Février 1971

LRP 46/71

GENERALIZED ONE DIMENSIONAL MAGNETOHYDRODYNAMIC COMPUTER CODE FOR PARTIALLY IONIZED HYDROGEN OR HELIUM PLASMAS

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A b s t r a c t

A computer program has been developed to solve the "generalized" MHD equations (including electron inertia and drift velocity effects) for four fluids in one dimension. The physical model and numerical methods are discussed. Results of a typical calculation are presented.

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Table of Contents

I. Introduction

Numerical solutions for one-dimensional time-dependent problems in magnetohydrodynamics are widely used for the interpretation and planning of plasma experiments. Several computer programs have been developed for this purpose[1,2,3], and, in general, the agreement between calculation and experiment has been surprisingly good[1,4].

There are experimental situations, however, where some of the basic assumptions of the standard MHD model are not satisfied. Consider, for example, the confinement of a high β plasma by rapidly oscillating magnetic fields (Rotating Magnetic Field Pinch[5]). In that experiment, current densities and electron drift velocities, and, especially their time derivatives, are so large that electron inertia becomes important. The standard MHD model assumes electron inertia to be negligible.

Therefore, in order to simulate experiments of this kind, we have extended the standard MHD model to include the effects of non-zero electron drift velocity and inertia. The resulting equations, which we shall call the "generalized" MHD equations, are derived in Section II of the present paper. Finite difference approximations and numerical methods are discussed in Section III, and typical results are presented in Section IV.

Two different versions of the program have been written: The first is a three fluid model for partially ionized hydrogen or deuterium, the second is a four fluid model designed for helium plasmas.

II. The Physical Model

As has been pointed out in the Introduction, we are considering two distinct models, one for hydrogen and one for helium. The two models are very similar, however, because in the helium version it is assumed that the temperatures and fluid velocities of the two ion species are equal ($T_{He^+} = T_{He^{++}}$, $\vec{v}_{He^+} = \vec{v}_{He^{++}}$). The derivation of the hydromagnetic equations is, therefore, completely analogous in the two cases.

In the first part of this section, "generalized" MHD equations are derived for three fluids, i.e., electrons, ions and neutral atoms. MKS units are used throughout the derivation. Then, at the end of the section, a complete list of equations, in terms of machine units, is given. It includes the hydromagnetic equations and transport coefficients for both the hydrogen and the helium model.

1. Basic Assumptions

Cylindrical symmetry, quasi-neutrality and ideal gases ($\gamma = 5/3$) are assumed. Ion motion in the azimuthal and axial directions is neglected, i.e., ion currents are assumed to be negligibly small compared to electron currents. Momentum and energy transfer between electrons, ions and neutrals due to ionization, recombination, charge exchange and elastic collisions are taken into account. The transport and rate coefficients are functions of the local plasma parameters[6-11].

2. Conservation of Mass

For every particle specie, there is a conservation equation of the form

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\vec{v}) = P \qquad (1)$$

where n is the number of particles per unit volume, v is the velocity and P describes creation and destruction of particles as a result of ionization and recombination. In cylindrical symmetry we have

for electrons: $$\frac{\partial n_e}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\, n_e v_{er}) = n_e(-n_e\alpha + n_o S) \tag{2}$$

for ions: $$n_i = n_e \text{ , due to quasi-neutrality} \tag{3}$$

for neutrals: $$\frac{\partial n_o}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\, n_o v_{or}) = n_e(n_e\alpha - n_o S) \tag{4}$$

where S and α are the ionization and recombination rate coefficients, the subscript r designates radial vector components and the subscripts e, i and o refer to electrons, ions and neutrals, respectively.

3. Conservation of Momentum

The equation of conservation of momentum, for one fluid, may be written as[12]

$$\rho\left[\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right] = -\nabla\left[p + \frac{2}{3}\mu(\nabla\cdot\vec{v})\right] + \nabla\cdot\left[\mu(\nabla\vec{v} + \vec{v}\nabla)\right] + \vec{F} \tag{5}$$

where ρ is the mass density, p is the scalar pressure, μ is the viscosity and $\vec{F}$ is the external force. $\vec{F}$ is given for electrons, ions and neutrals by equations (6), (7) and (8) below

$$\begin{aligned}\vec{F}_e = &- n_e e(\vec{E} + \vec{v}_e x\vec{B}) - n_e n_i\ \sigma_{ei} v_{ei} m_e\ (\vec{v}_e - \vec{v}_i) \\ &- n_e n_o\ \sigma_{eo} v_{eo} m_e (\vec{v}_e - \vec{v}_o) - n_e n_o S\ m_e(\vec{v}_e - \vec{v}_o)\end{aligned} \tag{6}$$

$$\begin{aligned}\vec{F}_i = &\ n_i e(\vec{E} + \vec{v}_i x\vec{B}) + n_e n_i\ \sigma_{ei} v_{ei} m_e (\vec{v}_e - \vec{v}_i) \\ &- n_i n_o\ \sigma_{io} v_{io} m_i (\vec{v}_i - \vec{v}_o) - n_e n_o S m_i (\vec{v}_i - \vec{v}_o)\end{aligned} \tag{7}$$

$$\vec{F}_o = n_e n_o \sigma_{eo} v_{eo} m_e (\vec{v}_e - \vec{v}_o) + n_i n_o \sigma_{io} v_{io} m_i (\vec{v}_i - \vec{v}_o)$$
$$+ n_i n_e \alpha m_i (\vec{v}_i - \vec{v}_o) \qquad (8)$$

where $n_i \sigma_{ei} v_{ei}$ is the Coulomb collision frequency for an electron, σ_{eo} is the cross section for elastic collisions between electrons and neutral atoms, v_{eo} is the effective electron velocity relative to neutrals, σ_{io} is the cross section for elastic scattering and charge transfer between ions and neutrals and v_{io} is the effective ion velocity relative to neutrals. From equations (6), (7) and (8) we may compute the quantities

$$F_{er}+F_{ir} = - n_e e(\vec{v}_e x\vec{B})_r - n_i(v_{ir}-v_{or})\Big[m_i(n_o\sigma_{io}v_{io} + n_o S)$$
$$+ m_e(n_o\sigma_{eo}v_{eo} + n_o S)\Big] \qquad (9)$$

$$F_{e\theta,z} = - n_e e(\vec{E} + \vec{v}_e x\vec{B})_{\theta,z}$$
$$- n_e^2 v_{e\theta,z} e^2 \left[\eta_{Sp} + \frac{m_e}{n_e e^2} (n_o \sigma_{eo} v_{eo} + n_o S)\right] \qquad (10)$$

$$F_{or} = n_i(v_{ir} - v_{or})\left[m_i(n_o\sigma_{io}v_{io} + n_e\alpha) + m_e(n_o\sigma_{eo}v_{eo})\right] \qquad (11)$$

In equation (10), η_{Sp} is the Spitzer resistivity[11] ($\eta_{Sp} = n_i\sigma_{ei}v_{ei}m_e/n_e e^2$). The indices, r, θ and z refer to radial, azimuthal and axial vector components, respectively. The last term on the right hand side of equation (9) and equation (11) may be neglected because $m_e << m_i$, and we have

$$F_{er} + F_{ir} = - n_e e(\vec{v}_e x\vec{B})_r - n_i(v_{ir} - v_{or})m_i\lambda \qquad (12)$$

$$F_{e\theta,z} = - n_e e(\vec{E} + \vec{v}_e x\vec{B})_{\theta,z} - n_e^2 v_{e\theta,z} e^2 \eta_{Cl} \qquad (13)$$

$$F_{or} = n_o(v_{ir} - v_{or})m_i \lambda_o \tag{14}$$

where we have used the abbreviations

$$\lambda = n_o \sigma_{io} v_{io} + n_o S \tag{15}$$

$$\lambda_o = n_i \sigma_{io} v_{io} + \frac{n_e n_i}{n_o} \alpha \tag{16}$$

$$\eta_{Cl} = \eta_{Sp} + \frac{m_e}{n_e e^2} (n_o \sigma_{eo} v_{eo} + n_o S) \tag{17}$$

The radial components of the momentum conservation equations for electrons and ions are now summed and, using eq. (12), we obtain

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} = - \lambda(v-v_o) + \frac{1}{n_i m_i}\left[- \frac{\partial}{\partial r}\left\{(n_e T_e + n_i T_i) - \frac{4}{3}\mu \frac{\partial v}{\partial r} + \frac{2\mu v}{3r}\right\} - \frac{2\mu v}{r^2} + (j_\theta B_z - j_z B_\theta) + \left(\frac{m_e j_\theta^2}{n_e e^2 r}\right)\right] \tag{18}$$

Here, μ is the ion viscosity, electron viscosity has been neglected and we have introduced the definitions: $v = v_{ir} = v_{er}$, $v_o = v_{or}$, $T_e = p_e/n_e$, $T_i = p_i/n_i$, $j_\theta = - en_e v_{e\theta}$ and $j_z = - en_e v_{ez}$. The corresponding equation for the neutral atoms reads

$$\frac{\partial v_o}{\partial t} + v_o \frac{\partial v_o}{\partial r} = \lambda_o(v-v_o) + \frac{1}{n_o m_i}\left[- \frac{\partial}{\partial r}\left\{n_o T_o - \frac{4}{3} \mu_N \frac{\partial v_o}{\partial r} + \frac{2\mu_N v_o}{3r}\right\} - \frac{2\mu_N v_o}{r^2}\right] \tag{19}$$

where $T_o = p_o/n_o$ and μ_N is the neutral viscosity. The azimuthal and axial components of eq. (5), for electrons, are given by

$$\frac{\partial j_\theta}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v\, j_\theta) = \frac{n_e e^2}{m_e}(E_\theta - \eta_{C1} j_\theta - vB_z) \tag{20}$$

$$\frac{\partial j_z}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\, v\, j_z) = \frac{n_e e^2}{m_e}(E_z - \eta_{C1} j_z + vB_\theta) \tag{21}$$

If the resistivity, η_{C1}, is anisotropic, equations (20) and (21) must be generalized:

$$\frac{\partial j_\theta}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v\, j_\theta) = \frac{n_e e^2}{m_e}(E_\theta - \eta_{\theta\theta} j_\theta - \eta_{\theta z} j_z - vB_z) \tag{22}$$

$$\frac{\partial j_z}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\, v\, j_z) = \frac{n_e e^2}{m_e}(E_z - \eta_{z\theta} j_\theta - \eta_{zz} j_z + vB_\theta) \tag{23}$$

where

$$\eta_{\theta\theta} = \frac{1}{B^2}\left[\eta_{\parallel} B_\theta^2 + \eta_\perp B_z^2\right] \tag{24}$$

$$\eta_{\theta z} = \eta_{z\theta} = \frac{1}{B^2}\left[(\eta_{\parallel} - \eta_\perp) B_\theta B_z\right] \tag{25}$$

$$\eta_{zz} = \frac{1}{B^2}\left[\eta_{\parallel} B_z^2 + \eta_\perp B_\theta^2\right] \tag{26}$$

$\eta_{\parallel}$ and $\eta_\perp$ are the resistivities parallel and perpendicular to the magnetic field and $B^2 = B_\theta^2 + B_z^2$. Equations (22) and (23) may be regarded as a "generalized Ohm's law".

4. Conservation of Energy

For any one of the fluids, the equation of conservation of energy may be written in the form[13]

$$\frac{3}{2} n \left[\frac{\partial T}{\partial t} + (\vec{v}\cdot\nabla)T \right] = - p(\nabla\cdot\vec{v}) + \nabla\cdot(Q\,\nabla T) + W \tag{27}$$

where $T = p/n$ and Q is the thermal conductivity divided by the Boltzmann constant. W is an energy source (or sink) which describes the following phenomena: (a) energy equipartition between the different fluids, (b) heating due to dynamical friction(including Joule heating), (c) ionization and recombination, and (d) viscous heating. Eq. (27) is written for electrons, ions and neutrals in cylindrical coordinates:

$$\frac{\partial T_e}{\partial t} + v\frac{\partial T_e}{\partial r} = -\frac{2}{3}\frac{T_e}{r}\frac{\partial(rv)}{\partial r} + \frac{2}{3n_e r}\frac{\partial}{\partial r}\left[r\,Q_e\frac{\partial T_e}{\partial r}\right]$$
$$-\frac{2}{3n_e}\left[\frac{T_e-T_o}{\tau_{eo}} + \frac{T_e-T_i}{\tau_{ei}}\right] + \frac{2}{3n_e}\left[\eta_{\theta\theta}j_\theta^2 + 2\eta_{\theta z}j_\theta j_z + \eta_{zz}j_z^2\right]$$
$$-\frac{2}{3}\left[n_o S(\chi + \frac{3}{2}T_e) - n_e\,\alpha^c\,\chi\right] \tag{28}$$

$$\frac{\partial T_i}{\partial t} + v\frac{\partial T_i}{\partial r} = -\frac{2}{3}\frac{T_i}{r}\frac{\partial(rv)}{\partial r} + \frac{2}{3n_e r}\frac{\partial}{\partial r}\left[r\,Q_i\frac{\partial T_i}{\partial r}\right]$$
$$-\frac{2}{3n_e}\left[\frac{T_i-T_o}{\tau_{io}} - \frac{T_e-T_i}{\tau_{ei}}\right] + f_i(v-v_o)^2 - n_o S(T_i-T_o)$$
$$+\frac{2\mu}{3n_e}\left[\frac{4}{3}\left\{\frac{1}{r}\frac{\partial(rv)}{\partial r}\right\}^2 - 4\frac{v}{r}\left\{\frac{\partial v}{\partial r}\right\}\right] \tag{29}$$

$$\frac{\partial T_o}{\partial t} + v_o \frac{\partial T_o}{\partial r} = -\frac{2}{3}\frac{T_o}{r}\frac{\partial (rv_o)}{\partial r} + \frac{2}{3n_o r}\frac{\partial}{\partial r}\left[r\, Q_o \frac{\partial T_o}{\partial r}\right]$$

$$+ \frac{2}{3n_o}\left[\frac{T_i - T_o}{\tau_{io}} + \frac{T_e - T_o}{\tau_{eo}}\right] + f_o (v - v_o)^2 + \frac{n_e^2 \alpha}{n_o}(T_i - T_o)$$

$$+ \frac{2\mu_N}{3n_o}\left[\frac{4}{3}\left\{\frac{1}{r}\frac{\partial (rv_o)}{\partial r}\right\}^2 - 4\frac{v_o}{r}\left\{\frac{\partial v_o}{\partial r}\right\}\right] \qquad (30)$$

where τ_{eo}, τ_{ei} and τ_{io} are the coefficients for energy equipartition, χ is the ionization potential (13.6 ev for hydrogen), α^c is the collisional part of the recombination coefficient[7], f_i and f_o are the coefficients of frictional heating[10].

5. Maxwell's Equations

Under the conditions stated at the beginning of this section, Maxwell's equations may be written as

$$\frac{\partial B_\theta}{\partial t} = \frac{\partial E_z}{\partial r} \qquad (31)$$

$$\frac{\partial B_z}{\partial t} = \frac{-1}{r}\frac{\partial}{\partial r}(rE_\theta) \qquad (32)$$

$$\mu_o j_\theta = -\frac{\partial B_z}{\partial r} \qquad (33)$$

$$\mu_o j_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) \qquad (34)$$

Combining eqns. (22), (23), (31), (32), (33), (34) and eliminating j_θ, j_z, E_θ and E_z, yields

$$\frac{d(rB_\theta)}{dt} + (rB_\theta)\left[\frac{\partial v}{\partial r} - \frac{v}{r}\right] = \frac{r}{\mu_o}\frac{\partial}{\partial r}\left[-\eta_{\theta z}\frac{\partial B_z}{\partial r} + \eta_{zz}\frac{1}{r}\frac{\partial(rB_\theta)}{\partial r}\right.$$

$$\left. + \frac{C_e}{n_e r}\frac{\partial}{\partial r}\left\{\frac{d(rB_\theta)}{dt}\right\}\right] \tag{35}$$

$$\frac{dB_z}{dt} + B_z\left[\frac{1}{r}\frac{\partial(rv)}{\partial r}\right] = \frac{1}{r\mu_o}\frac{\partial}{\partial r}\left[r\left\{\left(\eta_{\theta\theta} + \frac{2C_e v}{n_e r}\right)\frac{\partial B_z}{\partial r}\right.\right.$$

$$\left.\left. - \eta_{\theta z}\frac{1}{r}\frac{\partial(rB_\theta)}{\partial r} + \frac{C_e}{n_e}\frac{\partial}{\partial r}\left[\frac{dB_z}{dt}\right]\right\}\right] \tag{36}$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial r}$, and $C_e = \frac{m_e}{e^2}$

6. Summary of Equations in Machine Units

For convenience, we shall use the following set of units:

Name	Symbol	Unit
distance	r, z	10^{-2} m
mass	m	10^{-27} kg
time	t	10^{-6} sec
number density	n	10^{21} m^{-3}
velocity	v	10^{4} m sec^{-1}
temperature	T	ev
magnetic field	B	Gauss
ionization and recombination coeff.	S, α	10^{-15} m^3 sec^{-1}
friction coefficient	λ	10^{6} sec^{-1}
thermal conduction coefficient	Q	10^{23} m^{-1} sec^{-1}
energy equipartition coefficient	τ	10^{-27} m^3 sec
coefficient of frictional heating	f	10^{-2} ev m^{-2} sec
resistivity	η	10^{-2} Ohm m
viscosity	μ	10^{-4} kg m^{-1} sec^{-1}

The MHD equations for three fluids, eqns. (2), (4), (18), (19), (28), (29), (30), (35) and (36), written in these units, are listed below:

$$\frac{dn_e}{dt} = - \frac{n_e}{r} \frac{\partial (rv)}{\partial r} + n_e \left[- n_e \alpha + n_o S \right] \quad (37)$$

$$\frac{dn_o}{dt} = -\frac{n_o}{r}\frac{\partial(rv_o)}{\partial r} + n_e\left[n_e\alpha - n_o S\right] \tag{38}$$

$$\frac{dv}{dt} = \frac{1}{n_e m_i}\left[-1.6\frac{\partial}{\partial r}\left(n_e\left[T_e+T_i\right]\right) + \frac{\partial}{\partial r}\left\{\frac{4}{3}\mu\frac{\partial v}{\partial r} - \frac{2\mu v}{3r}\right\} - \frac{2\mu v}{r^2}\right.$$

$$\left. - \frac{1}{4000\pi}\left\{\left(B_z - \frac{C_1}{n_e r}\frac{\partial B_z}{\partial r}\right)\frac{\partial B_z}{\partial r} + \frac{B_\theta}{r}\frac{\partial(rB_\theta)}{\partial r}\right\}\right] - \lambda\left(v - v_o\right) \tag{39}$$

$$\frac{dv_o}{dt} = \frac{1}{n_o m_i}\left[-1.6\frac{\partial}{\partial r}\left(n_o T_o\right) + \frac{\partial}{\partial r}\left\{\frac{4}{3}\mu_N\frac{\partial v_o}{\partial r} - \frac{2\mu_N v}{3r}\right\} - \frac{2\mu_N v_o}{r^2}\right]$$

$$+ \lambda_o\left(v - v_o\right) \tag{40}$$

$$\frac{dT_e}{dt} = -\frac{2}{3}\frac{T_e}{r}\frac{\partial(rv)}{\partial r} + \frac{2}{3n_e r}\frac{\partial}{\partial r}\left[rQ_e\frac{\partial T_e}{\partial r}\right] - \frac{2}{3n_e}\left[\frac{T_e - T_o}{\tau_{eo}} + \frac{T_e - T_i}{\tau_{ei}}\right]$$

$$+ \frac{2}{3n_e}\frac{1}{1.6\,(4\pi)^2}\left[\eta_{\theta\theta}\left\{\frac{\partial B_z}{\partial r}\right\}^2 - 2\eta_{\theta z}\left\{\frac{\partial B_z}{\partial r}\right\}\left\{\frac{1}{r}\frac{\partial(rB_\theta)}{\partial r}\right\} + \eta_{zz}\left\{\frac{1}{r}\frac{\partial(rB_\theta)}{\partial r}\right\}^2\right]$$

$$-\frac{2}{3}\left[n_o S\left(\chi + \frac{3}{2}T_e\right)\right] + \frac{2}{3}\left[n_e\alpha^c\chi\right] \tag{41}$$

$$\frac{dT_i}{dt} = -\frac{2}{3}\frac{T_i}{r}\frac{\partial(rv)}{\partial r} + \frac{2}{3n_e r}\frac{\partial}{\partial r}\left[rQ_i\frac{\partial T_i}{\partial r}\right] - \frac{2}{3n_e}\left[\frac{T_i - T_o}{\tau_{io}} - \frac{T_e - T_i}{\tau_{ei}}\right]$$

$$+ f_i\left(v - v_o\right)^2 - n_o S\left(T_i - T_o\right) + \frac{5}{9n_e}\mu\left[\left\{\frac{1}{r}\frac{\partial(rv)}{\partial r}\right\}^2 - 3\frac{v}{r}\left\{\frac{\partial v}{\partial r}\right\}\right] \tag{42}$$

$$\frac{dT_o}{dt} = -\frac{2}{3}\frac{T_o}{r}\frac{\partial(rv_o)}{\partial r} + \frac{1}{3n_o r}\frac{\partial}{\partial r}\left[rQ_o\frac{\partial T_o}{\partial r}\right] + \frac{2}{3n_o}\left[\frac{T_i - T_o}{\tau_{io}} + \frac{T_e - T_o}{\tau_{eo}}\right]$$

$$+ f_o(v - v_o)^2 + \frac{n_e^2\alpha}{n_o}(T_i - T_o) + \frac{5}{9n_o}\mu_N\left[\left\{\frac{1}{r}\frac{\partial(rv_o)}{\partial r}\right\}^2 - 3\frac{v_o}{r}\left\{\frac{\partial v_o}{\partial r}\right\}\right] \tag{43}$$

$$\frac{d(rB_\theta)}{dt} = -rB_\theta\left[\frac{\partial v}{\partial v} - \frac{v}{r}\right] + \frac{10^3}{4\pi}r\frac{\partial}{\partial r}\left[-\eta_{\theta z}\frac{\partial B_z}{\partial r} + \frac{\eta_{zz}}{r}\frac{\partial(rB_\theta)}{\partial r} + \frac{c_2}{n_e r}\frac{\partial}{\partial r}\left\{\frac{d(rB_\theta)}{dt}\right\}\right] \tag{44}$$

$$\frac{dB_z}{dt} = -\frac{B_z}{r}\left[\frac{\partial(rv)}{\partial r}\right] + \frac{10^3}{4\pi r}\frac{\partial}{\partial r}\left[r\left\{\left(\eta_{\theta\theta} + \frac{2c_2 v}{n_e r}\right)\frac{\partial B_z}{\partial r} - \frac{\eta_{\theta z}}{r}\frac{\partial(rB_\theta)}{\partial r}\right.\right.$$

$$\left.\left. + \frac{c_2}{n_e}\frac{\partial}{\partial r}\left\{\frac{dB_z}{dt}\right\}\right\}\right] \tag{45}$$

where $c_1 = (m_e/0.1024\ \pi)$, $c_2 = (m_e/256\ \pi)$, $m_e = 9.108 \times 10^{-4}$, χ is the ionization potential of hydrogen, $\chi = 13.6$ ev, m_i is the ion mass in machine units, and $\frac{d}{dt} = \frac{\partial}{\partial t} + v_o\frac{\partial}{\partial r}$ or $\frac{d}{dt} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial r}$ depending on whether it operates on a quantity with or without subscript o, respectively.

The corresponding transport coefficients [6, 7, 8, 10, 11], for hydrogen or deuterium, are given by:

$$\alpha = \left[0.8\,T_e^{-\frac{9}{2}}n_e + 0.355\,T_e^{-\frac{3}{4}}\right] * 10^{-3} \tag{46}$$

$$\alpha^c = \left[0.8\,T_e^{-\frac{9}{2}}n_e\right] * 10^{-3} \tag{47}$$

$$S = S_o + \left[S_\infty - S_o\right]\frac{\sqrt{n}}{3+\sqrt{n}} \tag{48}$$

where $$S_o = 25\, e^{-\left(\frac{16.95}{T_e}\right)}$$

$$S_\infty = 55\, e^{-\left(\frac{10.73}{T_e}\right)}$$

$$\lambda = n_o S + n_o v_{io}\left[0.983 + \frac{2.73}{m_i v_{io}^2} - 0.106 \log_{10} T_i\right] \tag{49}$$

where $$v_{io} = \left[\frac{4.07}{m_i}\,(T_i + T_o) + (v - v_o)^2\right]^{1/2}$$

$$\lambda_o = \frac{n_e^2 \alpha}{n_o} + n_e v_{io}\left[0.983 + \frac{2.73}{m_i v_{io}^2} - 0.106 \log_{10} T_i\right] \tag{50}$$

$$\frac{1}{\tau_{eo}} = \frac{0.244}{m_i}\, n_e n_o T_e^{1/2}\left[\frac{4.74}{T_e + 1.3} + 0.214\right] \tag{51}$$

$$\frac{1}{\tau_{io}} = n_e n_o v_{io}\left[1.437 + \frac{5.31}{m_i v_{io}^2} - 0.159 \log_{10} T_i\right] \tag{52}$$

$$\frac{1}{\tau_{ie}} = \frac{7.95}{m_i}\, n_e^2 \ln\Lambda\, T_e^{-3/2} \tag{53}$$

$$\eta_{\parallel ce} = 0.00522 \ln\Lambda\, T_e^{-3/2} + 3.56 * 10^{-6}\,\frac{n_o}{n_e}\left[S + 67\left\{\frac{4.74}{T_e + 1.3} + 0.214\right\} T_e^{1/2}\right] \tag{54}$$

$$\eta_{\perp ce} = 1.97\, \eta_{\parallel ce} \tag{54a}$$

$$\frac{1}{Q_e} = \frac{1.23 \ln\Lambda}{T_e^{5/2}}\left[1 + 2.04 * 10^{-5}\left\{\frac{B\, T_e^{3/2}}{\ln\Lambda\, n_e}\right\}^2\right] + \frac{n_o}{83.8\, n_e T_e^{1/2}}\left[\frac{4.74}{T_e + 1.3} + 0.214\right] \tag{55}$$

$$\frac{1}{Q_i} = \frac{13.7 \ln\Lambda\, m_i^{1/2}}{T_i^{5/2}}\left[1 + \left\{\frac{2.92 * 10^{-4} B T_i^{3/2}}{m_i^{1/2} \ln\Lambda\, n_e}\right\}^2\right] + \frac{n_o m_i^{1/2}}{2.23\, n_e T_i^{1/2}}\left[1.057 + \frac{5.31}{m_i v_{io}^2} - 0.106 \log_{10} T_i\right] \tag{56}$$

$$\frac{1}{Q_o} = \frac{m_i^{1/2}}{2.28\, T_o^{1/2}}\left[0.445 + \frac{n_e}{n_o}\left\{1.057 + \frac{5.31}{m_i v_{io}^2} - 0.106 \log_{10} T_i\right\}\right] \tag{57}$$

$$f_i = 0.0052\ m_i\ n_o\ v_{io}\left[2.97 + \frac{53.1}{m_i v_{io}^2}\right] \tag{58}$$

$$f_o = 0.0052\ m_i\ n_e\ v_{io}\left[2.97 + \frac{53.1}{m_i v_{io}^2}\right] \tag{59}$$

$$\mu = \frac{4}{15}\ Q_i\ m_i \tag{60}$$

$$\mu_N = \frac{4}{15}\ Q_o\ m_i \tag{61}$$

The MHD equations for four fluids (Helium) are:

$$\frac{dn_e}{dt} = -\frac{n_e}{r}\frac{\partial(rv)}{\partial r} + n_e\left[-n_s\alpha_1 + n_o S_1 - n_i\alpha_2 + n_s S_2\right] \tag{62}$$

$$\frac{dn_i}{dt} = -\frac{n_i}{r}\frac{\partial(rv)}{\partial r} + n_e\left[-n_i\alpha_2 + n_s S_2\right] \tag{63}$$

$$\frac{dn_o}{dt} = -\frac{n_o}{r}\frac{\partial(rv_o)}{\partial r} + n_e\left[n_s\alpha_1 - n_o S_1\right] \tag{64}$$

$$\frac{dv}{dt} = \frac{1}{n_t m_i}\left[-1.6\frac{\partial}{\partial r}(n_e T_e + n_t T_i) + \frac{\partial}{\partial r}\left\{\frac{4}{3}\mu\frac{\partial v}{\partial r} - \frac{2\mu v}{3r}\right\} - \frac{2\mu v}{r^2}\right.$$
$$\left. - \frac{1}{4000\pi}\left\{\left(B_z - \frac{C_1}{n_e r}\frac{\partial B_z}{\partial r}\right)\frac{\partial B_z}{\partial r} + \frac{B_\theta}{r}\frac{\partial(rB_\theta)}{\partial r}\right\}\right] - \lambda(v - v_o) \tag{65}$$

$$\frac{dT_e}{dt} = -\frac{2}{3}\frac{T_e}{r}\frac{\partial(rv)}{\partial r} + \frac{2}{3n_e r}\frac{\partial}{\partial r}\left[rQ_e\frac{\partial T_e}{\partial r}\right] - \frac{2}{3n_e}\left[\frac{T_e - T_o}{\tau_{eo}} + \frac{T_e - T_i}{\tau_{ei}}\right]$$
$$+ \frac{2}{3n_e}\frac{1}{1.6(4\pi)^2}\left[\eta_{\theta\theta}\left\{\frac{\partial B_z}{\partial r}\right\}^2 - 2\eta_{\theta z}\left\{\frac{\partial B_z}{\partial r}\right\}\left\{\frac{1}{r}\frac{\partial(rB_\theta)}{\partial r}\right\} + \eta_{zz}\left\{\frac{1}{r}\frac{\partial(rB_\theta)}{\partial r}\right\}^2\right] \tag{66}$$
$$- \frac{2}{3}\left[n_o S_1\left(\chi_1 + \frac{3}{2}T_e\right) + n_s S_2\left(\chi_2 + \frac{3}{2}T_e\right)\right] - \frac{2}{3}\left[n_s\alpha_1^c\chi_1 + n_i\alpha_2^c\chi_2\right]$$

$$\frac{dT_i}{dt} = -\frac{2}{3}\frac{T_i}{r}\frac{\partial(rv)}{\partial r} + \frac{2}{3n_t r}\frac{\partial}{\partial r}\left[r Q_i \frac{\partial T_i}{\partial r}\right] - \frac{2}{3n_t}\left[\frac{T_i - T_o}{\tau_{io}} - \frac{T_e - T_i}{\tau_{ei}}\right]$$

$$+ f_i (v - v_0)^2 - \frac{n_o n_e}{n_t} S_1 (T_i - T_o) + \frac{5}{9 n_t} \mu \left[\left\{\frac{1}{r}\frac{\partial(rv)}{\partial r}\right\}^2 - 3\frac{v}{r}\left\{\frac{\partial v}{\partial r}\right\}\right] \quad (67)$$

$$\frac{dT_o}{dt} = -\frac{2}{3}\frac{T_o}{r}\frac{\partial(rv_o)}{\partial r} + \frac{2}{3n_o r}\frac{\partial}{\partial r}\left[r Q_o \frac{\partial T_o}{\partial r}\right] + \frac{2}{3n_o}\left[\frac{T_i - T_o}{\tau_{io}} + \frac{T_e - T_o}{\tau_{eo}}\right]$$

$$+ f_o (v - v_o)^2 + \frac{n_s n_e}{n_o} \alpha_1 (T_i - T_o) + \frac{5}{9 n_o} \mu_N \left[\left\{\frac{1}{r}\frac{\partial(rv_o)}{\partial r}\right\}^2 - 3\frac{v_o}{r}\left\{\frac{\partial v_o}{\partial r}\right\}\right] \quad (68)$$

where n_i is the number density of doubly ionized He, n_s is the number density of singly ionized He ($n_s = n_e - 2n_i$), n_t is the total ion number density ($n_t = n_s + n_i$), α_1 is the recombination coefficient for the reaction $He^+ + e^- \rightarrow He$, α_2 is the recombination coefficient for $He^{++} + e^- \rightarrow He^+$, S_1 is the coefficient for ionization of He, S_2 is the coefficient for ionization of He^+, χ_1 is the first ionization potential of He (24.46 ev), and χ_2 is the second ionization potential of He(54.14 ev). The equations for the neutral velocity, v_o, and for the magnetic field components, B_θ and B_z, have been omitted because they are identical with the corresponding three fluid equations (eqns. (40), (44) and (45)).

The transport coefficients for helium[6,7,9,10,11] are listed below:

$$\alpha_1 = \left[5.6\, T_e^{-9/2} n_e + 0.27\, T_e^{-3/4}\right] * 10^{-3} \quad (69)$$

$$\alpha_1^c = 5.6 * 10^{-3}\, T_e^{-9/2} n_e \quad (70)$$

$$\alpha_2 = \left[44.8\, T_e^{-9/2} n_e + 1.53\, T_e^{-3/4}\right] * 10^{-3} \quad (71)$$

$$\alpha_2^c = 44.8 * 10^{-3}\, T_e^{-9/2} n_e \quad (72)$$

$$S_1 = 40 \; e^{-\chi_1/T_e} \tag{73}$$

$$S_2 = 12.5 \; e^{-\chi_2/T_e} \tag{74}$$

$$\lambda = \frac{n_e n_o}{n_t} S_1 + n_o v_{io}\left[2.39 + \frac{0.218}{v_{io}^2} - 0.5 \log_{10} T_i\right] \tag{75}$$

where $v_{io} = \left[0.614\,(T_i + T_o) + (v - v_o)^2\right]^{1/2}$

$$\lambda_0 = \frac{n_e n_s}{n_o} \alpha_1 + n_t v_{io}\left[2.39 + \frac{0.218}{v_{io}^2} - 0.5 \log_{10} T_i\right] \tag{76}$$

$$\frac{1}{\tau_{eo}} = 0.365\, n_e n_o T_e^{1/2}\left[\frac{7.35}{T_e + 34.3} + 0.027\right]^2 \tag{77}$$

$$\frac{1}{\tau_{io}} = n_t n_o v_{io}\left[3.56 - 0.75 \log_{10} T_i + \frac{0.425}{v_{io}^2}\right] \tag{78}$$

$$\frac{1}{\tau_{ie}} = n_e (n_s + 4 n_i) \ln \Lambda \, T_e^{-3/2} * 1.20 \tag{79}$$

$$\eta_{\parallel ce} = 0.00522 \, \frac{n_s + 1.71\, n_i}{n_t} \ln \Lambda \, T_e^{-3/2} + 3.56 * 10^{-6} \frac{n_o}{n_e}\left[S_1 + \frac{n_s}{n_o} S_2 + 67\left\{\frac{23.2}{T_e + 34.3} + 0.085\right\}^2 T_e^{1/2}\right] \tag{80}$$

$$\eta_{\perp ce} = 1.97\, \eta_{\parallel ce} \tag{80a}$$

$$\frac{1}{Q_e} = \frac{2.46 \ln\Lambda (n_e + n_i)}{n_e T_e^{5/2}}\left[1 + 0.511 * 10^{-5}\left\{\frac{B T_e^{3/2}}{\ln\Lambda (n_e + n_i)}\right\}^2\right] + 0.119 \frac{n_o}{n_e T_e^{1/2}}\left\{\frac{7.35}{T_e + 34.3} + 0.027\right\}^2 \tag{81}$$

$$\frac{1}{Q_i} = \frac{33.8 \ln\Lambda (n_s + 4 n_i)^2}{n_t^2 \, T_i^{5/2}} + \frac{0.115\, n_o}{n_t T_i^{1/2}}\left\{24.3 - 5 \log_{10} T_i + \frac{4.25}{v_{io}^2}\right\} \tag{82}$$

$$\frac{1}{Q_0} = \frac{1}{8.87 \, T_o^{1/2}}\left[2.6 + \frac{n_t}{n_o}\left\{24.3 - 5 \log_{10} T_i + \frac{4.25}{v_{io}^2}\right\}\right] \tag{83}$$

$$f_i = 0.0346\, n_o v_{io} \left[1.73 + \frac{4.25}{v_{io}^2} \right] \tag{84}$$

$$f_o = 0.0346\, n_t v_{io} \left[1.73 + \frac{4.25}{v_{io}^2} \right] \tag{85}$$

$$\mu = 1.77\, Q_i \tag{86}$$

$$\mu_N = 1.77\, Q_o \tag{87}$$

III. Numerical Methods

For the sake of simplicity, the discussion will again be confined to the three fluid model. The extension to four fluids is straightforward and is readily obtained from the equations at the end of Section II.

1. Finite Difference Equations

Let us first write down the finite difference approximations for equations (37) through (45), in Lagrangian form:

$$\frac{1}{\Delta t}\left[A^{vp}_{j+\frac{1}{2}} - A^{o}_{j+\frac{1}{2}} \right] = (1-\varepsilon)\left[\frac{dA}{dt}\right]^{o}_{j+\frac{1}{2}} + \varepsilon \left[\frac{dA}{dt}\right]^{v}_{j+\frac{1}{2}} \tag{88}$$

$$\frac{1}{\Delta t}\left[C^{wp}_{j+\frac{1}{2}} - C^{o}_{j+\frac{1}{2}} \right] = (1-\varepsilon)\left[\frac{dC}{dt}\right]^{o}_{j+\frac{1}{2}} + \varepsilon \left[\frac{dC}{dt}\right]^{w}_{j+\frac{1}{2}} \tag{89}$$

$$\frac{1}{\Delta t}\left[V^{vp}_{j} - V^{o}_{j} \right] = (1-\varepsilon)\left[\frac{dV}{dt}\right]^{o}_{j} + \varepsilon \left[\frac{dV}{dt}\right]^{vp}_{j} \tag{90}$$

$$\frac{1}{\Delta t}\left[W^{wp}_{j} - W^{o}_{j} \right] = (1-\varepsilon)\left[\frac{dW}{dt}\right]^{o}_{j} + \varepsilon \left[\frac{dW}{dt}\right]^{wp}_{j} \tag{91}$$

$$\frac{1}{\Delta t}\left[E^{vp}_{j+\frac{1}{2}} - E^{o}_{j+\frac{1}{2}}\right] = (1-\varepsilon)\left[\frac{dE}{dt}\right]^{o}_{j+\frac{1}{2}} + \varepsilon\left[\frac{dE}{dt}\right]^{vp}_{j+\frac{1}{2}} \tag{92}$$

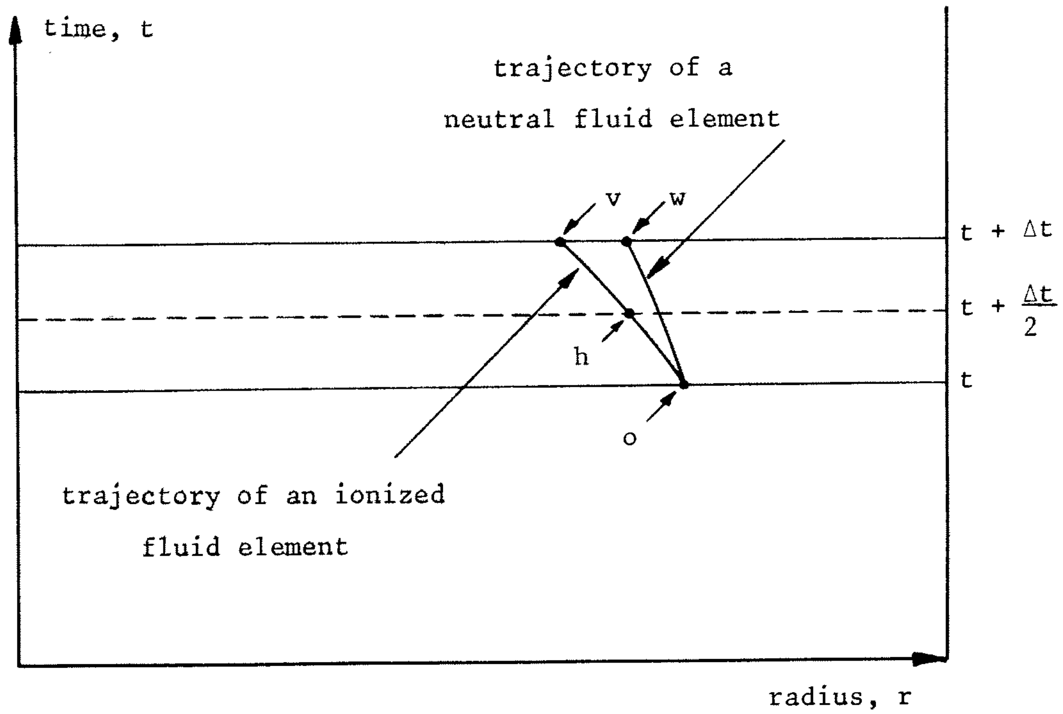
$$\frac{1}{\Delta t}\left[F^{vp}_{j+\frac{1}{2}} - F^{o}_{j+\frac{1}{2}}\right] = (1-\varepsilon)\left[\frac{dF}{dt}\right]^{o}_{j+\frac{1}{2}} + \varepsilon\left[\frac{dF}{dt}\right]^{vp}_{j+\frac{1}{2}} \tag{93}$$

$$\frac{1}{\Delta t}\left[G^{wp}_{j+\frac{1}{2}} - G^{o}_{j+\frac{1}{2}}\right] = (1-\varepsilon)\left[\frac{dG}{dt}\right]^{o}_{j+\frac{1}{2}} + \varepsilon\left[\frac{dG}{dt}\right]^{wp}_{j+\frac{1}{2}} \tag{94}$$

$$\frac{1}{\Delta t}\left[Y^{vp}_{j+\frac{1}{2}} - Y^{o}_{j+\frac{1}{2}}\right] = (1-\varepsilon)\left[\frac{dY}{dt}\right]^{o}_{j+\frac{1}{2}} + \varepsilon\left[\frac{dY}{dt}\right]^{vp}_{j+\frac{1}{2}} + X^{hp}_{j+\frac{1}{2}} \tag{95}$$

$$\frac{1}{\Delta t}\left[Z^{vp}_{j+\frac{1}{2}} - Z^{o}_{j+\frac{1}{2}}\right] = (1-\varepsilon)\left[\frac{dZ}{dt}\right]^{o}_{j+\frac{1}{2}} + \varepsilon\left[\frac{dZ}{dt}\right]^{vp}_{j+\frac{1}{2}} + U^{hp}_{j+\frac{1}{2}} \tag{96}$$

where $A = n_e$, $C = n_o$, $V = v_{er} = v_{ir}$, $W = v_o$, $E = T_e$, $F = T_i$, $G = T_o$, $Y = rB_\theta$, $Z = B_z$, and the subscript j is the spatial index. Note that the velocities are taken at integral space points, whereas all other variables are defined at half integral space points[14]. The meaning of the superscripts o, v, w and h is explained in the figure below:



The superscript p appearing in equations (88) through (96) refers to the p^{th} iteration; if it is omitted, the $(p-1)^{th}$ iteration is understood. Note, however, that expressions such as $\left[\frac{dE}{dt}\right]^{vp}$ contain quantities from the p^{th} <u>and</u> the $(p-1)^{th}$ iteration, as indicated in equations (99) through (107). The iteration parameter ε is usually taken as $\varepsilon = 0.5$.

The subscripted quantities appearing on the right hand side of equations (88) through (96) are listed below:

$$\left[\frac{dA}{dt}\right]_{j+\frac{1}{2}} = -\frac{A_{j+\frac{1}{2}}}{R_{j+\frac{1}{2}}\,H_{j+\frac{1}{2}}}\left[R_{j+1}\,V_{j+1} - R_j\,V_j\right] + A_{j+\frac{1}{2}}\left[-A_{j+\frac{1}{2}}\,\alpha_{j+\frac{1}{2}} + C_{j+\frac{1}{2}}\,S_{j+\frac{1}{2}}\right] \tag{97}$$

$$\left[\frac{dC}{dt}\right]_{j+\frac{1}{2}} = -\frac{C_{j+\frac{1}{2}}}{R_{j+\frac{1}{2}}\,H_{j+\frac{1}{2}}}\left[R_{j+1}\,W_{j+1} - R_j\,W_j\right] + A_{j+\frac{1}{2}}\left[A_{j+\frac{1}{2}}\,\alpha_{j+\frac{1}{2}} - C_{j+\frac{1}{2}}\,S_{j+\frac{1}{2}}\right] \tag{98}$$

$$\begin{aligned}\left[\frac{dV}{dt}\right]_j^p = \frac{1}{A_j m_i}\Bigg\{ & -\frac{1.6}{H_j}\left[A_{j+\frac{1}{2}}\left(E_{j+\frac{1}{2}} + F_{j+\frac{1}{2}}\right) - A_{j-\frac{1}{2}}\left(E_{j-\frac{1}{2}} + F_{j-\frac{1}{2}}\right)\right] \\ & + \frac{4}{3H_j}\left[\frac{\mu_{j+\frac{1}{2}}}{H_{j+\frac{1}{2}}}\left(V_{j+1}^p - V_j^p\right) - \frac{\mu_{j-\frac{1}{2}}}{H_{j-\frac{1}{2}}}\left(V_j^p - V_{j-1}^p\right)\right] \\ & - \frac{2}{3H_j}\left[\frac{\mu_{j+\frac{1}{2}}}{2R_{j+\frac{1}{2}}}\left(V_{j+1}^p + V_j^p\right) - \frac{\mu_{j-\frac{1}{2}}}{2R_{j-\frac{1}{2}}}\left(V_j^p + V_{j-1}^p\right)\right] - \frac{2\mu_j V_j^p}{R_j^2} \\ & - \frac{1}{4000\pi}\left[\left(Z_j - \frac{c_1 ZD_j}{A_j R_j}\right) ZD_j + \frac{Y_j}{R_j^2}\,YD_j\right]\Bigg\} - \lambda_j\left(V_j - W_j\right)\end{aligned} \tag{99}$$

$$\left[\frac{dW}{dt}\right]_j^P = \frac{1}{c_j\, m_i}\Bigg\{ -\frac{1.6}{H_j}\left[C_{j+\frac{1}{2}}\, G_{j+\frac{1}{2}} - C_{j-\frac{1}{2}}\, G_{j-\frac{1}{2}}\right]$$
$$+\frac{4}{3H_j}\left[\frac{\mu_{N,j+\frac{1}{2}}}{H_{j+\frac{1}{2}}}\left(W^P_{j+1} - W^P_j\right) - \frac{\mu_{N,j-\frac{1}{2}}}{H_{j-\frac{1}{2}}}\left(W^P_j - W^P_{j-1}\right)\right]$$
$$-\frac{2}{3H_j}\left[\frac{\mu_{N,j+\frac{1}{2}}}{2R_{j+\frac{1}{2}}}\left(W^P_{j+1} + W^P_j\right) - \frac{\mu_{N,j-\frac{1}{2}}}{2R_{j-\frac{1}{2}}}\left(W^P_j + W^P_{j-1}\right)\right] - \frac{2\mu_{N,j} W^P_j}{R^2_j}\Bigg\}$$
$$+\lambda_{o,j}\left(V_j - W_j\right) \tag{100}$$

$$\left[\frac{dE}{dt}\right]^P_{j+\frac{1}{2}} = -\frac{2}{3}\,\frac{E_{j+\frac{1}{2}}}{R_{j+\frac{1}{2}}\; H_{j+\frac{1}{2}}}\left[R_{j+1} V_{j+1} - R_j V_j\right]$$
$$+\frac{2}{3A_{j+\frac{1}{2}} R_{j+\frac{1}{2}} H_{j+\frac{1}{2}}}\left[\frac{R_{j+1} Q_{e,j+1}}{H_{j+1}}\left(E^P_{j+\frac{3}{2}} - E^P_{j+\frac{1}{2}}\right) - \frac{R_j Q_{e,j}}{H_j}\left(E^P_{j+\frac{1}{2}} - E^P_{j-\frac{1}{2}}\right)\right]$$
$$-\frac{2}{3A_{j+\frac{1}{2}}}\left[\frac{E_{j+\frac{1}{2}} - G_{j+\frac{1}{2}}}{\tau_{eo,j+\frac{1}{2}}} + \frac{E_{j+\frac{1}{2}} - F_{j+\frac{1}{2}}}{\tau_{ei,j+\frac{1}{2}}}\right]$$
$$+\frac{\frac{2}{3}}{25.6\pi^2 A_{j+\frac{1}{2}}}\Bigg\{\frac{\eta_{\theta\theta,j+\frac{1}{2}}}{4}\left[ZD_{j+1} + ZD_j\right]^2 - \frac{\eta_{\theta z,j+\frac{1}{2}}}{2R_{j+\frac{1}{2}}}\left[ZD_{j+1} + ZD_j\right]\left[YD_{j+1} + YD_j\right]$$
$$+\frac{\eta_{zz,j+\frac{1}{2}}}{4R^2_{j+\frac{1}{2}}}\left[YD_{j+1} + YD_j\right]^2\Bigg\}$$
$$-\frac{2}{3}\left[C_{j+\frac{1}{2}}\, S_{j+\frac{1}{2}}\left(\chi + \frac{3}{2}E_{j+\frac{1}{2}}\right)\right] + \frac{2}{3}\left[A_{j+\frac{1}{2}}\, \alpha^c_{j+\frac{1}{2}}\, \chi\right] \tag{101}$$

$$\left[\frac{dF}{dt}\right]^{p}_{j+\frac{1}{2}} = -\frac{2}{3}\,\frac{F_{j+\frac{1}{2}}}{R_{j+\frac{1}{2}}\;H_{j+\frac{1}{2}}}\left[R_{j+1}V_{j+1}-R_{j}V_{j}\right]$$

$$+\frac{2}{3\,A_{j+\frac{1}{2}}R_{j+\frac{1}{2}}H_{j+\frac{1}{2}}}\left[\frac{R_{j+1}Q_{i,j+1}}{H_{j+1}}\left(F^{p}_{j+\frac{3}{2}}-F^{p}_{j+\frac{1}{2}}\right)-\frac{R_{j}Q_{i,j}}{H_{j}}\left(F^{p}_{j+\frac{1}{2}}-F^{p}_{j-\frac{1}{2}}\right)\right]$$

$$-\frac{2}{3\,A_{j+\frac{1}{2}}}\left[\frac{F_{j+\frac{1}{2}}-G_{j+\frac{1}{2}}}{\tau_{io,j+\frac{1}{2}}}-\frac{E_{j+\frac{1}{2}}-F_{j+\frac{1}{2}}}{\tau_{ei,j+\frac{1}{2}}}\right]$$

$$+\frac{f_{i,j+\frac{1}{2}}}{4}\left[V_{j+1}+V_{j}-W_{j+1}-W_{j}\right]^{2}-C_{j+\frac{1}{2}}\;S_{j+\frac{1}{2}}\left[F_{j+\frac{1}{2}}-G_{j+\frac{1}{2}}\right]$$

$$+\frac{5\mu_{j+\frac{1}{2}}}{9\,A_{j+\frac{1}{2}}}\left[\left\{\frac{R_{j+1}V_{j+1}-R_{j}V_{j}}{R_{j+\frac{1}{2}}\;H_{j+\frac{1}{2}}}\right\}^{2}-\frac{3(V_{j+1}+V_{j})}{2R_{j+\frac{1}{2}}}\left\{\frac{V_{j+1}-V_{j}}{H_{j+\frac{1}{2}}}\right\}\right] \qquad (102)$$

$$\left[\frac{dG}{dt}\right]^{p}_{j+\frac{1}{2}} = -\frac{2}{3}\,\frac{G_{j+\frac{1}{2}}}{R_{j+\frac{1}{2}}\;H_{j+\frac{1}{2}}}\left[R_{j+1}W_{j+1}-R_{j}W_{j}\right]$$

$$+\frac{2}{3C_{j+\frac{1}{2}}R_{j+\frac{1}{2}}H_{j+\frac{1}{2}}}\left[\frac{R_{j+1}Q_{o,j+1}}{H_{j+1}}\left(G^{p}_{j+\frac{3}{2}}-G^{p}_{j+\frac{1}{2}}\right)-\frac{R_{j}Q_{o,j}}{H_{j}}\left(G^{p}_{j+\frac{1}{2}}-G^{p}_{j-\frac{1}{2}}\right)\right]$$

$$+\frac{2}{3C_{j+\frac{1}{2}}}\left[\frac{F_{j+\frac{1}{2}}-G_{j+\frac{1}{2}}}{\tau_{io,j+\frac{1}{2}}}-\frac{E_{j+\frac{1}{2}}-G_{j+\frac{1}{2}}}{\tau_{eo,j+\frac{1}{2}}}\right]$$

$$+\frac{f_{o,j+\frac{1}{2}}}{4}\left[V_{j+1}+V_{j}-W_{j+1}-W_{j}\right]^{2}+\frac{A^{2}_{j+\frac{1}{2}}\alpha_{j+\frac{1}{2}}}{C_{j+\frac{1}{2}}}\left[F_{j+\frac{1}{2}}-G_{j+\frac{1}{2}}\right]$$

$$+\frac{5\mu_{N,j+\frac{1}{2}}}{9C_{j+\frac{1}{2}}}\left[\left\{\frac{R_{j+1}W_{j+1}-R_{j}W_{j}}{R_{j+\frac{1}{2}}\;H_{j+\frac{1}{2}}}\right\}^{2}-\frac{3(W_{j+1}+W_{j})}{2R_{j+\frac{1}{2}}}\left\{\frac{W_{j+1}-W_{j}}{H_{j+\frac{1}{2}}}\right\}\right] \qquad (103)$$

$$\left[\frac{dY}{dt}\right]^{p}_{j+\frac{1}{2}} = -Y_{j+\frac{1}{2}}\left[\frac{V_{j+1}-V_{j}}{H_{j+\frac{1}{2}}} - \frac{V_{j+1}+V_{j}}{2R_{j+\frac{1}{2}}}\right]$$
$$+\frac{10^{3}R_{j+\frac{1}{2}}}{4\pi H_{j+\frac{1}{2}}}\left[\left\{-\eta_{\theta z,j+1}\frac{Z^{p}_{j+\frac{3}{2}}-Z^{p}_{j+\frac{1}{2}}}{H_{j+1}} + \eta_{zz,j+1}\frac{Y^{p}_{j+\frac{3}{2}}-Y^{p}_{j+\frac{1}{2}}}{R_{j+1}H_{j+1}}\right\}\right.$$
$$\left.-\left\{-\eta_{\theta z,j}\frac{Z^{p}_{j+\frac{1}{2}}-Z^{p}_{j-\frac{1}{2}}}{H_{j}} + \eta_{zz,j}\frac{Y^{p}_{j+\frac{1}{2}}-Y^{p}_{j-\frac{1}{2}}}{R_{j}H_{j}}\right\}\right] \qquad (104)$$

$$\left[\frac{dZ}{dt}\right]^{p}_{j+\frac{1}{2}} = -Z_{j+\frac{1}{2}}\left[\frac{R_{j+1}V_{j+1}-R_{j}V_{j}}{R_{j+\frac{1}{2}}\,H_{j+\frac{1}{2}}}\right]$$
$$+\frac{10^{3}}{4\pi R_{j+\frac{1}{2}}H_{j+\frac{1}{2}}}\left[R_{j+1}\left\{\left(\eta_{\theta\theta,j+1}+\frac{2c_{2}V_{j+1}}{A_{j+1}R_{j+1}}\right)\frac{Z^{p}_{j+\frac{3}{2}}-Z^{p}_{j+\frac{1}{2}}}{H_{j+1}} - \eta_{\theta z,j+1}\frac{Y^{p}_{j+\frac{3}{2}}-Y^{p}_{j+\frac{1}{2}}}{R_{j+1}H_{j+1}}\right\}\right.$$
$$\left.-R_{j}\left\{\left(\eta_{\theta\theta,j}+\frac{2c_{2}V_{j}}{A_{j}R_{j}}\right)\frac{Z^{p}_{j+\frac{1}{2}}-Z^{p}_{j-\frac{1}{2}}}{H_{j}} - \eta_{\theta z,j}\frac{Y^{p}_{j+\frac{1}{2}}-Y^{p}_{j-\frac{1}{2}}}{R_{j}H_{j}}\right\}\right] \qquad (105)$$

$$X^{hp}_{j+\frac{1}{2}} = \frac{10^{3}c_{2}}{8\pi}\left[\frac{R^{o}_{j+\frac{1}{2}}}{H^{o}_{j+\frac{1}{2}}} + \frac{R^{v}_{j+\frac{1}{2}}}{H^{v}_{j+\frac{1}{2}}}\right] *$$
$$\left[\frac{1}{2}\left\{\frac{1}{R^{o}_{j+1}A^{o}_{j+1}H^{o}_{j+1}} + \frac{1}{R^{v}_{j+1}A^{v}_{j+1}H^{v}_{j+1}}\right\}\frac{1}{\Delta t}\left\{\left(Y^{vp}_{j+\frac{3}{2}}-Y^{o}_{j+\frac{3}{2}}\right)-\left(Y^{vp}_{j+\frac{1}{2}}-Y^{o}_{j+\frac{1}{2}}\right)\right\}\right.$$
$$\left.-\frac{1}{2}\left\{\frac{1}{R^{o}_{j}A^{o}_{j}H^{o}_{j}} + \frac{1}{R^{v}_{j}A^{v}_{j}H^{v}_{j}}\right\}\frac{1}{\Delta t}\left\{\left(Y^{vp}_{j+\frac{1}{2}}-Y^{o}_{j+\frac{1}{2}}\right)-\left(Y^{vp}_{j-\frac{1}{2}}-Y^{o}_{j-\frac{1}{2}}\right)\right\}\right] \qquad (106)$$

$$U^{hp}_{j+\frac{1}{2}} = \frac{10^3 c_2}{8\pi} \left[\frac{1}{A^o_{j+\frac{1}{2}} \; H^o_{j+\frac{1}{2}}} + \frac{1}{A^v_{j+\frac{1}{2}} \; H^v_{j+\frac{1}{2}}} \right] *$$

$$\left[\frac{1}{2} \left\{ \frac{R^o_{j+1}}{A^o_{j+1} \; H^o_{j+1}} + \frac{R^v_{j+1}}{A^v_{j+1} \; H^v_{j+1}} \right\} \frac{1}{\Delta t} \left\{ \left(Z^{vp}_{j+\frac{3}{2}} - Z^o_{j+\frac{3}{2}} \right) - \left(Z^{vp}_{j+\frac{1}{2}} - Z^o_{j+\frac{1}{2}} \right) \right\} \right.$$

$$\left. - \frac{1}{2} \left\{ \frac{R^o_j}{A^o_j \; H^o_j} + \frac{R^v_j}{A^v_j \; H^v_j} \right\} \frac{1}{\Delta t} \left\{ \left(Z^{vp}_{j+\frac{1}{2}} - Z^o_{j+\frac{1}{2}} \right) - \left(Z^{vp}_{j-\frac{1}{2}} - Z^o_{j-\frac{1}{2}} \right) \right\} \right] \qquad (107)$$

In order to simplify the notation, the superscripts o, v and w have been omitted in equations (97) through (105). It is understood that if the quantity on the left hand side carries one of these superscripts, all quantities on the right hand side must carry the same superscript.

In equations (97) through (107) we have used the following abbreviations:

$$A_j = \tfrac{1}{2}(A_{j+\frac{1}{2}} + A_{j-\frac{1}{2}})$$

$$C_j = \tfrac{1}{2}(C_{j+\frac{1}{2}} + C_{j-\frac{1}{2}})$$

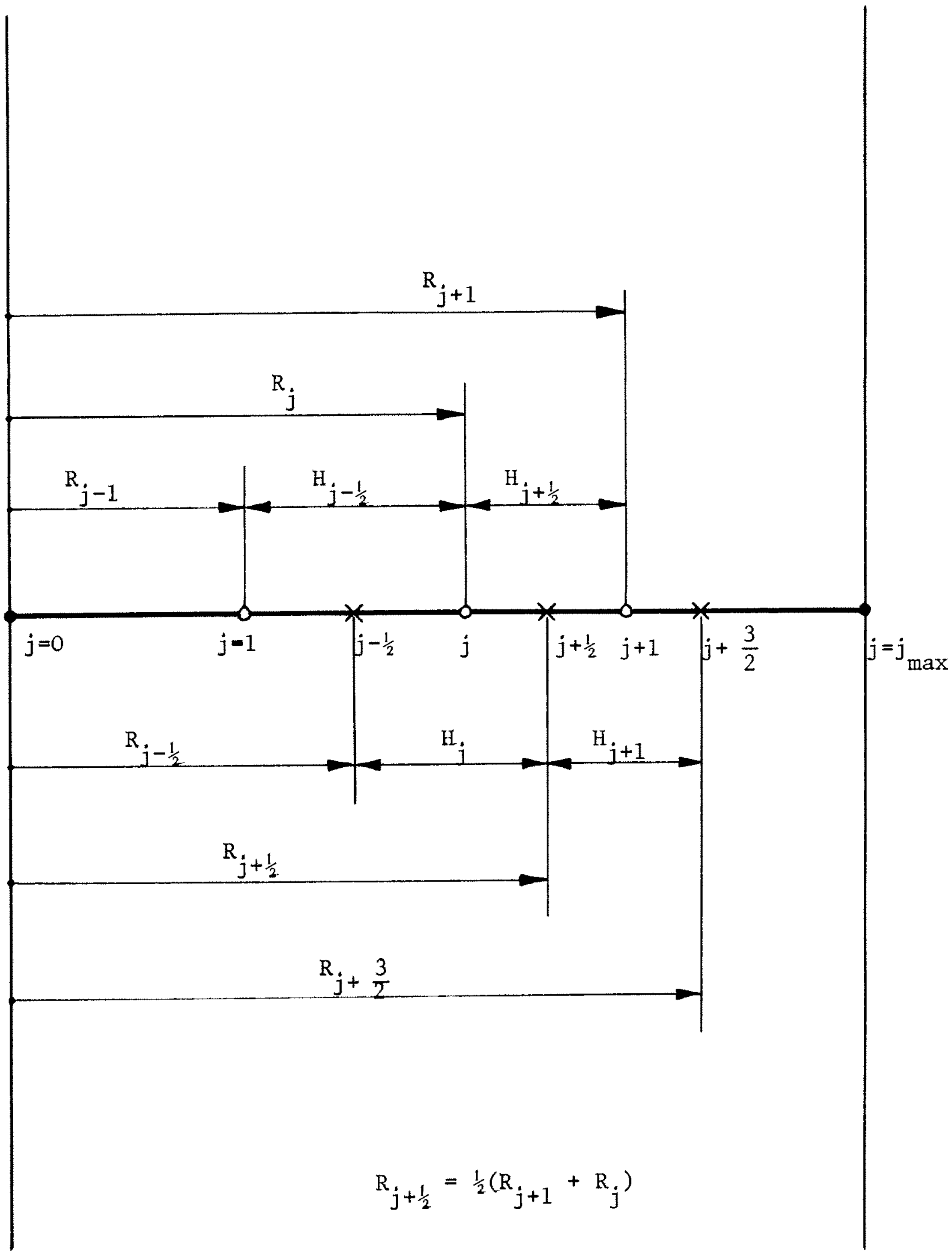
$$Y_j = \tfrac{1}{2}(Y_{j+\frac{1}{2}} + Y_{j-\frac{1}{2}})$$

$$Z_j = \tfrac{1}{2}(Z_{j+\frac{1}{2}} + Z_{j-\frac{1}{2}})$$

$$YD_j = \frac{1}{H_j} (Y_{j+\frac{1}{2}} - Y_{j-\frac{1}{2}})$$

$$ZD_j = \frac{1}{H_j} (Z_{j+\frac{1}{2}} - Z_{j-\frac{1}{2}})$$

The spatial mesh is shown in the Figure below:


R j+1
R j
R j-1
H j-½
H j+½
j=0
j=1
j-½
j
j+½
j+1
j+ 3/2
j=j max
R j-½
H j
H j+1
R j+½
R j+ 3/2
$R_{j+\frac{1}{2}} = \frac{1}{2}(R_{j+1} + R_j)$

2. Method of Solution

Let us assume that all variables are known at time t (superscript o) and that we are looking for the solution at time $t + \Delta t$. The unknowns are the quantities $A^v_{j+\frac{1}{2}}$, $C^w_{j+\frac{1}{2}}$, V^v_j, W^w_j, $E^v_{j+\frac{1}{2}}$, $F^v_{j+\frac{1}{2}}$, $G^w_{j+\frac{1}{2}}$, $Y^v_{j+\frac{1}{2}}$ and $Z^v_{j+\frac{1}{2}}$, for all values of j. The problem is solved by matrix inversion and iteration: A first approximation ($A^{v1}_{j+\frac{1}{2}}$, etc.) is obtained from the difference equations (88)-(96) by putting $\varepsilon = 0$ and using $X^h_{j+\frac{1}{2}}$ and $U^h_{j+\frac{1}{2}}$ from the previous time step. Then, this first approximation becomes the $(p-1)^{th}$ approximation and the difference equations are solved for the p^{th} approximation, using $\varepsilon = 0.5$, for example. This is done by tridiagonal matrix inversion[14]. At this stage, the p^{th} approximation is called the $(p-1)^{th}$ approximation and the procedure is repeated until a convergence criterion is satisfied. It should be noted that, after every iteration, the quantities $A^w_{j+\frac{1}{2}}$, $C^v_{j+\frac{1}{2}}$, V^w_j, W^v_j, $E^w_{j+\frac{1}{2}}$, $F^w_{j+\frac{1}{2}}$ and $G^v_{j+\frac{1}{2}}$ must be computed by interpolation from the quantities $A^v_{j+\frac{1}{2}}$, $C^w_{j+\frac{1}{2}}$, V^v_j, W^w_j, $E^v_{j+\frac{1}{2}}$, $F^v_{j+\frac{1}{2}}$ and $G^w_{j+\frac{1}{2}}$. Furthermore, the spatial mesh at time $t + \Delta t$ must be recomputed at every iteration, according to the prescription:

$$R^{vp}_j = R^o_j + \frac{\Delta t}{2}\left[(V^v_j)^{p-1} + V^o_j \right] \tag{108}$$

$$R^{wp}_j = R^o_j + \frac{\Delta t}{2}\left[(W^w_j)^{p-1} + W^o_j \right] \tag{109}$$

The time step, Δt, must be chosen such that $\frac{\partial v}{\partial r} \Delta t < 1$. This is equivalent to the condition that the mesh size should not change by a large amount in one time step, $(H^v_{j+\frac{1}{2}} - H^o_{j+\frac{1}{2}})/H^o_{j+\frac{1}{2}} < 1$, which is an obvious requirement for a Lagrangian system. The iteration procedure described above converges quite rapidly, usually in about five iterations.

3. Boundary Conditions

Consider first the inner boundary where symmetry about the point $r = 0$ is desired. Here we have

$$v_i = v_o = 0 \tag{110}$$

$$\frac{\partial T_e}{\partial r} = \frac{\partial T_i}{\partial r} = \frac{\partial T_o}{\partial r} = 0 \tag{111}$$

$$\frac{\partial B_z}{\partial r} = \frac{\partial (rB_\theta)}{\partial r} = 0 \tag{112}$$

No boundary condition is required for the density because its spatial derivative at the end points, $j = 0$ and $j = j_{max}$, does not appear in the equations.

The conditions applied at the outer boundary, i.e., at the wall, are not quite so obvious. We are using the following scheme:

$$v_i = 0 \tag{113}$$

$$v_o = - \Phi/n_o \tag{114}$$

$$Q_{e,i,o} \frac{\partial T_{e,i,o}}{\partial r} = - \gamma_{e,i,o} \; T_{e,i,o} \; n_{e,i,o} \left[\frac{2T_{e,i,o}}{m_{e,i,o}} \right]^{\frac{1}{2}} \tag{115}$$

$$\begin{aligned} B_\theta &= B_{\theta,wall} \\ B_z &= B_{z,wall} \end{aligned} \tag{116}$$

where Φ, γ_e, γ_i, γ_o, $B_{\theta,wall}$ and $B_{z,wall}$ are prescribed functions of time. The first of the above conditions, $v_i = 0$, might seem incompatible with a Lagrangian difference scheme. This is not the case, however, because all densities are kept above a certain minimum, typically 10^{-4} times the initial filling density (see paragraph 4., below). Φ is a given flux of neutral atoms being injected from the wall into the plasma. The temperature boundary condition, eq. (115), states that the energy flux, at the wall, due to thermal conduction must equal the energy lost to the wall, per unit time, as a result of inelastic collisions. The fraction of particles that collide inelastically with the wall is expressed in the quantities γ_e, γ_i, γ_o (typically, we assume $\gamma_e = \gamma_i = \gamma_o = 0.1$). The magnetic field at the outer boundary is either given as a predetermined function of time or it is computed from external circuit equations.

4. Minimum Density

It is clear that the density cannot be allowed to approach zero because certain terms in the equations are divided by n. Therefore, some minimum density must be specified. As soon as the density tends to go below that value, at a certain space-time point, it is artificially restored to the specified minimum. This means, of course, that matter is being introduced into the plasma. We have investigated the effect of this "source term" and found that if the minimum is chosen sufficiently low ($\sim 10^{-4}$ times initial density), then it has practically no influence at all.

IV. Results

In this section, some of the results of a typical calculation are presented. The following input parameters were used:

discharge tube radius	2.45 cm
tube length	50 cm
filling pressure	80 mTorr He
initial ionization	30 % singly ionized
magnetic field at the wall	as shown in Fig. 1.
initial temperatures	$T_e = T_i = T_o = 1$ ev
iteration parameter	$\varepsilon = 0.5$
resistivity	classical (eq. (80))
flux of neutral atoms from the wall	10^{20} cm^{-2} sec^{-1}
flux of ions and electrons from the wall	0
number of radial mesh points	20

Fig. 1. shows the magnetic field components, B_θ and B_z, at the wall. They were obtained from measurements in an actual experiment[5]. Fig.2. and Fig. 3. are "three-dimensional" plots of electron and neutral densities, as functions of position and time. Fig. 4. compares the average densities of electrons, singly and doubly charged ions, and neutral Helium atoms, as functions of time. Electron and ion temperature distributions are shown in Fig. 5. and the average temperatures of electrons, ions and neutrals are plotted in Fig. 6. Fig. 7. shows the magnitude of the magnetic field, $B = (B_\theta^2 + B_z^2)^{\frac{1}{2}}$ as a function of radius, at various times, and Fig. 8. shows some of the current distributions $(j = (j_\theta^2 + j_z^2)^{\frac{1}{2}})$. Velocity profiles of ions and neutrals are shown in Fig. 9. Finally, we have plotted the thermal, kinetic, ionization and magnetic energies as functions of time in Fig. 10.

V. Acknowledgments

The author wishes to thank Dr. K.V. Roberts of Culham Laboratory and Drs. E.S. Weibel and F.S. Troyon of Lausanne for a number of discussions on the subject of this paper. He also wishes to thank the staff of the "Centre de Calcul de l'Ecole Polytechnique Fédérale, Lausanne" and the staff of the "Rechenzentrum der Eidgenössischen Technischen Hochschule, Zürich" for their helpfulness and cooperation.

This work was supported by the "Fonds National Suisse de la Recherche Scientifique".

The computer programs described in this report are available upon request from the author.

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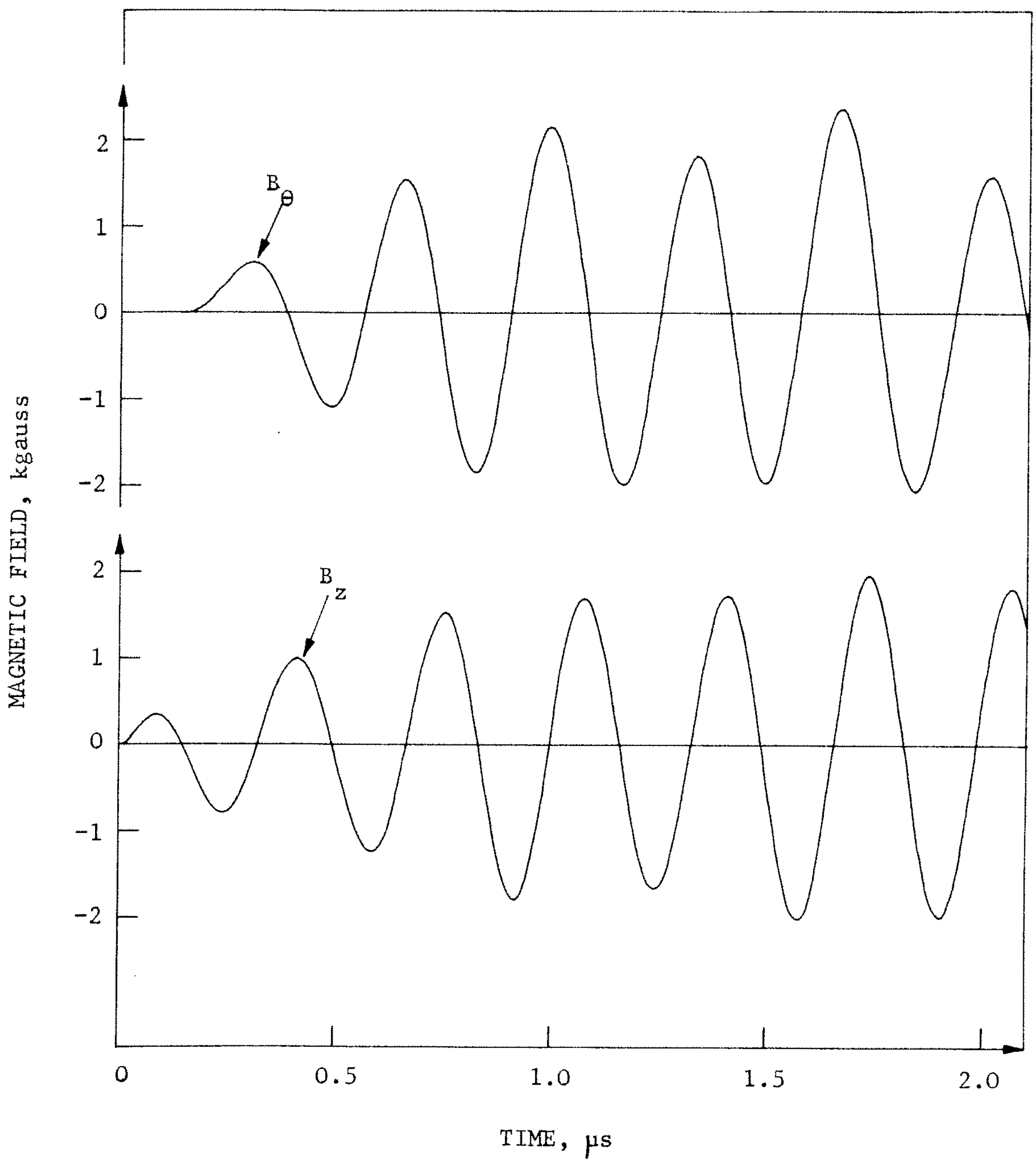

FIG.1. MAGNETIC FIELDS AT THE WALL

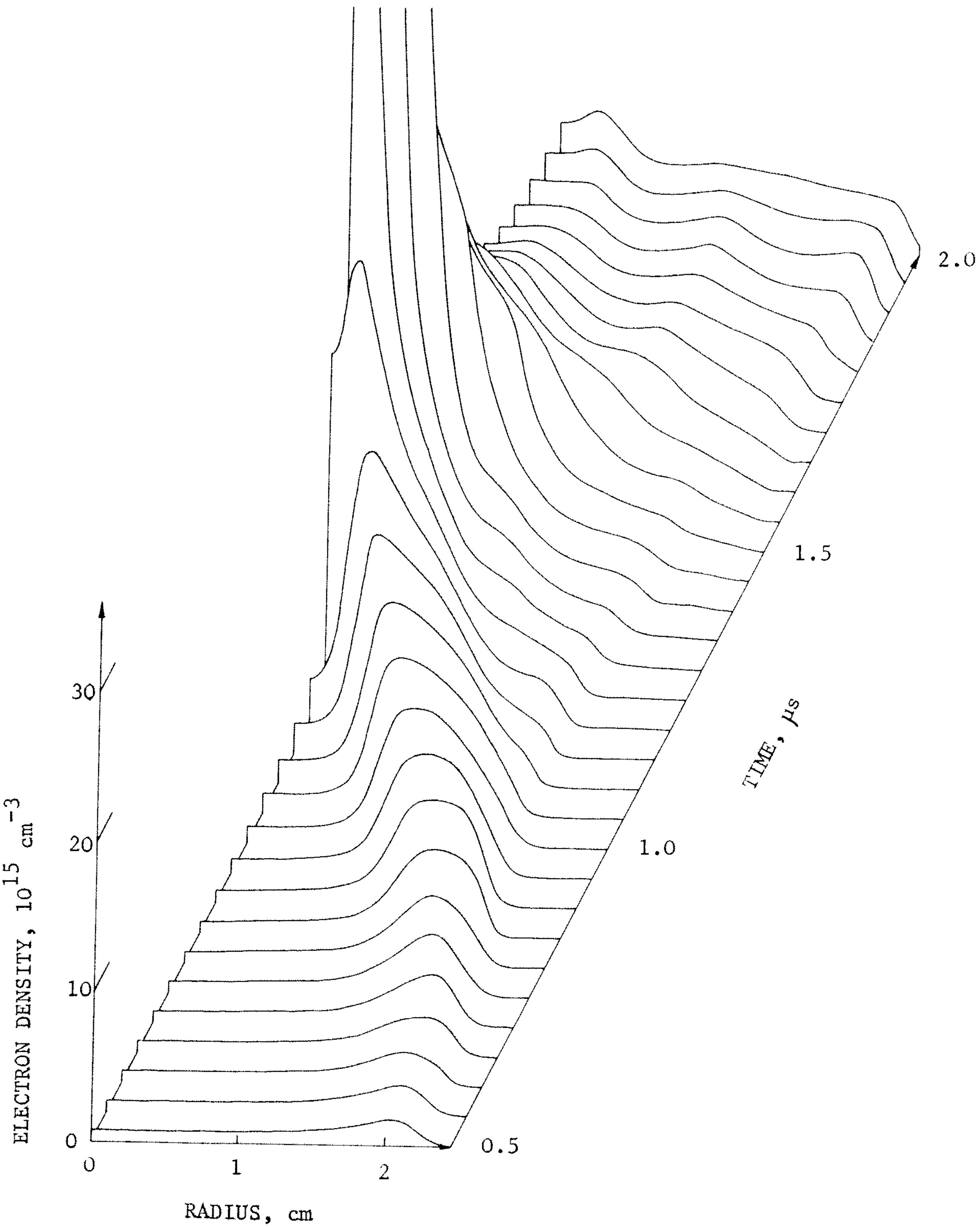

FIG.2. ELECTRON DENSITY

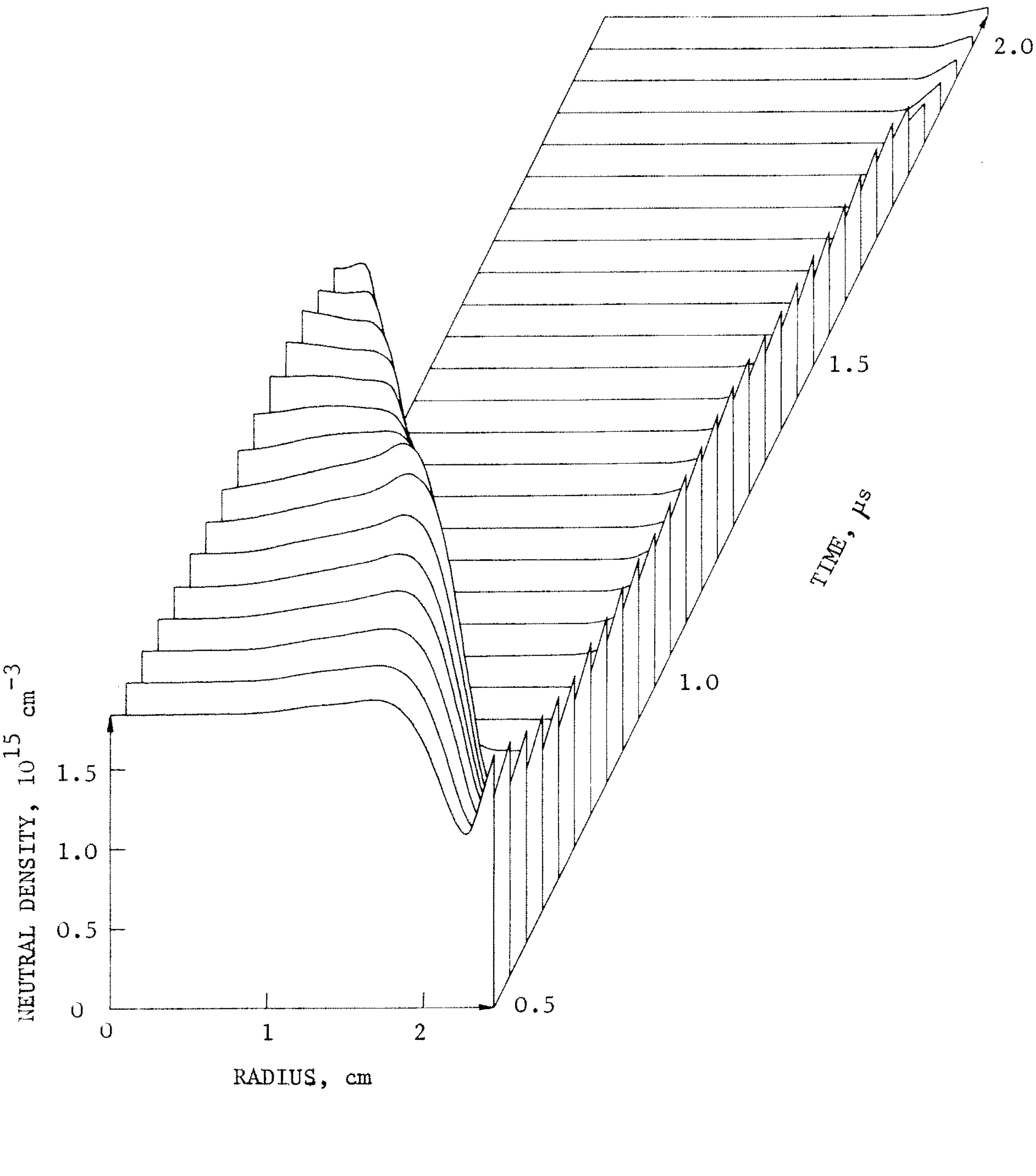

FIG.3. NEUTRAL DENSITY

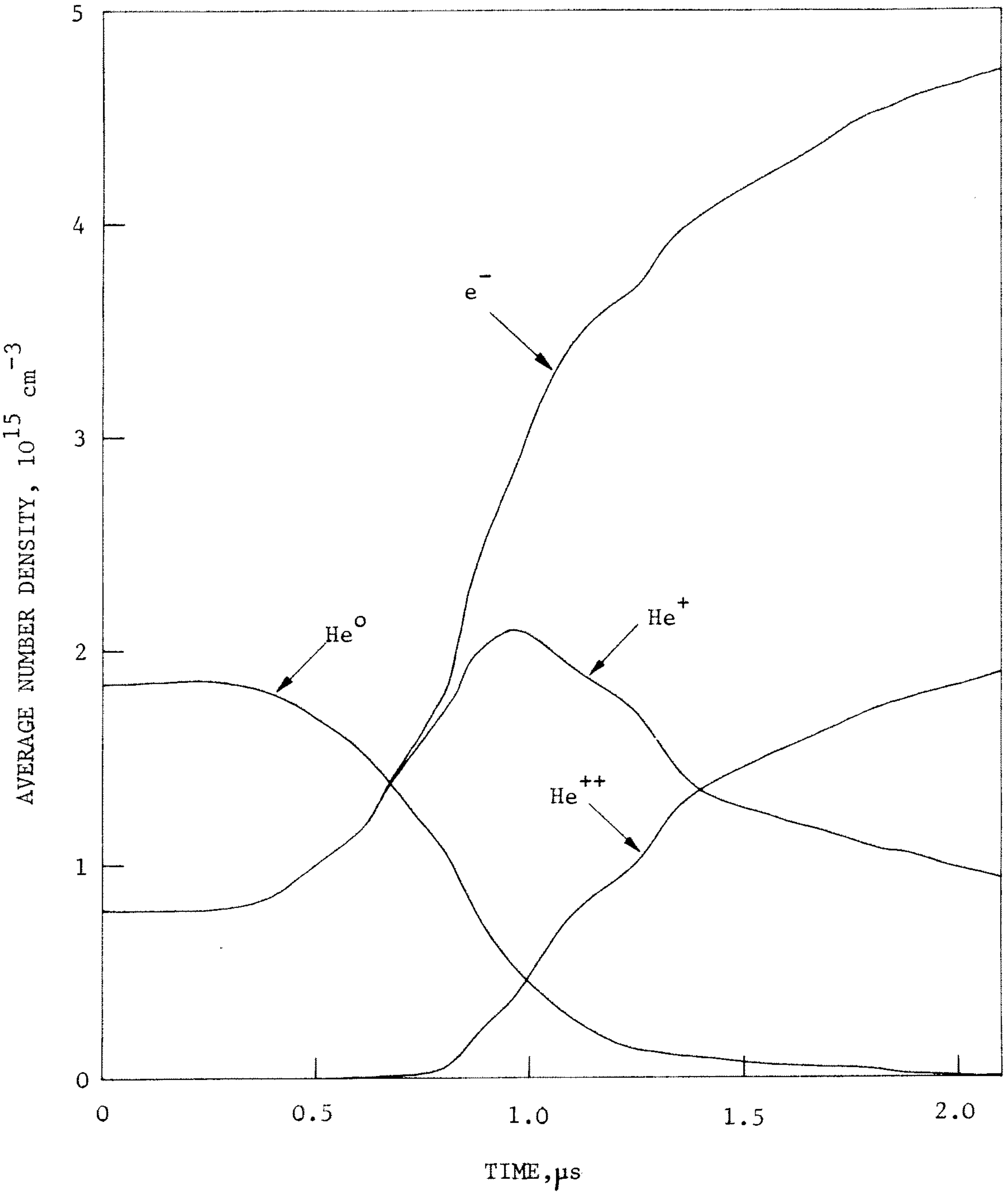

FIG.4. AVERAGE DENSITY OF ELECTRONS, IONS AND NEUTRALS

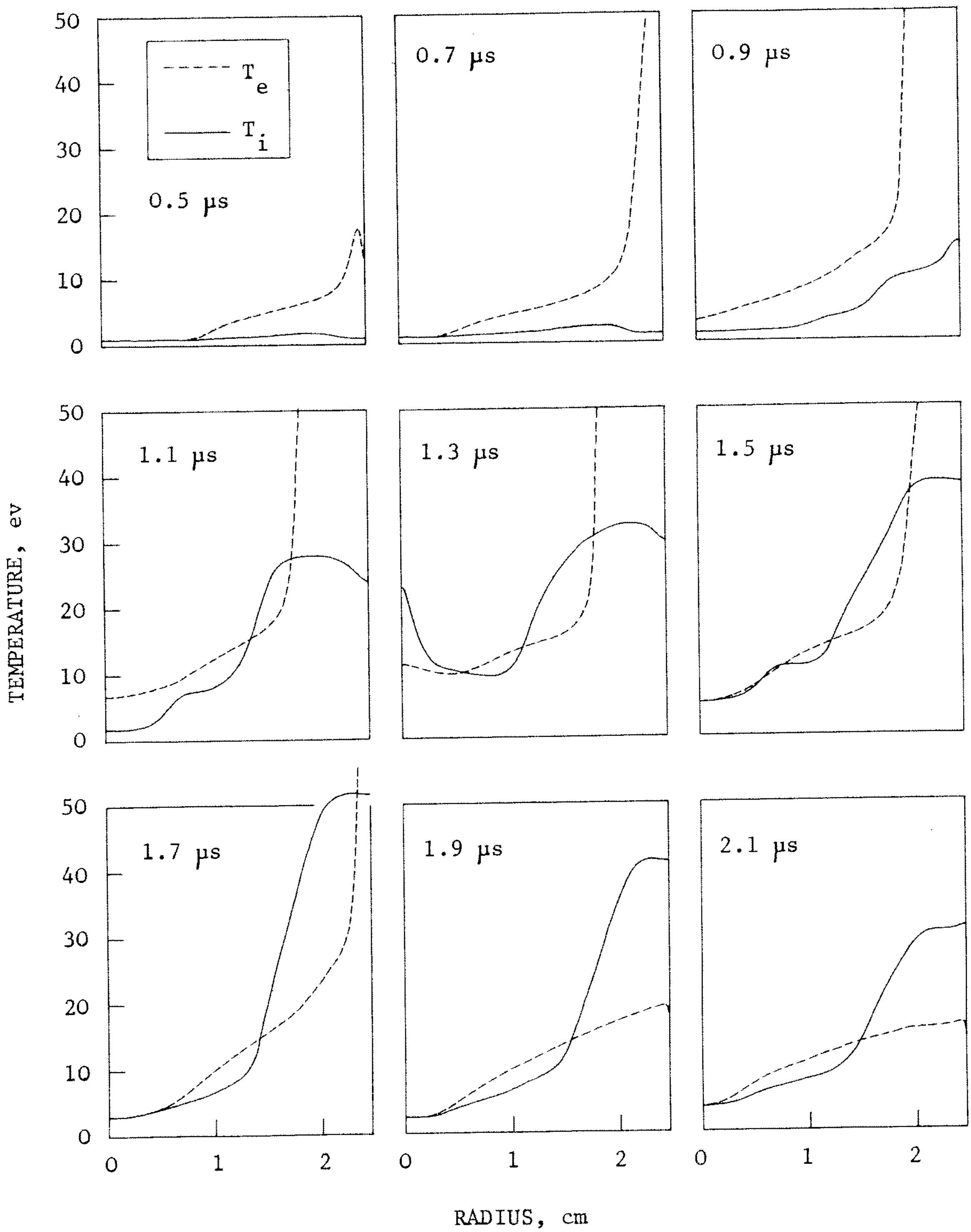

FIG.5. ELECTRON AND ION TEMPERATURES

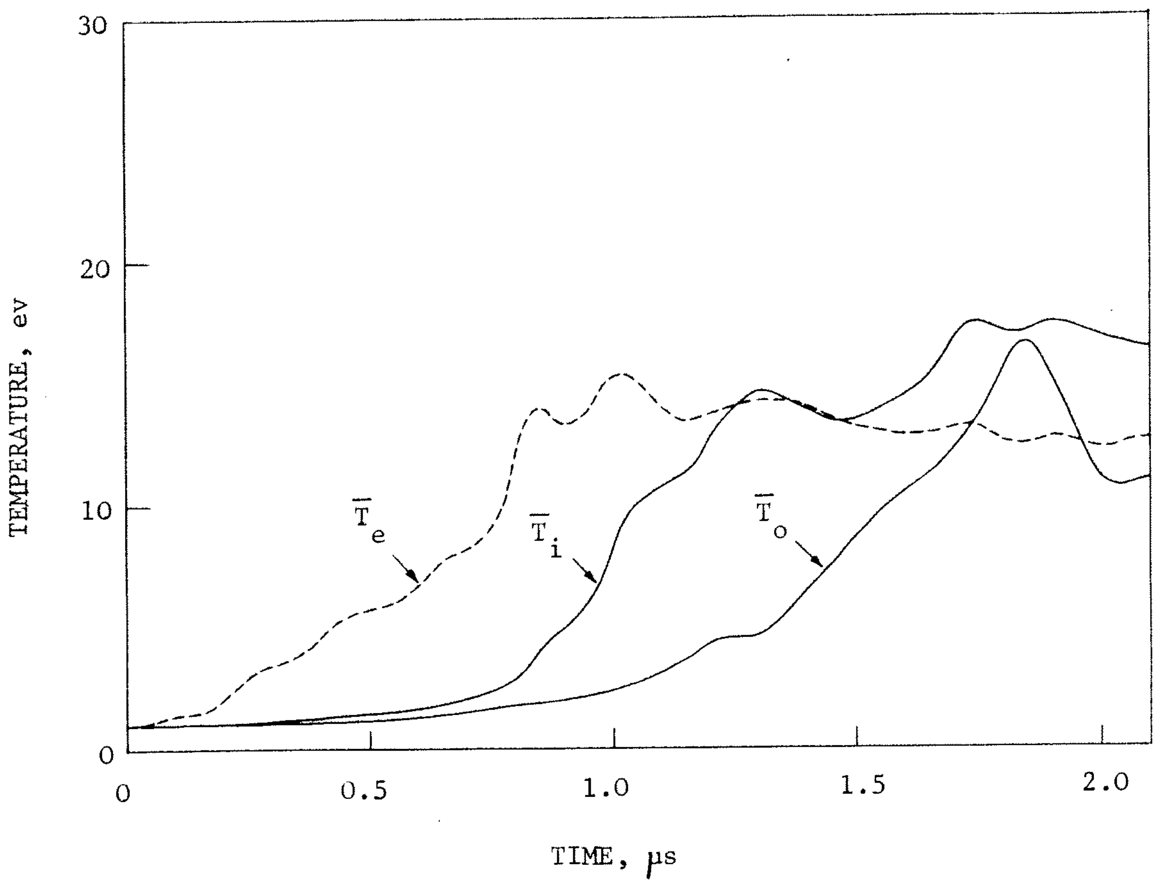

FIG.6. AVERAGE TEMPERATURE OF ELECTRONS, IONS AND NEUTRALS

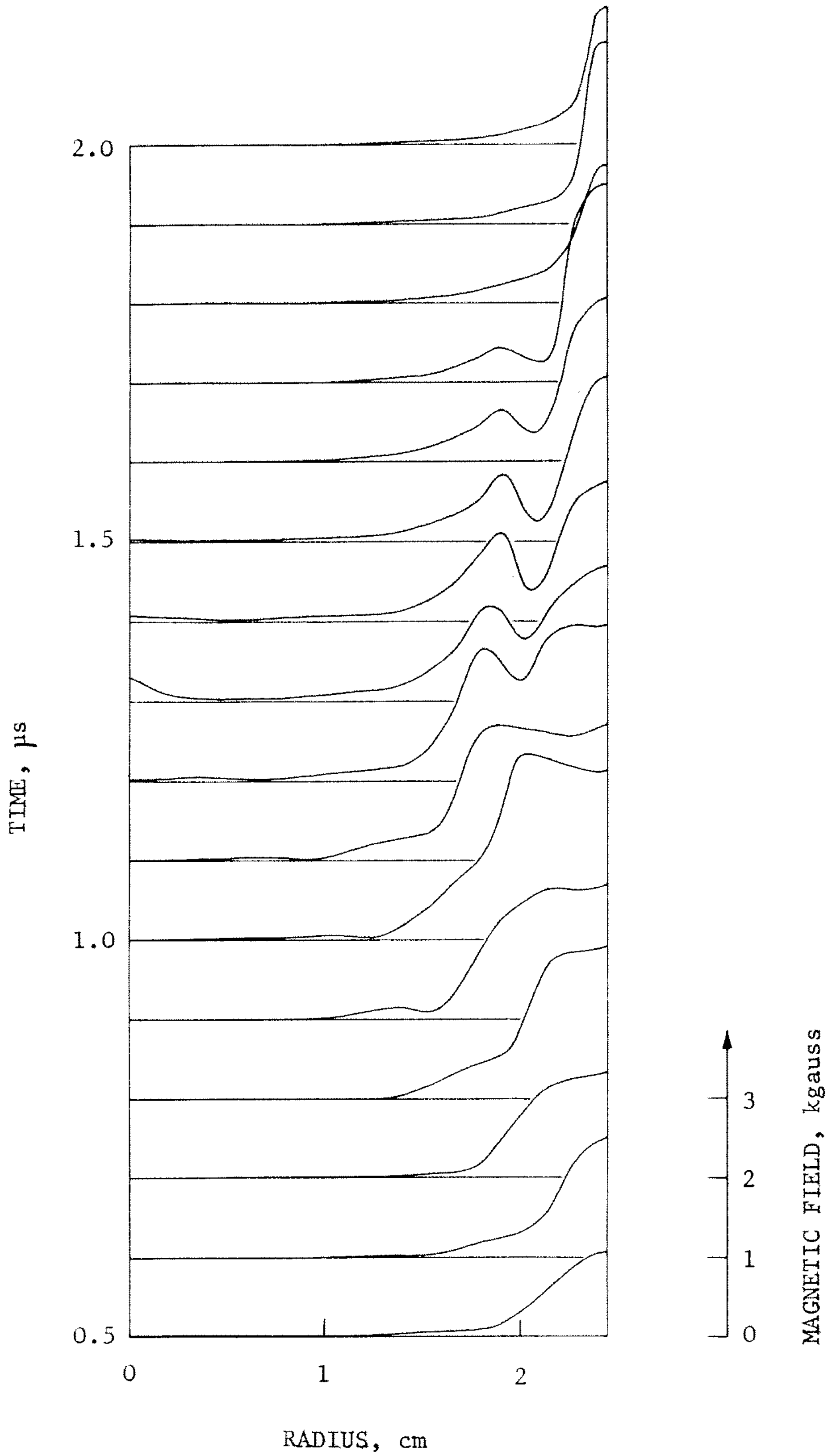

FIG.7. MAGNITUDE OF MAGNETIC FIELD VECTOR

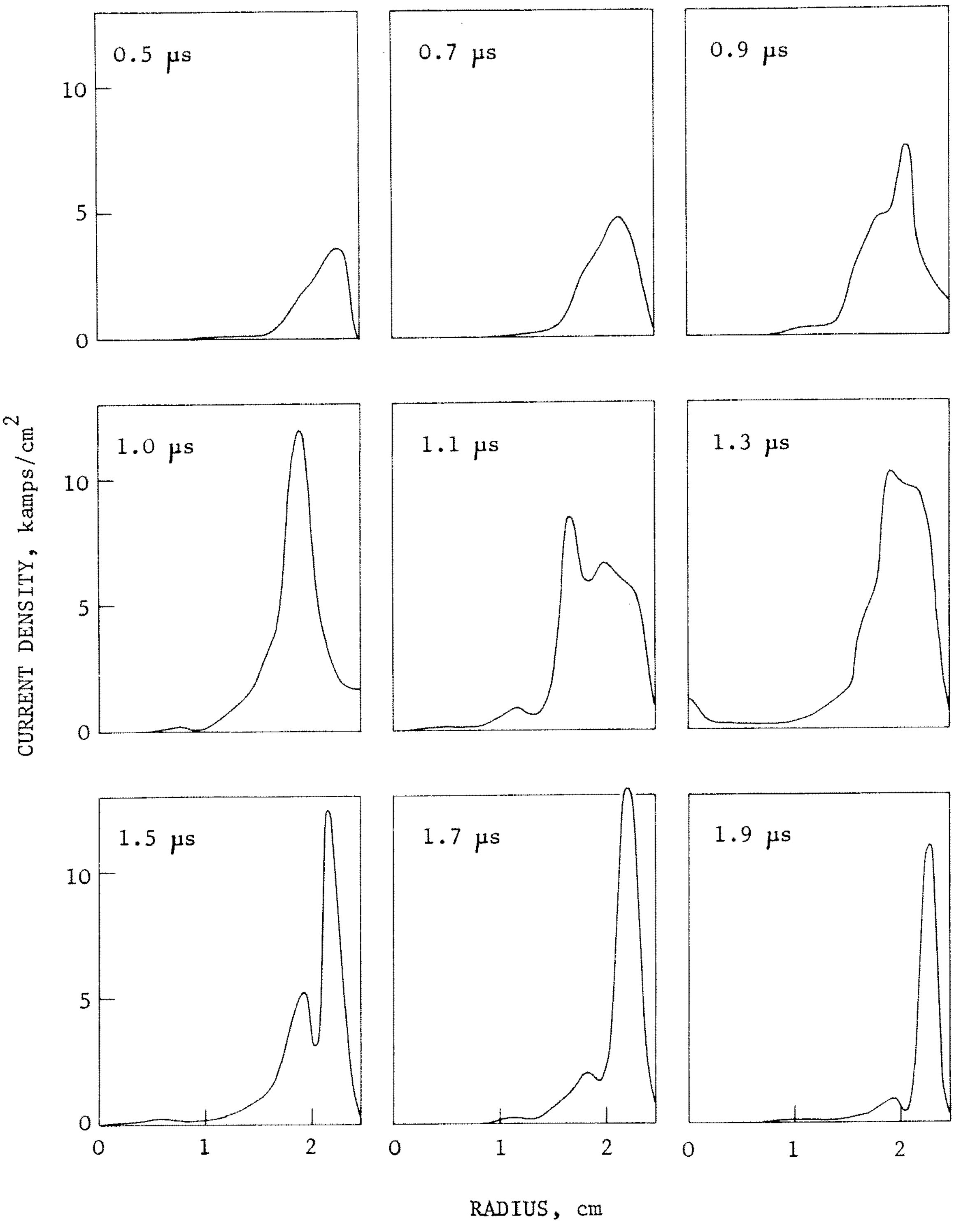

FIG.8. MAGNITUDE OF CURRENT DENSITY VECTOR

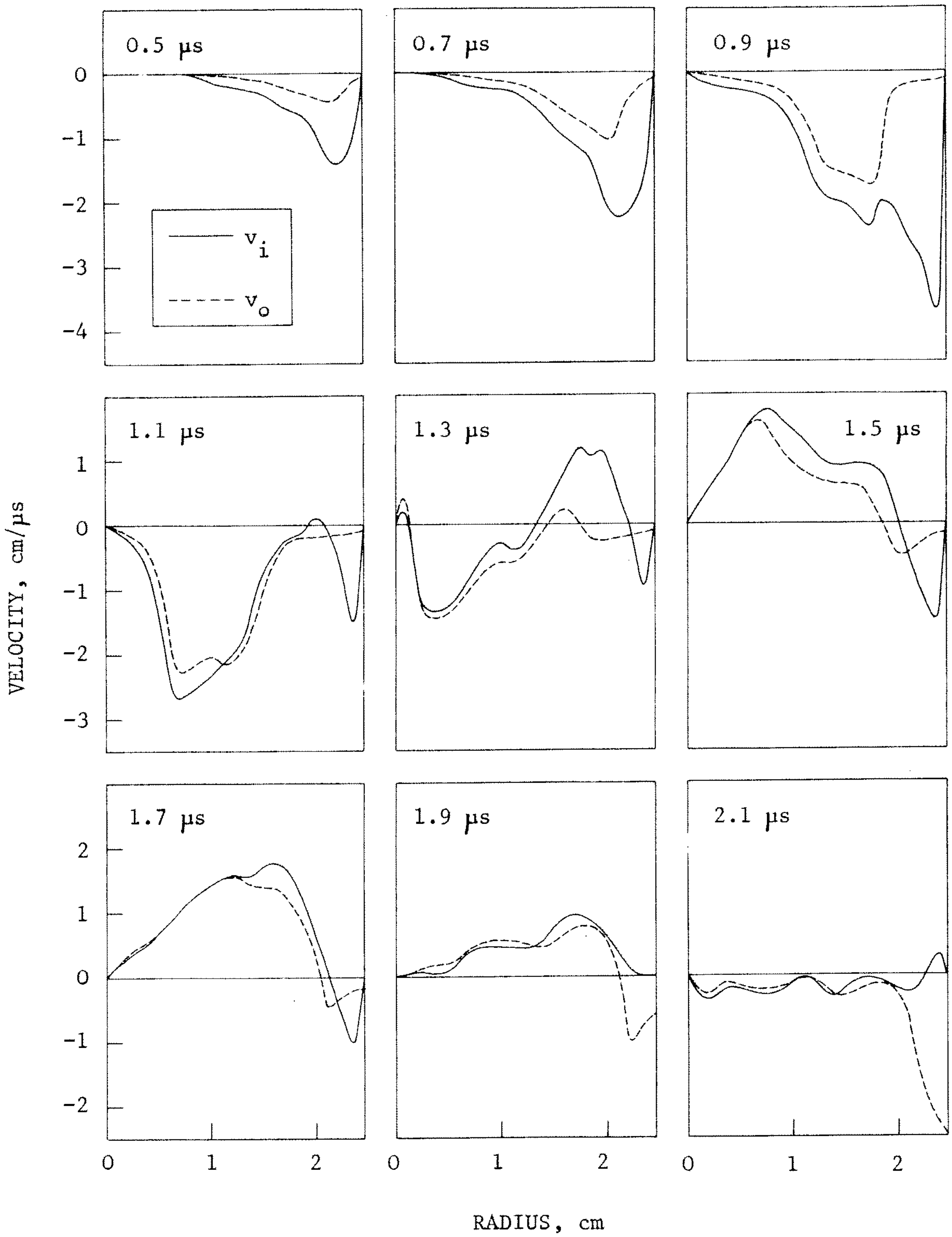

FIG.9. FLUID VELOCITIES OF IONS AND NEUTRALS

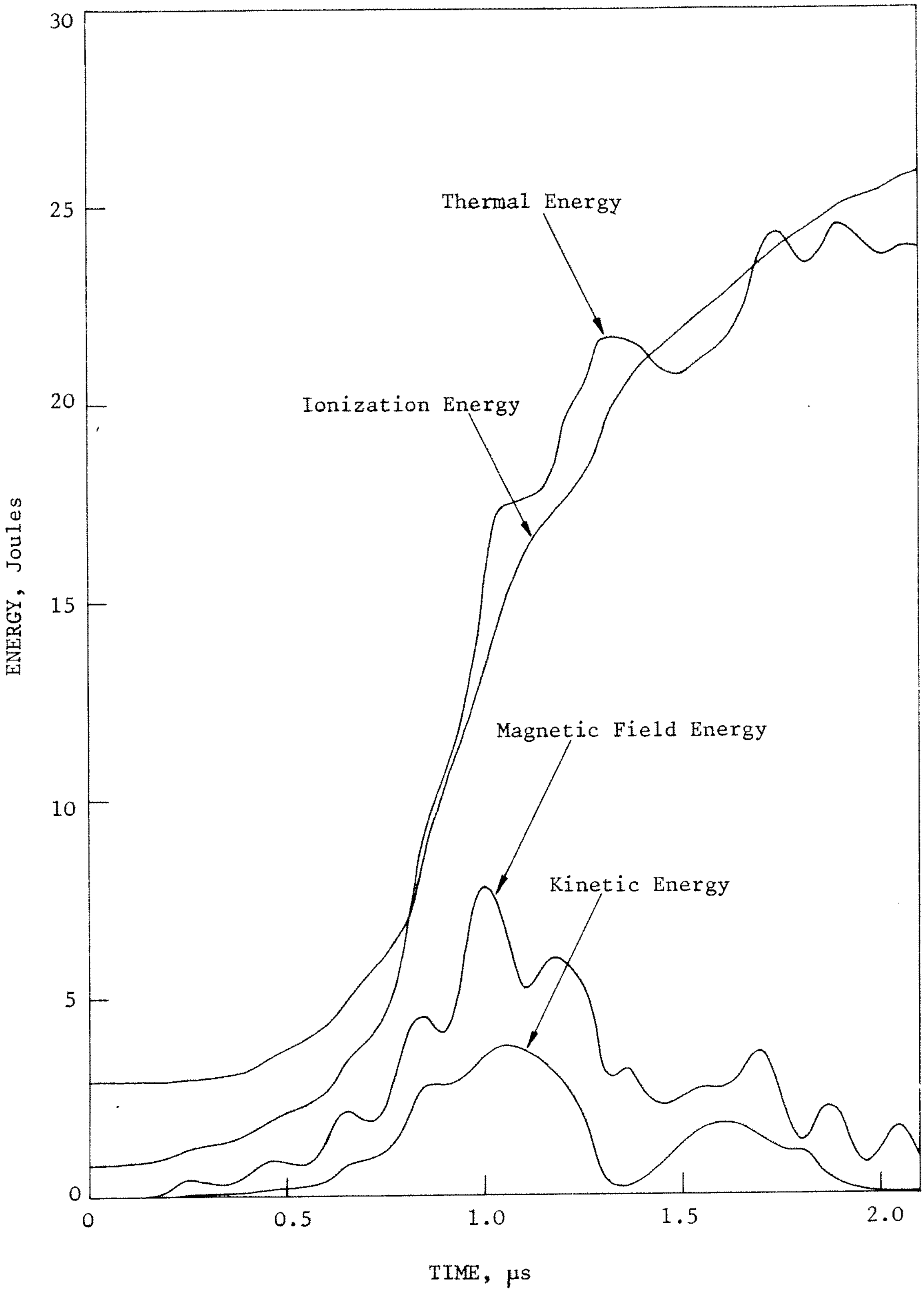

FIG.10. ENERGY BALANCE