = j/\sigma$ and because the total current through each cross section of the wire is zero.(This special path is of course very close to the axis of the wire). By definition B_o is strictly proportional to I so that we may define a constant inductance L by the equation

$$\int B_o \, dA = LI$$

thus

$$V_1 = i\omega LI \left[1 + \text{order} \ \mu_o \sigma \omega \ \frac{a^3 \ell}{A} \right]$$

Acknowledgement

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R e f e n c e s

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2) J. Cooper, J. Nuclear Energy, Pt. C 5, 285 (1963)

3) R. Keller, Rev. of Sci. Instr. 35, 1057 (1964)

4) Always assuming that the input is integrated perfectly, see introduction.

Figures

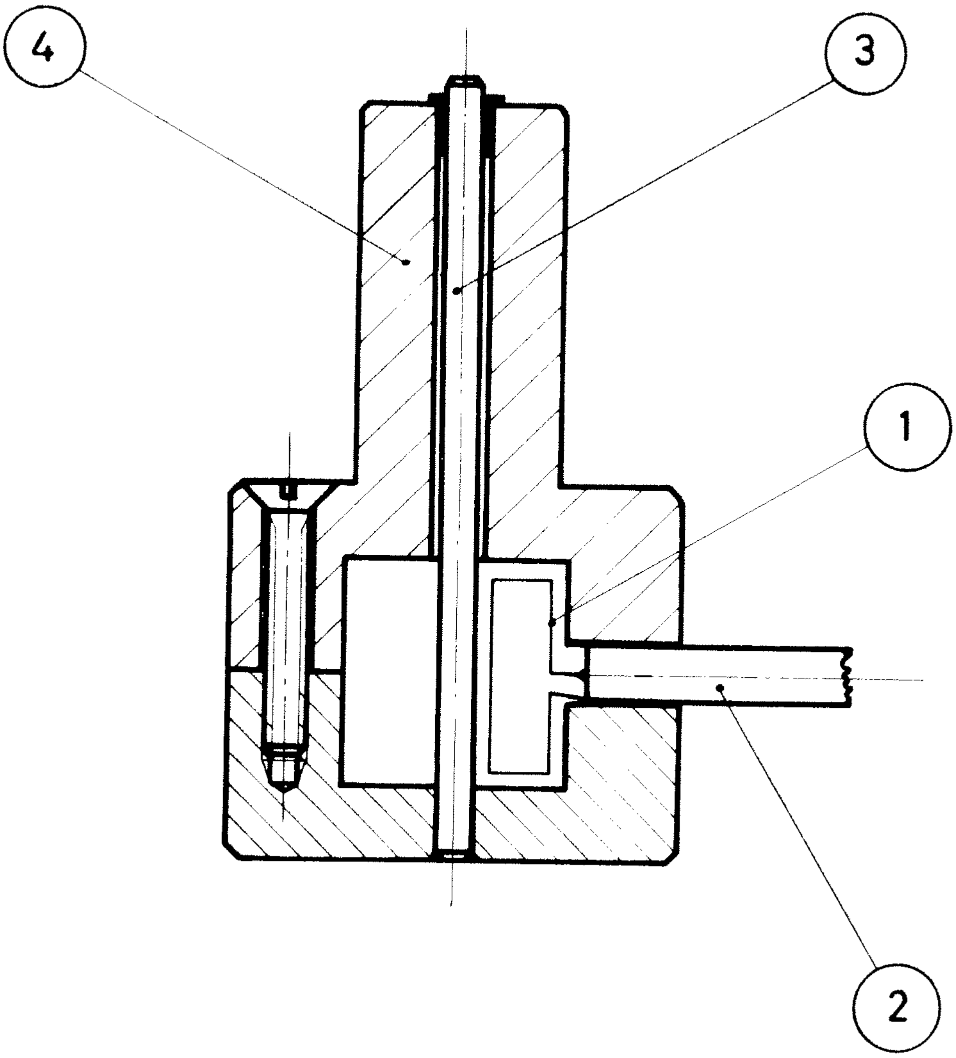

Fig. 1

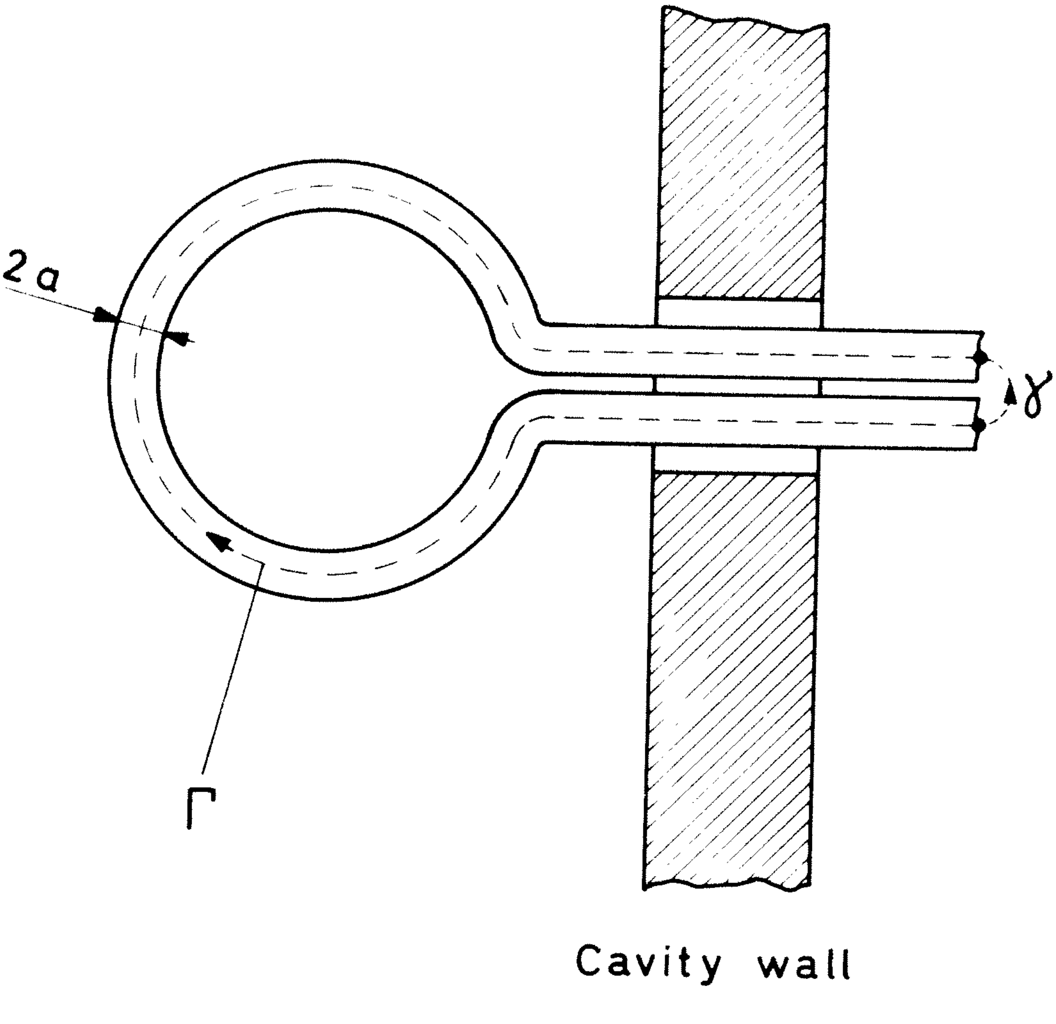

Fig. 2

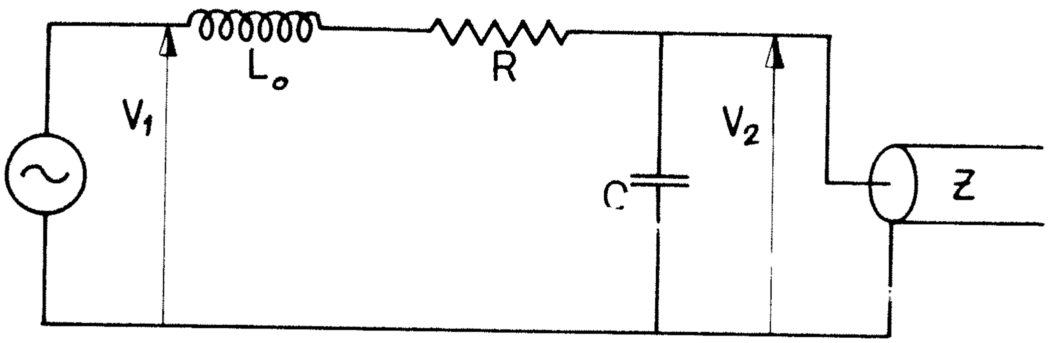

Fig. 3

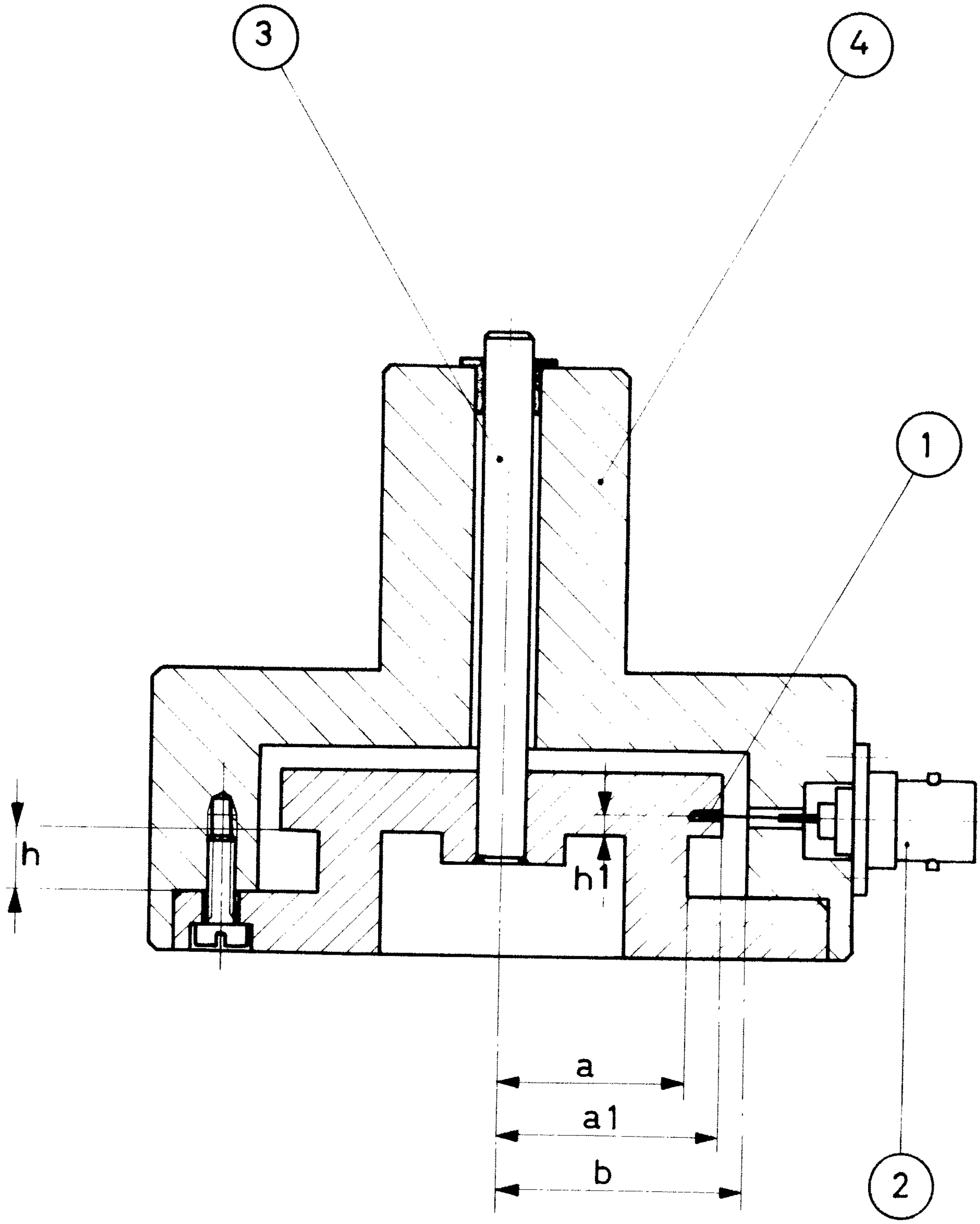

Fig. 4

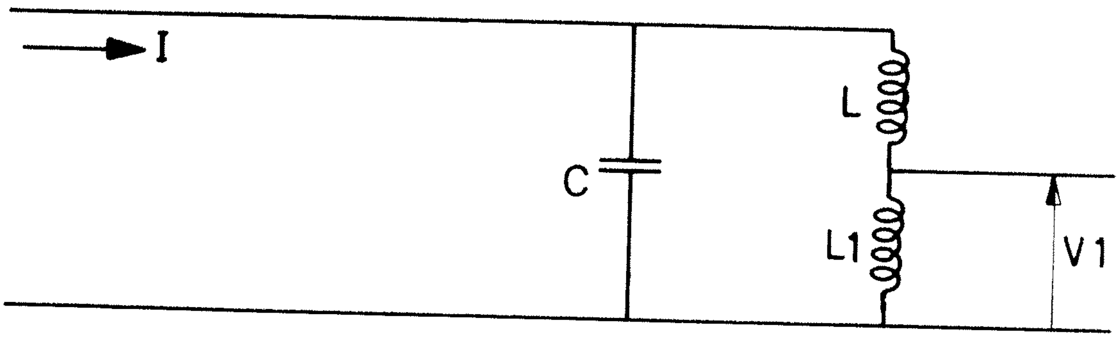

Fig. 5

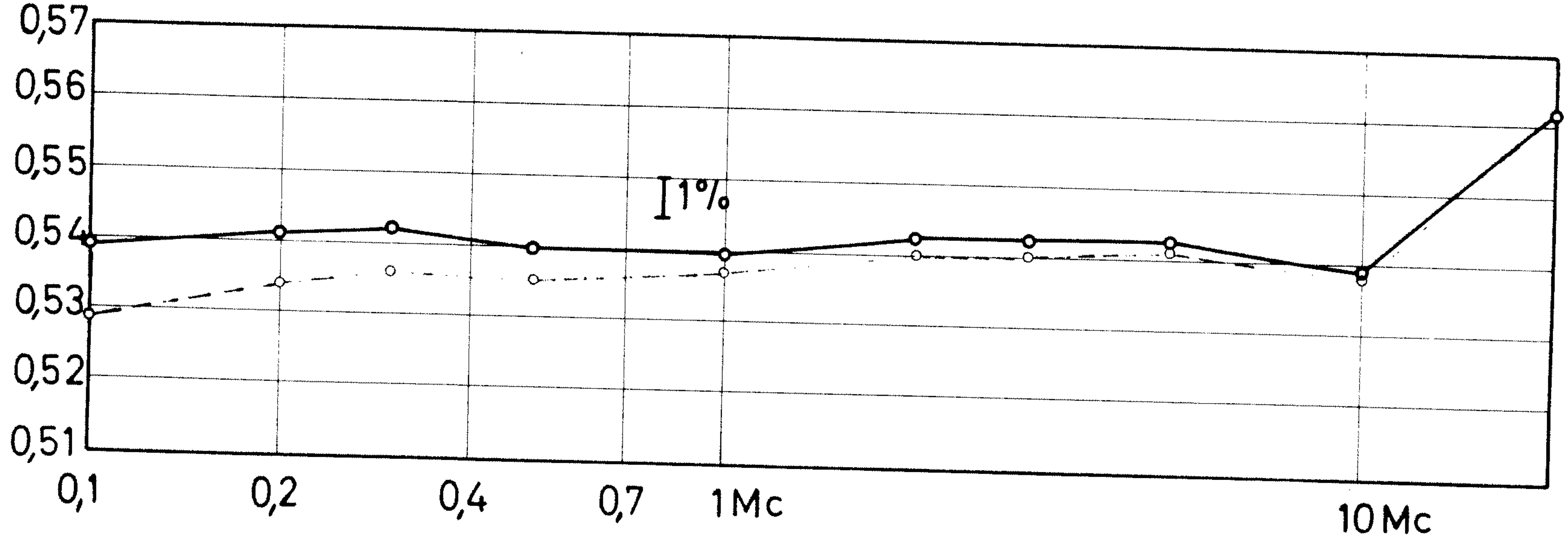

Fig. 6